Optimal pair–trade execution with generalized cross–impact*

Masamitsu OHNISHI†‡ and Makoto SHIMOSHIMIZU†¶

†Graduate School of Economics, Osaka University, Osaka, 560–0043, Japan

‡Center for Mathematical Modeling and Data Science, Osaka University, Osaka, 560–8531, Japan

¶Graduate School of Management, Tokyo Metropolitan University, Tokyo, 100–0005, Japan

Abstract

We examine a discrete–time optimal pair–trade execution problem with generalized cross–impact. This research is an extension of [14], which consider the price impact of aggregate random orders posed by small traders with a Markovian dependence. We focus on how a risk–averse large trader optimally executes two correlated assets to maximize his/her expected utility from the final wealth over a finite horizon. A stochastic dynamic programming modeling constitutes the basis for the formulation of the optimal pair–trade execution problem. Then, under some regularity conditions, the backward induction method of dynamic programming enables us to derive the optimal pair–trade execution strategy and its associated optimal value function. Besides, we reveal that the trading orders of each risky asset posed by small traders do affect the optimal execution volume of both risky assets.

1 Introduction

A considerable number of empirical researches conducted in the last decade show that considering the cross–impact of multi assets is important when constructing a portfolio or an optimal execution strategy (e.g., [3], [36]). Along with these empirical researches, a multitude of theoretical studies analyze the optimal portfolio liquidation strategies of multi assets for a single or multiple large investors (e.g., [5], [9], [17], [18], [28]). These works are vulnerable from the practitioners' point of view.

This paper addresses an optimal pair–trade execution problem for a single large trader. We the following situations which the institutional trader (or large trader) may face in a real marketplace: Institutional traders manage their trading with multiple assets all or some of which are correlated with each other to mitigate the price risk. The so–called 'pair trading' has already been discussed in much empirical literature, clarifying the importance of trading multiple assets (e.g., in [15]). In line with the significant insights from other researches, we consider an optimal pair–trade strategy for an institutional trader. Also, the market model includes the effect of temporary, permanent, and transient price impacts caused by both the large trader and the (aggregate) trading volume submitted by small traders. This research is, to the best of our knowledge, the first paper to incorporate the transient price impact as well as the effect of small traders' submission into the optimal pair–trade execution strategy.

The organization of this paper is as follows. Section 2 describes the market model and performance criteria. A stochastic dynamic programming approach leads us to derive the optimal pair–trade execution strategy. We present the closed–form solution of the optimal pair–trade execution strategy and its associated value function. Section 3 concludes.

*This work was supported by Japan Society for the Promotion of Science under KAKENHI [Grant Numbers 17K01255, and 19J10501].

2 Price impact model with cross–impact

In a discrete time framework $t \in \{1, \dots, T, T+1\}, (T \in \mathbb{Z}_+ := \{1, 2, \dots\})$, we assume that one large trader purchases two risky assets in a trading market. It is also supposed that he/she has a Constant Absolute Risk Aversion (CARA) von Neumann-Morgenstern (vN-M) utility (or negative exponential utility) utility function with the absolute risk aversion rate $\gamma > 0$.

2.1 Market model

For each asset $i \in \{1, 2\} =: \mathcal{I}$, he/she must purchase $\mathfrak{Q}^i (\in \mathbb{R})$ volume by the time $T+1$. Let $q_t^i (\in \mathbb{R})$ for $i \in \mathcal{I}$ represent large amount of orders of asset i submitted by the large trader at time $t \in \{1, \dots, T\} =: \mathcal{T}$. Then, we denote by $\overline{Q}_t^i$ the remained execution volume of asset i, that is, the number of shares remained to purchase by the large trader at time $t \in \{1, \dots, T, T+1\}$.[1] From this assumption, we have $\overline{Q}_1^i = \mathfrak{Q}^i$, $\overline{Q}_T^i = 0$ and $\overline{Q}_{t+1}^i = \overline{Q}_t^i - q_t^i$ for $i \in \mathcal{I}$. In a stacked form,

$$\overline{\mathbf{Q}}_{t+1} := \begin{pmatrix} \overline{Q}_{t+1}^1 \\ \overline{Q}_{t+1}^2 \end{pmatrix} = \overline{\mathbf{Q}}_t - \mathbf{q}_t := \begin{pmatrix} \overline{Q}_t^1 - q_t^1 \\ \overline{Q}_t^2 - q_t^2 \end{pmatrix} (\in \mathbb{R}^2), \quad t = 1, \dots, T. \tag{2.1}$$

with the initial and terminal conditions:

$$\overline{\mathbf{Q}}_1 = \begin{pmatrix} \mathfrak{Q}_1^1 \\ \mathfrak{Q}_1^2 \end{pmatrix} (\in \mathbb{R}^2); \quad \overline{\mathbf{Q}}_{T+1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} =: \mathbf{0} \ (\in \mathbb{R}^2). \tag{2.2}$$

Then, the (aggregate) trading volumes submitted by small traders at time $t \in \mathcal{T}$ are assumed to have a Markov dependence described as follows:

$$\begin{aligned} &\mathbf{v}_0 \equiv \mathbf{0}; \\ &\mathbf{v}_{t+1}|_{\mathbf{v}_t} \sim N_{\mathbb{R}^2}\left(\mathbf{a}_{t+1}^v + \mathbf{b}_{t+1}^v \mathbf{v}_t, \boldsymbol{\Sigma}_{t+1}^v\right), \quad t = 0, \dots, T-1, \end{aligned} \tag{2.3}$$

where

$$\mathbf{a}_t := \begin{pmatrix} a_t^1 \\ a_t^2 \end{pmatrix} \in \mathbb{R}^2; \quad \mathbf{b}_t := \begin{pmatrix} b_t^{11} & b_t^{12} \\ b_t^{21} & b_t^{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}; \quad \boldsymbol{\Sigma}_t^v := \begin{pmatrix} \sigma_t^{11} & \sigma_t^{12} \\ \sigma_t^{12} & \sigma_t^{22} \end{pmatrix} \in \mathbb{R}^{2\times 2}. \tag{2.4}$$

Note that $\mathbf{a}_t^v, \mathbf{b}_t^v$, and $\boldsymbol{\Sigma}_t^v$ are deterministic functions of time t. The dynamics of $\mathbf{v}_t$ can be rewritten as follows:

$$\begin{aligned} &\mathbf{v}_0 \equiv \mathbf{0}; \\ &\mathbf{v}_{t+1} = (\mathbf{a}_{t+1}^{\mathbf{v}} + \mathbf{b}_{t+1}^{\mathbf{v}} \mathbf{v}_t) + \boldsymbol{\sigma}_{t+1}^{\mathbf{v}} \boldsymbol{\omega}_{t+1}, \quad t = 1, \dots, T-1, \end{aligned} \tag{2.5}$$

where $\boldsymbol{\omega}_t \sim N_{\mathbb{R}^2}(0, \mathbf{I}_2)$ for all $t \in \mathcal{T}$ and $\boldsymbol{\sigma}_{t+1}^{\mathbf{v}} (\boldsymbol{\sigma}_{t+1}^v)^\top := \boldsymbol{\Sigma}_t^{\mathbf{v}}$ is a Cholesky decomposition of $\boldsymbol{\Sigma}_t^{\mathbf{v}}$ for all $t \in \mathcal{T}$. In the rest of this paper, $\mathbf{I}_2$ indicates the 2×2 identity matrix.

Remark 2.1 (Effect of $\mathbf{b}^{\mathbf{v}}$)**.** Eq. (2.5) becomes a covariance–stationary VAR (1) process when all values z satisfying $|\mathbf{I}_2 - z\mathbf{b}_t^{\mathbf{v}}| = 0$ lies outside of the unit circle. Equivalent conditoins are:

$$|z| = \left| \frac{\mathrm{Tr}\,(\mathbf{b}_t^{\mathbf{v}}) \pm \sqrt{\mathrm{Tr}\,(\mathbf{b}_t^{\mathbf{v}})^2 - 4\,|\mathbf{b}_t^{\mathbf{v}}|}}{2\,|\mathbf{b}_t^{\mathbf{v}}|} \right| > 1, \tag{2.6}$$

where for any squared matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ $|\mathbf{A}|$ stands for the determinant of $\mathbf{A}$ and $\mathrm{Tr}\,(\mathbf{A})$ is the trace of $\mathbf{A}$ defined as a map from $\mathbb{R}^{n\times n}$ to $\mathbb{R}$ such that

$$\mathrm{Tr}\,(\mathbf{A}) = \mathrm{Tr} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} := \sum_{i=1}^n a_{ii}. \tag{2.7}$$

[1] The positive q_t^i for $t \in \mathcal{T}$ stand for the acquisition and negative q_t^i the liquidation of the risky asset $i \in \mathcal{I}$. This setting allows us to establish a similar setup for a selling problem of a large trader.

Moreover, if the following determinant: $|\mathbf{I}_2 - \mathbf{b}_t^{\mathbf{v}}|$ is zero, then the Eq (2.5) contains at least one unit root. For the details, see, e.g., [21]. The continuous–time version, that is, the multivariate OU process is discussed in [30].

We assume the market price (or quoted price) of the risky asset $i \in \mathcal{I}$ at time $t \in \{1, \ldots, T, T+1\}$ is set as P_t^i. Then, the execution price of the asset i becomes $\widehat{P}_t^i$ since the large trader submits a large number of orders, influencing the asset price at which he/she execute the transaction. In the rest of this paper, we assume that for each asset $i \in \mathcal{I}$ submitting one unit of (large) order at time $t \in \mathcal{T}$ causes the instantaneous price impact denoted as $\lambda_t^i (> 0)$. We also assume that the aggregate trading volume posed by small traders also has some impact on the execution price. $\kappa_t^i (> 0)$ represents the price impact per unit at time $t \in \mathcal{T}$ caused by small traders for asset $i \in \mathcal{I}$. The (aggregate) trading volume submitted by small traders at time $t \in \mathcal{T}$ is assumed to be a sequence of random variables $\mathbf{v}_t$, which has a Markovian dependence and follows a normal distribution with the following mean and variance:

In the sequel of this paper, the buy–trade and sell–trade of a large trader are supposed to induce the same (instantaneous) linear price impact.[2] We consider the cross–impact caused by the order submission of both the large trader and small traders. From this assumption, we define the execution price, $\widehat{\mathbf{P}}_t := \begin{pmatrix} \widehat{P}_t^1 \\ \widehat{P}_t^2 \end{pmatrix}$, in the form of a linear price impact model as follows:

$$\widehat{\mathbf{P}}_t = \mathbf{P}_t + (\mathbf{\Lambda}_t \mathbf{q}_t + \boldsymbol{\kappa}_t \mathbf{v}_t), \quad t = 1, \ldots, T. \tag{2.8}$$

where

$$\mathbf{P}_t := \begin{pmatrix} P_t^1 \\ P_t^2 \end{pmatrix}; \quad \mathbf{\Lambda}_t := \begin{pmatrix} \lambda_t^1 & \lambda_t^{12} \\ \lambda_t^{21} & \lambda_t^2 \end{pmatrix}; \quad \boldsymbol{\kappa}_t := \begin{pmatrix} \kappa_t^1 & \kappa_t^{12} \\ \kappa_t^{21} & \kappa_t^2 \end{pmatrix}, \tag{2.9}$$

and λ_t^{ij} represents the cross–impact of asset j's order execution on asset i per unit caused by the large trader, and κ_t^{ij} caused by small traders at time $t \in \mathcal{T}$. [36] show that, in order for the market not to allow the dynamic arbitrage opportunities, the cross–impact must take the form of a symmetric matrix. From this viewpoint, we have the following assumption.

Assumption 2.1. $\mathbf{\Lambda}_t$ and $\boldsymbol{\kappa}_t$ are symmetric for all $t \in \mathcal{T}$.

Adding to the above assumption, we restrict our analysis of the price impact with cross-impacts on the following case.

Assumption 2.2. $\mathbf{\Lambda}_t$ and $\boldsymbol{\kappa}_t$ are positive definite matrices for all $t \in \mathcal{T}$.

Remark 2.2 (Symmetricity of a positive definite matrix)**.** We can always regard a positive definite $n \times n$ matrix as a symmetric one since for each positive definite matrix $\mathbf{\Lambda} \in \mathbb{R}^{n \times n}$ a symmetric and positive definite matrix $\mathbf{\Lambda}^* (:= \frac{1}{2}\left(\mathbf{\Lambda} + \mathbf{\Lambda}^\top\right))$ such that $\mathbf{x}^\top \mathbf{\Lambda} \mathbf{x} = \mathbf{x}^\top \mathbf{\Lambda}^* \mathbf{x}$.

Remark 2.3 (Diagonally dominant matrices)**.** For a complex– (or real–)valued $n \times n$ matrix $\mathbf{\Lambda}$, we say that $\mathbf{\Lambda}$ is a **strictly diagonally dominant matrix** if

$$|\lambda_{ii}| > \sum_{j \neq i} |a_{ij}|, \tag{2.10}$$

for all $i \in \{1, \ldots, n\}$. Referring to this definition, we can replace the Assumption 2.2 as follows.

[2]This assumption would be inconsistent with the situation observed in a real market. [7] and [8] conduct a linear regression of price changes on net order–flow using trading data obtained from Nasdaq to estimate the permanent and temporary price impact. Then they reveal that assuming linear price impact is compatible with the real stock market and that the price impact caused by both buy and sell trades are deemed as same from a statistical analysis point of view.

Assumption 2.2'. $\mathbf{\Lambda}_t$ and $\boldsymbol{\kappa}_t$ are strictly diagonally dominant matrices.

If $\mathbf{\Lambda}_t$ is a strictly diagonally dominant matrix, regardless of the symmetricity of the matrix, the fact that $\lambda_t^{11} > 0$ and

$$|\mathbf{\Lambda}_t| = \lambda_t^{11}\lambda_t^{22} - \lambda_t^{12}\lambda_t^{21} > \lambda_t^{11}\lambda_t^{22} - \left|\lambda_t^{12}\right|\left|\lambda_t^{21}\right| > 0 \tag{2.11}$$

reveals the positive definiteness of $\mathbf{\Lambda}_t$, and the same holds for $\boldsymbol{\kappa}_t$. If we weaken the assumption 2.3 to weakly diagonally dominant matrices, then both $\mathbf{\Lambda}_t$ and $\boldsymbol{\kappa}_t$ become positive semidefinite matrices. Moreover, we can further say that if $\mathbf{\Lambda}_t$ is a strictly diagonally dominant matrix, then it also becomes a generalized diagonally dominant matrix. For more detail, see, e.g., [37].

Remark 2.4 (Positive definiteness of the price impact coefficient matrix)**.** We can interpret the assumption above as follows. Consider the case $\lambda_t^1 = \lambda_t^2 = \lambda_t$ (> 0) and $\lambda_t^{12} = \lambda_t^*$ for all $t \in \mathcal{T}$. Then, using $\xi \in \mathbb{R}$, the characteristic equation results in

$$|\mathbf{\Lambda}_t - \xi\mathbf{I}_2| = 0 \Longleftrightarrow (\lambda_t - \xi)^2 - (\lambda_t^*)^2 = 0, \tag{2.12}$$

and solving the above equation with respect to ξ yields the eigen value:

$$\xi = \lambda_t \pm \lambda_t^*. \tag{2.13}$$

By the assumption $\lambda_t > 0$, the condition $\lambda_t \pm \lambda_t^* > 0$, i.e., $\lambda_t^1 = \lambda_t^2 > \left|\lambda_t^{12}\right|$ must hold for $\mathbf{\Lambda}_t$ to be a positive definite matrix for all $t \in \mathcal{T}$. This means that the price impact of the asset $i \in \{1, 2\}$ on the execution price on asset $j(\neq i) \in \{1, 2\}$ must be less than that on asset i itself, which is rather a realistic situation considering the real marketplace.

Remark 2.5 (An extension from two assets to n assets $(n \geq 3)$)**.** We can extend this model to the case of n risky assets $(n \geq 3)$. Consider, for example, the case that a large trader trades three risky assets and the price impact caused by the large trader is the following form:

$$\mathbf{\Lambda}_t = \begin{pmatrix} \lambda_t^1 & 0 & \lambda_t^{13} \\ 0 & \lambda_t^2 & 0 \\ \lambda_t^{31} & 0 & \lambda_t^3 \end{pmatrix}, \tag{2.14}$$

meaning that the execution of asset 1 causes a cross impact on the priced of asset 3 and vice versa, and no cross impacts exist between assets 1 and 2 as well as assets 2 and 3. Then, by a similar examination as in Remark 2.2 with the assumption $\lambda_t^1 = \lambda_t^2 = \lambda_t^3 = \lambda_t$ for all $t \in \{1, \ldots, T\}$ and $\lambda_t^{13} = \lambda_t^{31} = \lambda_t^*$, the eigenvalue of $\boldsymbol{\lambda}_t$, denoted by ξ, becomes $\xi = \lambda_t, \lambda_t \pm \lambda_t^*$. Thus, the condition $\lambda_t^1 = \lambda_t^2 = \lambda_t^3 > \left|\lambda_t^{13}\right|$ is the necessary and sufficient condition for $\mathbf{\Lambda}_t$ for all $t \in \{1, \ldots, T\}$ to be positive definite.

We subsequently define the residual effect of past price impact at time $t \in \mathcal{T}$, represented by $\mathbf{R}_t \in \mathbb{R}^2$, which characterizes the discounted sum of past transient price impact. Many existing researches, conducted from both theoretical and empirical viewpoint, highlight the significance of the transient nature of price impacts (e.g., [6], [31]). By means of the following exponential decay kernel function $\mathbf{G}\colon \mathbb{R} \to \mathbb{R}^{2\times 2}$:

$$\mathbf{G}(t) := \mathrm{e}^{-\rho t}\mathbf{I}_2 = \mathrm{e}^{-\rho t \mathbf{I}_2} = \begin{pmatrix} \mathrm{e}^{-\rho t} & 0 \\ 0 & \mathrm{e}^{-\rho t} \end{pmatrix}, \quad t = 1, \ldots, T, \tag{2.15}$$

where ρ $(\in [0, \infty))$ stands for the deterministic resilience speed, we formulate the residual effect of the past orders posed by both the large trader and small traders.

Remark 2.6. [1] show that the decay kernel $\mathbf{G}(t)\colon [0, \infty) \to \mathbb{R}^{n\times n}$ for multiple (n) risky assets with cross–impact must satisfy

1. **nonnegative** matrix for all $t \in [0, \infty)$;
2. for all $\mathbf{x} \in \mathbb{R}^n$ the function $t \longmapsto \mathbf{x}^\top \mathbf{G}(t)\mathbf{x}$ is **nonincreasing**;
3. for all $\mathbf{x} \in \mathbb{R}^n$ the function $t \longmapsto \mathbf{x}^\top \mathbf{G}(t)\mathbf{x}$ is **convex**;
4. **commuting**, i.e., $\mathbf{G}(t)\mathbf{G}(s) = \mathbf{G}(s)\mathbf{G}(t)$ for all $t \in [0, \infty)$.

Eq. (2.15) clearly satisfies all the conditions.

Remark 2.7 (Extension of deterministic resilience speed)**.** We can extend the exponential decay kernel function in several ways. For example, the time dependency for the resilience speed, i.e., ρ_t, is consistent with much of empirical analysis. Another extension includes different resilience speeds for each asset, which [38] considers as follows:

$$\begin{aligned}
\mathbf{G}(t) = \mathrm{e}^{-t\boldsymbol{\rho}} &:= \exp\left\{\begin{pmatrix} -\rho^1 t & 0 \\ 0 & -\rho^2 t \end{pmatrix}\right\} \\
&= \begin{pmatrix} \displaystyle\sum_{k=0}^{\infty} \frac{(\rho^1 t)^k}{k!} & 0 \\ 0 & \displaystyle\sum_{k=0}^{\infty} \frac{(\rho^2 t)^k}{k!} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\rho^1 t} & 0 \\ 0 & \mathrm{e}^{-\rho^2 t} \end{pmatrix} \in \mathbb{R}^{2\times 2},
\end{aligned} \tag{2.16}$$

where ρ^i for $i \in \mathcal{I}$ is the deterministic resilience speed of asset i and $\boldsymbol{\rho} := \begin{pmatrix} \rho^1 & 0 \\ 0 & \rho^2 \end{pmatrix}$.

The dynamics of the residual effect of past price impact $\mathbf{R}_t$ is given as follows:

$$\begin{aligned}
\mathbf{R}_1 &= \mathbf{0}; \\
\mathbf{R}_{t+1} &= \sum_{k=1}^{t} \mathbf{A}_k \left(\boldsymbol{\Lambda}_k \mathbf{q}_t + \boldsymbol{\kappa}_k \mathbf{v}_k\right) \mathrm{e}^{-\rho((t+1)-k)} \\
&= \mathrm{e}^{-\rho} \sum_{k=1}^{t-1} \mathbf{A}_t \left(\boldsymbol{\Lambda}_k \mathbf{q}_k + \boldsymbol{\kappa}_k \mathbf{v}_k\right) \mathrm{e}^{-\rho(t-k)} + \mathbf{A}_t \left(\boldsymbol{\Lambda}_t \mathbf{q}_t + \boldsymbol{\kappa}_t \mathbf{v}_t\right) \mathrm{e}^{-\rho} \\
&= \mathrm{e}^{-\rho} \left[\mathbf{R}_t + \mathbf{A}_t \left(\boldsymbol{\Lambda}_t \mathbf{q}_t + \boldsymbol{\kappa}_t \mathbf{v}_t\right)\right], \quad t = 1, \ldots, T,
\end{aligned} \tag{2.17}$$

where $\mathbf{A}_t := \begin{pmatrix} \alpha_t^1 & \alpha_t^{12} \\ \alpha_t^{21} & \alpha_t^2 \end{pmatrix}$ represents the price impact coefficients representing the temporary price impacts with cross–impact. Eq. (2.17) indicates that $\mathbf{R}_t$ has a Markov property in this settings, which stems from the assumption of the exponantial decay kernel. For simplicity, we assume that $\mathbf{A}_t$ is symmetric.

Furthermore, we define a sequence of independent random variables $\boldsymbol{\varepsilon}_t$ at time $t \in \mathcal{T}$ as the effect of the public news/information about the economic situation between t and $t+1$ since some public news or information about the economic situation affect the price. $\boldsymbol{\varepsilon}_t$ for $t \in \mathcal{T}$ are assumed to follow a bivariate normal distribution with mean $\boldsymbol{\mu}_t^{\boldsymbol{\varepsilon}} \in \mathbb{R}^2$ and variance $\boldsymbol{\Sigma}_t^{\boldsymbol{\varepsilon}} \in \mathbb{R}^{2\times 2}$, i.e.,[3]

$$\boldsymbol{\varepsilon}_t \sim N_{\mathbb{R}^2}\left(\boldsymbol{\mu}_t^{\boldsymbol{\varepsilon}}, \boldsymbol{\Sigma}_t^{\boldsymbol{\varepsilon}}\right), \quad t = 1, \ldots, T. \tag{2.18}$$

The 'fundamental price' at time $t \in \mathcal{T}$, denoted by $\mathbf{P}_t^f$, must be carefully constructed. The fact that the residual effect of the past execution dissipates over the course of the trading horizon allows us to define $\mathbf{P}_t - \mathbf{R}_t$ as the fundamental price of the risky asset. We assume that the permanent

[3] In the rest of this paper, we suppose that the two stochastic processes, $\boldsymbol{\omega}_t$ and $\boldsymbol{\varepsilon}_t$ for $t \in \mathcal{T}$ are mutually independent for simplicity.

price impact is represented by $\mathbf{B}_t(\boldsymbol{\lambda}_k\mathbf{q}_k+\boldsymbol{\kappa}_k\mathbf{v}_k)$ where $\mathbf{B}_t := \begin{pmatrix}\beta_t^1 & \beta_t^{12}\\ \beta_t^{21} & \beta_t^2\end{pmatrix}$. We assume that $\mathbf{B}_t$ is symmetric for all $t\in\mathcal{T}$ for simplicity. By the definition of $\boldsymbol{\varepsilon}_t$, we can set the fundamental price $\mathbf{P}_t^f := \mathbf{P}_t - \mathbf{R}_t$ with a permanent price impact as follows:

$$\begin{aligned}\mathbf{P}_{t+1}^f &= \mathbf{P}_{t+1}-\mathbf{R}_{t+1}\\ &= \mathbf{P}_t-\mathbf{R}_t+\mathbf{B}_t(\boldsymbol{\Lambda}_t\mathbf{q}_t+\boldsymbol{\kappa}_t\mathbf{v}_t)+\boldsymbol{\varepsilon}_t\\ &= \mathbf{P}_t^f+\mathbf{B}_t(\boldsymbol{\Lambda}_t\mathbf{q}_t+\boldsymbol{\kappa}_t\mathbf{v}_t)+\boldsymbol{\varepsilon}_t,\quad t=1,\ldots,T.\end{aligned} \tag{2.19}$$

This relation indicates that the permanent price impact caused by large traders and small traders and the public news or information about an economic situation are assumed to affect the fundamental price. This assumption also reveals that the permanent price impact may give a non–zero trend to the fundamental price, even if the mean of $\boldsymbol{\varepsilon}_t$ is zero vector for all $t\in\mathcal{T}$. According to Eq. (2.8), (2.17), and (2.19), the dynamics of market price are described as

$$\begin{aligned}\mathbf{P}_{t+1} &= \mathbf{P}_t+(\mathbf{R}_{t+1}-\mathbf{R}_t)+\mathbf{B}_t(\boldsymbol{\Lambda}_t\mathbf{q}_t+\boldsymbol{\kappa}_t\mathbf{v}_t)+\boldsymbol{\varepsilon}_t\\ &= \mathbf{P}_t-(1-\mathrm{e}^{-\rho})\mathbf{R}_t+\left(\mathrm{e}^{-\rho}\mathbf{A}_t+\mathbf{B}_t\right)(\boldsymbol{\Lambda}_t\mathbf{q}_t+\boldsymbol{\kappa}_t\mathbf{v}_t)+\boldsymbol{\varepsilon}_t,\quad t=1,\ldots,T,\end{aligned} \tag{2.20}$$

Remark 2.8. In this context, $\mathbf{B}_t(\boldsymbol{\Lambda}_t\mathbf{q}_t+\boldsymbol{\kappa}_t\mathbf{v}_t), \mathbf{A}_t(\boldsymbol{\Lambda}_t\mathbf{q}_t+\boldsymbol{\kappa}_t\mathbf{v}_t)$, and $\mathrm{e}^{-\rho}\mathbf{A}_t(\boldsymbol{\Lambda}_t\mathbf{q}_t+\boldsymbol{\kappa}_t\mathbf{v}_t)$ represent the permanent impact, temporary impact, and transient impact, respectively. Moreover, if $\rho\to\infty$, the residual effect of past price impact becomes zero for all $t\in\mathcal{T}$ since $\mathbf{R}_1=\mathbf{0}$ and from Eq. (2.17)

$$\lim_{\rho\to\infty}\mathbf{R}_{t+1} = \lim_{\rho\to\infty}\mathrm{e}^{-\rho}\left[\mathbf{R}_t+\mathbf{A}_t(\boldsymbol{\Lambda}_t\mathbf{q}_t+\boldsymbol{\kappa}_t\mathbf{v}_t)\right]=\mathbf{0},\quad t=1,\ldots,T, \tag{2.21}$$

and therefore,

$$\mathbf{P}_{t+1}=\mathbf{P}_t+\mathbf{B}_t(\boldsymbol{\Lambda}_t\mathbf{q}_t+\boldsymbol{\kappa}_t\mathbf{v}_t)+\boldsymbol{\varepsilon}_t,\quad t=1,\ldots,T, \tag{2.22}$$

that is, we have a permanent impact model.

In the sequel of this paper, we assume the following regularity condition holds.

Assumption 2.3. $\boldsymbol{\Lambda}_t(\mathbf{I}_2-(\mathrm{e}^{-\rho}\mathbf{A}_t+\mathbf{B}_t))+(\mathbf{I}_2-(\mathrm{e}^{-\rho}\mathbf{A}_t+\mathbf{B}_t))\boldsymbol{\Lambda}_t$ is positive semidefinite.

Remark 2.9. The assumption above is equivalent to the following conditions:

1. $\lambda_t^1\left(1-\mathrm{e}^{-\rho}\alpha_t^1-\beta_t^1\right)+\lambda_t^2\left(1-\mathrm{e}^{-\rho}\alpha_t^2-\beta_t^2\right)-2\lambda_t^{12}\left(\mathrm{e}^{-\rho}\alpha_t^{12}+\beta_t^{12}\right)\geq 0$,
2. $\prod_{i=1}^{2}\left\{\lambda_t^i\left(1-\mathrm{e}^{-\rho}\alpha_t^i-\beta_t^i\right)-\lambda_t^{12}\left(\mathrm{e}^{-\rho}\alpha_t^{12}+\beta_t^{12}\right)\right\}\geq\left\{\lambda_t^{12}\left(\mathrm{e}^{-\rho}\alpha_t^{12}+\beta_t^{12}\right)\right\}^2$.

Both conditions imply that the cross–impact must be small enough for the optimal execution strategy to exist. A close analysis of condition 1 reveals that $\lambda_t^i\left(1-\mathrm{e}^{-\rho}\alpha_t^i-\beta_t^i\right)$ for $i\in\mathcal{I}$ is positive (except when $\rho=0$) if the temporary and permanent price impacts are proportional (e.g., considered in [33]), i.e., $\beta_t^i := 1-\alpha_t^i$ for all $t\in\mathcal{T}$, since in this case

$$1-\mathrm{e}^{-\rho}\alpha_t^i-\beta_t^i = 1-\mathrm{e}^{-\rho}\alpha_t^i-\left(1-\alpha_t^i\right)=\alpha_t^i\left(1-\mathrm{e}^{-\rho}\right)>0, \tag{2.23}$$

holds for all $\rho\in(0,\infty)$.

Remark 2.10 (Alternative assumption)**.** We can replace Assumption 2.3 as follows.

Assumption 2.3'. $\boldsymbol{\Lambda}_t(\mathbf{I}_2-(\mathrm{e}^{-\rho}\mathbf{A}_t+\mathbf{B}_t))$ is normal in the sense of [29], i.e.,

$$\left\{\boldsymbol{\Lambda}_t\left(\mathbf{I}_2-\left(\mathrm{e}^{-\rho}\mathbf{A}_t+\mathbf{B}_t\right)\right)\right\}^\top\boldsymbol{\Lambda}_t\left(\mathbf{I}_2-\left(\mathrm{e}^{-\rho}\mathbf{A}_t+\mathbf{B}_t\right)\right)=\boldsymbol{\Lambda}_t\left(\mathbf{I}_2-\left(\mathrm{e}^{-\rho}\mathbf{A}_t+\mathbf{B}_t\right)\right)\left\{\boldsymbol{\Lambda}_t\left(\mathbf{I}_2-\left(\mathrm{e}^{-\rho}\mathbf{A}_t+\mathbf{B}_t\right)\right)\right\}^\top. \tag{2.24}$$

This statement follows from Theorem 3 of [29]: If $\boldsymbol{\Lambda}_t(\mathbf{I}_2-(\mathrm{e}^{-\rho}\mathbf{A}_t+\mathbf{B}_t))$ is normal, then $\boldsymbol{\Lambda}_t(\mathbf{I}_2-(\mathrm{e}^{-\rho}\mathbf{A}_t+\mathbf{B}_t))$ is also positive semidefinite.

From the definition of the execution price, the wealth process W_t $(\in\mathbb{R})$ evolves as follows:

$$W_{t+1}=W_t-\widehat{\mathbf{P}}_t^\top\mathbf{q}_t=W_t-\left\{\mathbf{P}_t+(\boldsymbol{\Lambda}_t\mathbf{q}_t+\boldsymbol{\kappa}_t\mathbf{v}_t)\right\}^\top\mathbf{q}_t,\quad t=1,\ldots,T. \tag{2.25}$$

2.2 Formulation as a Markov decision process

In a discrete–time window $t \in \{1, \dots, T, T+1\}$, we define the state of the decision process at time $t \in \{1, \dots, T, T+1\}$ as 5–tuple and denote it as

$$s_t = (W_t, \mathbf{P}_t, \overline{\mathbf{Q}}_t, \mathbf{R}_t, \mathbf{v}_{t-1}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 =: S. \tag{2.26}$$

For $t \in \mathcal{T}$, an allowable action chosen at state s_t is an execution volume $\mathbf{q}_t \in \mathbb{R} =: A$ so that the set A of admissible actions is independent of the current state s_t.

When an action $\mathbf{q}_t$ is chosen in a state s_t at time $t \in \mathcal{T}$, a transition to a next state

$$s_{t+1} = (W_{t+1}, \mathbf{P}_{t+1}, \overline{\mathbf{Q}}_{t+1}, \mathbf{R}_{t+1}, \mathbf{v}_t) \in S \tag{2.27}$$

occurs according to the law of motion which we have precisely described in the previous subsection. We symbolically describe the transition by a (Borel measurable) system dynamics function h_t (: $S \times A \times (\mathbb{R} \times \mathbb{R}) \longrightarrow S$):

$$s_{t+1} = h_t(s_t, \mathbf{q}_t, (\boldsymbol{\omega}_t, \boldsymbol{\varepsilon}_t)), \quad t = 1, \dots, T. \tag{2.28}$$

A utility payoff (or reward) arises only in a terminal state s_{T+1} at the end of horizon $T+1$ as

$$g_{T+1}(s_{T+1}) := \begin{cases} -\exp\{-\gamma W_{T+1}\} & \text{if } \overline{Q}_{T+1} = 0; \\ -\infty & \text{if } \overline{Q}_{T+1} \neq 0, \end{cases} \tag{2.29}$$

where $\gamma > 0$ represents the risk aversion. The term $-\infty$ means a hard constraint enforcing the large trader to execute all of the remaining volume $\overline{\mathbf{Q}}_T$ at the maturity T, that is, $q_T = \overline{Q}_T$.

If we define a (history–independent) one–stage decision rule $\mathbf{f}_t$ at time $t \in \mathcal{T}$ by a Borel measurable map from a state $s_t \in S = \mathbb{R}^5$ to an action

$$\mathbf{q}_t = \mathbf{f}_t(s_t) \in A = \mathbb{R}, \tag{2.30}$$

then a Markov execution strategy π is defined as a sequence of one–stage decision rules

$$\pi := (\mathbf{f}_1, \dots, \mathbf{f}_t, \dots, \mathbf{f}_T). \tag{2.31}$$

We denote the set of all Markov execution strategies as Π_{M}. Further, for $t \in \mathcal{T}$, we define the sub–execution strategy after time t of a Markov execution strategy $\pi \in \Pi_{\mathrm{M}}$ as

$$\pi_t := (\mathbf{f}_t, \dots, \mathbf{f}_T), \tag{2.32}$$

and the entire set of π_t as $\Pi_{\mathrm{M},t}$.

By definition (2.29), the value function under an execution strategy π becomes an expected utility payoff arising from the terminal wealth W_{T+1} of the large trader with the absolute risk aversion γ:

$$V_1^\pi\big[s_1\big] = \mathbb{E}_1^\pi\Big[g_{T+1}(s_{T+1})\Big|s_1\Big] = \mathbb{E}_1^\pi\Big[-\exp\big\{-\gamma W_{T+1}\big\} \cdot 1_{\{\overline{Q}_{T+1}=0\}} + (-\infty) \cdot 1_{\{\overline{Q}_{T+1}\neq 0\}}\Big|s_1\Big], \tag{2.33}$$

where 1_A is the indicator function of the event A and, for $t \in \{1, \dots, T\}$, $\mathbb{E}_t^\pi$ is a conditional expectation given a condition at time t under π.

Then, for $t \in \{1, \dots, T, T+1\}$ and $s_t \in S$, we further let

$$V_t^\pi\big[s_t\big] = \mathbb{E}_t^\pi\Big[g_{T+1}(s_{T+1})\Big|s_t\Big] = \mathbb{E}_t^\pi\Big[-\exp\big\{-\gamma W_{T+1}\big\} \cdot 1_{\{\overline{Q}_{T+1}=0\}} + (-\infty) \cdot 1_{\{\overline{Q}_{T+1}\neq 0\}}\Big|s_t\Big] \tag{2.34}$$

be the expected utility payoff at time t under the strategy π. It should be noted that the expected utility payoff $V_t^\pi\big[s_t\big]$ depends on the Markov execution policy $\pi = (\mathbf{f}_1, \dots, \mathbf{f}_t, \dots, \mathbf{f}_T)$ only through the sub–execution policy $\pi_t := (\mathbf{f}_t, \ldots, \mathbf{f}_T)$ after time t. The Markov property of residual effect (and hence the price dynamics) and the path independency of the large trader's utility at the terminal period makes the optimal value function V_t the function of decision process $(W_t, \mathbf{P}_t, \overline{\mathbf{Q}}_t, \mathbf{R}_t, \mathbf{v}_{t-1})$. Thus, from the principle of optimality, the optimality equation (Bellman equation, or dynamic programming equation) becomes

$$V_t\big[W_t, \mathbf{P}_t, \overline{\mathbf{Q}}_t, \mathbf{R}_t, \mathbf{v}_{t-1}\big] = \sup_{\mathbf{q}_t \in \mathbb{R}^2} \mathbb{E}\left[V_{t+1}\big[W_{t+1}, \mathbf{P}_{t+1}, \overline{\mathbf{Q}}_{t+1}, \mathbf{R}_{t+1}, \mathbf{v}_t\big] \Big| W_t, \mathbf{P}_t, \overline{\mathbf{Q}}_t, \mathbf{R}_t, \mathbf{v}_{t-1}\right], \quad t = 1, \ldots, T. \tag{2.35}$$

Therefore, we obtain the optimal execution volume $\mathbf{q}_t^*$ for $t \in \mathcal{T}$ which attains V_1 from the maturity T by backward induction method (of dynamic programming) in terms of time t.

2.3 Optimal pair–trade execution

The optimal dynamic execution strategy π is acquired by solving the above equation (2.35) backwardly in time t from maturity T. Then, we obtain the following theorem.

Theorem 2.1 (Optimal Execution Strategy and Optimal Value Function)**.** Under a set of regularity conditions,

1. The optimal execution volume at time $t \in \mathcal{T}$, denoted as $\mathbf{q}_t^*$, becomes an affine function of the aggregate volume submitted by small traders at time $t-1$, $\mathbf{v}_{t-1}$, as well as the remaining execution volume $\overline{\mathbf{Q}}_t$ and the cumulative residual effect $\mathbf{R}_t$: that is,

$$\mathbf{q}_t^* = \mathbf{a}_t + \mathbf{b}_t \overline{\mathbf{Q}}_t + \mathbf{c}_t \mathbf{R}_t + \mathbf{d}_t \mathbf{v}_{t-1}, \quad t = 1, \ldots, T. \tag{2.36}$$

2. The optimal value function $V_t\big[\mathbf{s}_t\big]$ at time $t \in \{1, \ldots, T, T+1\}$ is represented as a functional form shown as follows:

$$\begin{aligned} V_t\big[W_t, \mathbf{P}_t, \overline{\mathbf{Q}}_t, \mathbf{R}_t, \mathbf{v}_{t-1}\big] = -\exp\Big\{ -\gamma\Big[W_t - \mathbf{P}_t^\top \overline{\mathbf{Q}}_t + \overline{\mathbf{Q}}_t^\top \mathbf{G}_t \overline{\mathbf{Q}}_t + \mathbf{H}_t^\top \overline{\mathbf{Q}}_t + \overline{\mathbf{Q}}_t^\top \mathbf{I}_t \mathbf{R}_t \\ + \mathbf{R}_t^\top \mathbf{J}_t \mathbf{R}_t + \mathbf{L}_t^\top \mathbf{R}_t + \overline{\mathbf{Q}}_t^\top \mathbf{M}_t \mathbf{v}_{t-1} + \mathbf{R}_t^\top \mathbf{N}_t \mathbf{v}_{t-1} + \mathbf{v}_{t-1}^\top \mathbf{X}_t \mathbf{v}_{t-1} + \mathbf{Y}_t^\top \mathbf{v}_{t-1} + Z_t\Big]\Big\}, \end{aligned} \tag{2.37}$$

where $\mathbf{a}_t, \mathbf{b}_t, \mathbf{c}_t, \mathbf{d}_t$; $\mathbf{G}_t, \mathbf{H}_t, \mathbf{I}_t, \mathbf{J}_t, \mathbf{L}_t, \mathbf{M}_t, \mathbf{N}_t, \mathbf{X}_t, \mathbf{Y}_t, Z_t$ for $t \in \{1, \ldots, T, T+1\}$ are deterministic functions of time t which are dependent on the problem parameters, and can be computed backwardly in time t from maturity T.

Proof. See Appendix A. □

As the above theorem shows, the optimal execution volume $\mathbf{q}_t^*$ for $t \in \mathcal{T}$ depends on the state $\mathbf{s}_t = (W_t, \mathbf{P}_t, \overline{\mathbf{Q}}_t, \mathbf{R}_t, \mathbf{v}_{t-1})$ of the decision process through the total volume submitted by small traders at the previous time $\mathbf{v}_{t-1}$ in addition to the remaining execution volume $\overline{\mathbf{Q}}_t$ and the cumulative residual effect $\mathbf{R}_t$, and not through the wealth W_t or market price $\mathbf{P}_t$. Further, by the definition of the residual effect, the optimal execution volume $\mathbf{q}_t^*$ for $t \in \mathcal{T}$ includes a nondeterministic term (random variable) through $\mathbf{R}_t$ and $\mathbf{v}_{t-1}$. Thus, we have the following facts.

Corollary 2.1. If the aggregate trading volumes submitted by small traders, $\mathbf{v}_t$, for $t \in \mathcal{T}$ are deterministic, the optimal execution volumes $\mathbf{q}_t^*$ at time $t \in \mathcal{T}$ also become deterministic functions of time. This fact means that the optimal execution strategy is one in a class of the static (and non–randomized) execution strategy.

Not only does our analysis show that the optimal execution strategy becomes a stochastic one but also it reveals that the orders of asset 1 posed by small traders directly affect the execution volume of both assets for the large trader and vice versa. This is our contribution to the field of the optimal execution problem.

Corollary 2.2 (In the case without transient price impact)**.** If we consider only temporary and permanent price impact, the optimal execution volume for the large trader at time $t \in \mathcal{T}$ becomes

$$\mathbf{q}_t^* = \mathbf{a}_t + \mathbf{b}_t \overline{\mathbf{Q}}_t + \mathbf{c}_t \mathbf{v}_{t-1}. \tag{2.38}$$

In this case, the aggregate trading volume of each asset posed by small traders still directly affects the optimal execution volume of the large trader. However, if we further assume that $\mathbf{v}_t$ has no Markovian dependence and simply follows a bivariate normal distribution;

$$\mathbf{v}_t \sim N_{\mathbb{R}^2}\left(\boldsymbol{\mu}_t^{\mathbf{v}}, \boldsymbol{\Sigma}_t^{\mathbf{v}}\right), \tag{2.39}$$

then the optimal execution volume of the large trader at time $t \in \mathcal{T}$ takes the form as follows:

$$\mathbf{q}_t^* = \mathbf{a}_t + \mathbf{b}_t \overline{\mathbf{Q}}_t, \tag{2.40}$$

meaning that the aggregate trading volume posed by small traders do not affect the optimal execution strategy even if we incorporate the price impact caused by small traders' trading orders. Therefore, taking into account the transient price impact is significant when we consider the effect of price impact caused by small traders.

2.4 In the case with target close orders

In this subsection, we consider an execution model with a closing price. The time framework $t \in \{1, \ldots, T, T+1\}$ is same in the model mentioned above. However, we add an assumption that a large trader can execute his/her remaining execution volume at time $T+1$, $\overline{\mathbf{Q}}_{T+1}$, with closing price $\mathbf{P}_{T+1}$. We further assume that the trading at time $T+1$ impose the large trader to pay the additive cost $\boldsymbol{\chi}_{T+1}$ per unit of the remaining volume. As stated in the last section, we have the following proposition.

Theorem 2.2 (Optimal Value Function and Optimal Execution Strategy in the Case with Target Close Orders)**.** Under a set of regularity conditions assumed in Theorem 2.1 and the following additional assumption:

$$\boldsymbol{\Lambda}_T\left(\mathbf{I}_2 - \boldsymbol{\Pi}_T\right) + \left\{\boldsymbol{\Lambda}_T\left(\mathbf{I}_2 - \boldsymbol{\Pi}_T\right)\right\}^\top + \left(\boldsymbol{\chi}_{T+1} + \boldsymbol{\chi}_{T+1}^\top\right) \tag{2.41}$$

is positive definite,

1. The optimal execution volume at time $t \in \{1, \ldots, T, T+1\}$, denoted as $\mathbf{q}_t^{*'}$, becomes an affine function of the aggregate volume submitted by small traders at time $t-1$, $\mathbf{v}_{t-1}$, as well as the remaining execution volume $\overline{\mathbf{Q}}_t$ and the cumulative residual effect $\mathbf{R}_t$: that is,

$$\mathbf{q}_t^{*'} = \mathbf{a}_t^* + \mathbf{b}_t^* \overline{\mathbf{Q}}_t + \mathbf{c}_t^* \mathbf{R}_t + \mathbf{d}_t^* \mathbf{v}_{t-1}, \quad t = 1, \ldots, T, T+1. \tag{2.42}$$

2. The optimal value function $V_t\left[\mathbf{s}_t\right]$ at time $t \in \{1, \ldots, T, T+1\}$ is represented as follows:

$$\begin{aligned} V_t\left[W_t, \mathbf{P}_t, \overline{\mathbf{Q}}_t, \mathbf{R}_t, \mathbf{v}_{t-1}\right] = -\exp\Big\{ -\gamma\Big[& W_t - \mathbf{P}_t^\top \overline{\mathbf{Q}}_t + \overline{\mathbf{Q}}_t^\top \mathbf{G}_t^* \overline{\mathbf{Q}}_t + \mathbf{H}_t^{*\top} \overline{\mathbf{Q}}_t + \overline{\mathbf{Q}}_t^\top \mathbf{I}_t^* \mathbf{R}_t \\ & + \mathbf{R}_t^\top \mathbf{J}_t^* \mathbf{R}_t + \mathbf{L}_t^{*\top} \mathbf{R}_t + \overline{\mathbf{Q}}_t^\top \mathbf{M}_t^* \mathbf{v}_{t-1} + \mathbf{R}_t^\top \mathbf{N}_t^* \mathbf{v}_{t-1} + \mathbf{v}_{t-1}^\top \mathbf{X}_t^* \mathbf{v}_{t-1} + \mathbf{Y}_t^{*\top} \mathbf{v}_{t-1} + Z_t^* \Big]\Big\}, \end{aligned} \tag{2.43}$$

where $\mathbf{a}_t^*, \mathbf{b}_t^*, \mathbf{c}_t^*, \mathbf{d}_t^*$; $\mathbf{G}_t^*, \mathbf{H}_t^*, \mathbf{I}_t^*, \mathbf{J}_t^*, \mathbf{L}_t^*, \mathbf{M}_t^*, \mathbf{N}_t^*, \mathbf{X}_t^*, \mathbf{Y}_t^*, Z_t^*$ for $t \in \{1, \ldots, T, T+1\}$ are deterministic functions of time t which are dependent on the problem parameters, and can be computed backwardly in time t from maturity T.

Proof. See Appendix B. □

3 Conclusion

We constructed a two assets optimal execution problem of a single large trader in a (finite) discrete–time framework. The large trader maximizes the expected Constant Absolute Risk Aversion (CARA) von Neumann-Morgenstern (vN-M) utility that arises from his/her wealth at the end of the trading epoch in a market with small traders. By formulating a generalized price impact model, the backward induction method of dynamic programming based on the dynamic programming principle permitted us to derive the optimal pair–trade execution strategy. The most important result which emerged from this research is as follows: the aggregate volume of small traders has both direct and indirect impacts on the execution strategy of the large trader. Moreover, the small traders' orders do affect the execution volume of both assets. The simulation–based analysis is left for this research. Future research includes the continuous–time analog of this research and optimal VWAP strategy in this setting.

Acknowledgements

The authors would like to thank Dr. Rei Yamamoto for closely examining our preliminary draft and the comments at the Nippon Finance Conference 2020 Autumn.

Disclosure statement

No potential conflict of interest was reported by the authors.

References

[1] Alfonsi, A., Klöck, F. and Schied, A., Multivariate transient price impact and matrix-valued positive definite functions. *Math. Oper. Res.*, **41**(3), 914–934, 2016.

[2] Almgren, R. and Chriss, N., Optimal execution of portfolio transactions. *J. Risk*, **3**, 5–39, 2000.

[3] Benzaquen, M., Mastromatteo, I., Eisler, Z., and Bouchaud, J. P., Dissecting cross-impact on stock markets: An empirical analysis. *J. Stat Mech-Theory E.*, **2**, 023406, 2017.

[4] Bertsimas, D. and Lo, A. W., Optimal control of execution costs. *J. Financ. Mark.*, **1**, 1–50, 1998.

[5] Bouchard, B., Fukasawa, M., Herdegen, M. and Muhle-Karbe, J., Equilibrium returns with transaction costs. *Fin. Stoch.*, **22**, 569–601, 2018.

[6] Bouchaud, J. P., Gefen, Y., Potters, M. and Wyart, M., Fluctuations and response in financial markets: the subtle nature of 'random'price changes. *Quant. Financ.*, **4**(2), 176–190, 2004.

[7] Cartea, Á. and Jaimungal, S., Incorporating order-flow into optimal execution. *Math. and Financ. Econ.*, **10**, 339–364, 2016.

[8] Cartea, Á. and Jaimungal, S., A closed-form execution strategy to target volume weighted average price. *SIAM J. Financ. Math.*, **7**, 760–785, 2016.

[9] Cartea, Á., Gan, L. and Jaimungal, S., Trading co-integrated assets with price impact. *Math. Financ.*, **29**, 542–567, 2016.

[10] Chan, K., Menkveld, A. J., and Yang, Z., The informativeness of domestic and foreign investors' stock trades: Evidence from the perfectly segmented Chinese market. *J. Financ. Mark*, **10**, 391-415, 2007.

[11] Eisler, Z., Bouchaud, J. P. and Kockelkoren, J., The price impact of order book events: market orders, limit orders and cancellations. *Quant. Financ.*, **12**, 1395–1419, 2012.

[12] Fukasawa, M., Ohnishi, M., and Shimoshimizu, M., Optimal execution strategies with generalized price impacts in a continuous–time setting. Paper presented at 51st JAFEE (summer) Conference, Seijo University, 5-6 August 2019.

[13] Fukasawa, M., Ohnishi, M., and Shimoshimizu, M., Optimal execution strategies with generalized price impact in a discrete-time setting. *RIMS Kokyuroku*, **2158**, 66–79, 2020.

[14] Fukasawa, M., Ohnishi, M., and Shimoshimizu, M., Discrete-time optimal execution under a generalized price impact model with Markovian exogenous orders. Available at SSRN: abstract id=3714066.

[15] Gatev, E., Goetzmann, W. N. and Rouwenhorst, K. G., Pairs trading: Performance of a relative-value arbitrage rule. *Rev. Financ. Stud.*, **19**, 797–827, 2006.

[16] Gatheral, J., No-dynamic-arbitrage and market impact. *Quant. Financ.*, **10**, 749–759, 2010.

[17] Gârleanu, N. and Pedersen, L. H., Dynamic trading with predictable returns and transaction costs. *J. Financ.*, **68**, 2309–2340, 2013.

[18] Gârleanu, N. and Pedersen, L. H., Dynamic portfolio choice with frictions. *J. Econ. Theory*, **165**, 487–516, 2016.

[19] Hasbrouck, J., Measuring the information content of stock trades. *J. Financ.*, **46**, 179–207, 1991.

[20] Hasbrouck, J., and Seppi, D. J., Common factors in prices, order flows, and liquidity. *J. Financ. Econ.*, **59**, 383–411, 2001.

[21] Hamilton, J. D., *Time Series Analysis*. 2020 (Princeton university press).

[22] Huberman, G. and Stanzl, W., Price manipulation and quasi-arbitrage. *Econometrica*, **72**, 1247–1275, 2004.

[23] Kunou, S. and Ohnishi, M., Optimal execution strategies with price impact. *RIMS Kokyuroku*, **1675**, 234–247, 2010

[24] Kuno, S. and Ohnishi, M., Optimal execution in illiquid market with the absence of price manipulation. *J. Math. Fin.*, **5**, 1–14, 2015.

[25] Kuno, S., Ohnishi, M., and Shimizu, P., Optimal off-exchange execution with closing price. *J. Math. Fin.*, **7**, 54–64, 2017.

[26] Kuno, S., Ohnishi, M., and Shimoshimizu, M., Optimal execution strategies with generalized price impact models. *RIMS Kokyuroku*, **2078**, 77–83, 2018.

[27] Li, Y., Guo, J. E., Lai, K. K., and Shi, J., Optimal portfolio liquidation with cross-price impacts on trading. *Oper. Res.-GER*, 1-20, 2020.

[28] Ma, G., Siu, C. C. and Zhu, S.-P., Dynamic portfolio choice with return predictability and transaction costs. *Eur. J. Oper. Res.*, **278**, 976–988, 2019.

[29] Meenakshi, A. R., and Rajian, C., On a product of positive semidefinite matrices. *Linear Algebra. Appl.*, **295**(1-3), 3–6, 1999.

[30] Meucci, A., Review of statistical arbitrage, cointegration, and multivariate Ornstein-Uhlenbeck. Cointegration, and Multivariate Ornstein-Uhlenbeck (May 14, 2009). Available at http://www2.stat.duke.edu/ scs/Projects/StructuralPhylogeny/multivariateOU.pdf, 2009.

[31] Obizhaeva, A. A., and Wang, J., Optimal trading strategy and supply/demand dynamics. *J. Financ. Mark.*, **16**, 1–32, 2013.

[32] Ohnishi, M., and Shimoshimizu, M., Equilibrium execution strategy with generalized price impacts. *RIMS Kokyuroku*, **2111**, 84–106, 2019.

[33] Ohnishi, M., and Shimoshimizu, M., Optimal and equilibrium execution strategies with generalized price impact. *Quant. Financ.*, **20**(10), 1625–1644, 2020.

[34] Potters, M., and Bouchaud, J. P., More statistical properties of order books and price impact. *Physica A*, **324**, 133–140, 2003.

[35] Schied, A., Schöneborn, T., and Tehranchi, M., Optimal basket liquidation for CARA investors is deterministic. *Appl. Math. Fin.*, **17**, 471–489, 2010.

[36] Schneider, M., and Lillo, F., Cross-impact and no-dynamic-arbitrage. *Quant. Financ.*, **19**, 137–154, 2019.

[37] Tin-Zhu, H., A note on generalized diagonally dominant matrices. *Linear Algebra. Appl.*, **225**, 237-242, 1995.

[38] Tsoukalas, G., Wang, J., and Giesecke, K., Dynamic portfolio execution. *Manage. Sci.*, **65**(5), 2015–2040, 2019.

Appendix

A Proof of Theorem 2.1

We derive the optimal execution volume $\mathbf{q}_t^*$ at time $t \in \{1, \ldots, T\}$ by backward induction method of dynamic programming from the maturity T via the following steps.

Step 1 From the assumption that the large trader must unwind all the remainder of his/her position at time $t = T$,

$$\overline{\mathbf{Q}}_{T+1} = \overline{\mathbf{Q}}_T - \mathbf{q}_T = \mathbf{0}, \tag{A.1}$$

must hold, which yields $\overline{\mathbf{Q}}_T = \mathbf{q}_T$. Then, for $t = T$,

$$\begin{aligned} V_T\big[\mathbf{s}_T\big] &= \sup_{\mathbf{q}_T \in \mathbb{R}^d} \mathbb{E}\Big[V_{T+1}\big[\mathbf{s}_{T+1}\big]\Big|\mathbf{s}_T\Big] \\ &= \sup_{\mathbf{q}_T \in \mathbb{R}^d} \mathbb{E}\Big[V_{T+1}\big[W_{T+1}, \mathbf{P}_{T+1}, \overline{\mathbf{Q}}_{T+1}, \mathbf{R}_{T+1}, \mathbf{v}_T\big]\Big|W_T, \mathbf{P}_T, \overline{\mathbf{Q}}_T, \mathbf{R}_T, \mathbf{v}_{T-1}\Big] \\ &= \sup_{\mathbf{q}_T \in \mathbb{R}^d} \mathbb{E}\Big[-\exp\left\{-\gamma W_{T+1}\right\}\Big|W_T, \mathbf{P}_T, \overline{\mathbf{Q}}_T, \mathbf{R}_T, \mathbf{v}_{T-1}\Big] \\ &= \sup_{\mathbf{q}_T \in \mathbb{R}^d} \mathbb{E}\Big[-\exp\Big\{-\gamma\big[W_T - \left[\mathbf{P}_T + (\mathbf{\Lambda}_T\mathbf{q}_T + \boldsymbol{\kappa}_T\mathbf{v}_T)\right]^\top \mathbf{q}_T\big]\Big\}\Big|W_T, \mathbf{P}_T, \overline{\mathbf{Q}}_T, \mathbf{R}_T, \mathbf{v}_{T-1}\Big] \\ &= -\exp\Big\{-\gamma\big[W_T - \mathbf{P}_T^\top\overline{\mathbf{Q}}_T - \overline{\mathbf{Q}}_T^T\mathbf{\Lambda}_T\overline{\mathbf{Q}}_T\big]\Big\} \\ &\quad\times \mathbb{E}\Big[\exp\Big\{\gamma\mathbf{v}_T^\top\boldsymbol{\kappa}_T^\top\overline{\mathbf{Q}}_T\Big\}\Big|W_T, \mathbf{P}_T, \overline{\mathbf{Q}}_T, \mathbf{R}_T, \mathbf{v}_{T-1}\Big]. \end{aligned} \tag{A.2}$$

Using the fact that $\mathbf{v}_T^\top\boldsymbol{\kappa}_T^\top\overline{\mathbf{Q}}_T = \overline{\mathbf{Q}}_T^\top\boldsymbol{\kappa}_T\mathbf{v}_T (\in \mathbb{R})$ and the moment–generating function with respect to $\mathbf{v}_T$,

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big\{\gamma\overline{\mathbf{Q}}_T^\top\boldsymbol{\kappa}_T\mathbf{v}_T\Big\}\Big|W_T, \mathbf{P}_T, \overline{\mathbf{Q}}_T, \mathbf{R}_T, \mathbf{v}_{T-1}\Big] \\ &= \exp\left\{\gamma\overline{\mathbf{Q}}_T^\top\boldsymbol{\kappa}_T\left(\mathbf{a}_T^\mathbf{v} - \mathbf{b}_T^\mathbf{v}\mathbf{v}_{T-1}\right) + \frac{1}{2}\left(\gamma\overline{\mathbf{Q}}_T^\top\boldsymbol{\kappa}_T\right)\mathbf{\Sigma}_T^\mathbf{v}\left(\gamma\overline{\mathbf{Q}}_T^\top\boldsymbol{\kappa}_T\right)^\top\right\}, \end{aligned} \tag{A.3}$$

Eq. (2.35) becomes

$$\begin{aligned} V_T\big[\mathbf{s}_T\big] &= -\exp\Big\{-\gamma\big[W_T - \mathbf{P}_T^\top\overline{\mathbf{Q}}_T - \overline{\mathbf{Q}}_T^T\mathbf{\Lambda}_T\overline{\mathbf{Q}}_T\big]\Big\} \\ &\qquad\times\exp\left\{\gamma\overline{\mathbf{Q}}_T^\top\boldsymbol{\kappa}_T\left(\mathbf{a}_T^\mathbf{v} - \mathbf{b}_T^\mathbf{v}\mathbf{v}_{T-1}\right) + \frac{1}{2}\left(\gamma\overline{\mathbf{Q}}_T^\top\boldsymbol{\kappa}_T\right)\mathbf{\Sigma}_T^\mathbf{v}\left(\gamma\overline{\mathbf{Q}}_T^\top\boldsymbol{\kappa}_T\right)^\top\right\} \\ &= -\exp\Big\{-\gamma\Big[W_T - \mathbf{P}_T^\top\overline{\mathbf{Q}}_T + \overline{\mathbf{Q}}_T^\top\mathbf{G}_T\overline{\mathbf{Q}}_T + \mathbf{H}_T^\top\overline{\mathbf{Q}}_T + \overline{\mathbf{Q}}_T^\top\mathbf{M}_T\mathbf{v}_{T-1}\Big]\Big\}, \end{aligned} \tag{A.4}$$

where we have used the fact that $\overline{\mathbf{Q}}_T^\top\boldsymbol{\kappa}_T\mathbf{a}_T^\mathbf{v} = \left(\mathbf{a}_T^\mathbf{v}\right)^\top\boldsymbol{\kappa}_T^\top\overline{\mathbf{Q}}_T$ and

$$\begin{aligned} \mathbf{G}_T &:= -\mathbf{\Lambda}_T - \frac{1}{2}\gamma\boldsymbol{\kappa}_T\mathbf{\Sigma}_T^\mathbf{v}\boldsymbol{\kappa}_T^\top(\in \mathbb{R}^{2\times 2}); \\ \mathbf{H}_T^\top &:= -\left(\mathbf{a}_T^\mathbf{v}\right)^\top\boldsymbol{\kappa}_T(\in \mathbb{R}^{1\times 2}); \\ \mathbf{M}_T &:= \boldsymbol{\kappa}_T\mathbf{b}_T^\mathbf{v}(\in \mathbb{R}^{2\times 2}). \end{aligned} \tag{A.5}$$

Note that $\mathbf{G}_T$ is negative definite (by Assumptions 2.1 and 2.2).

Step 2 For $t = T-1$, we have

$$
\begin{aligned}
&V_{T-1}\left[\mathbf{s}_{T-1}\right] \\
&= \sup_{\mathbf{q}_{T-1}\in\mathbb{R}^2} \mathbb{E}\left[V_T\left[\mathbf{s}_T\right]\middle|\mathbf{s}_{T-1}\right] \\
&= \sup_{\mathbf{q}_{T-1}\in\mathbb{R}^2} \mathbb{E}\left[-\exp\left\{-\gamma\left[W_T - \mathbf{P}_T^\top\overline{\mathbf{Q}}_T + \overline{\mathbf{Q}}_T^\top\mathbf{G}_T\overline{\mathbf{Q}}_T + \mathbf{H}_T^\top\overline{\mathbf{Q}}_T + \overline{\mathbf{Q}}_T^\top\mathbf{M}_T\mathbf{v}_{T-1}\right]\right\}\middle|s_{T-1}\right] \\
&= \sup_{\mathbf{q}_{T-1}\in\mathbb{R}^2} \mathbb{E}\Big[-\exp\Big\{-\gamma\Big[W_{T-1} - \left\{\mathbf{P}_{T-1} + \left(\mathbf{\Lambda}_{T-1}\mathbf{q}_{T-1} + \boldsymbol{\kappa}_{T-1}\mathbf{v}_{T-1}\right)\right\}^\top\mathbf{q}_{T-1} \\
&\quad - \left\{\mathbf{P}_{T-1} - (1-\mathrm{e}^{-\rho})\mathbf{R}_{T-1} + (\mathbf{\Lambda}_{T-1}\mathbf{q}_{T-1} + \boldsymbol{\kappa}_{T-1}\mathbf{v}_{T-1})\left(\mathrm{e}^{-\rho}\mathbf{A}_{T-1} + \mathbf{B}_{T-1}\right) + \boldsymbol{\varepsilon}_{T-1}\right\}^\top\left(\overline{\mathbf{Q}}_{T-1} - \mathbf{q}_{T-1}\right) \\
&\quad + \left(\overline{\mathbf{Q}}_{T-1} - \mathbf{q}_{T-1}\right)^\top\mathbf{G}_T\left(\overline{\mathbf{Q}}_{T-1} - \mathbf{q}_{T-1}\right) + \left(\overline{\mathbf{Q}}_{T-1} - \mathbf{q}_{T-1}\right)^\top\mathbf{M}_T\mathbf{v}_{T-1}\Big]\Big\}\Big|\mathbf{s}_{T-1}\Big] \\
&= \sup_{q_{T-1}\in\mathbb{R}} -\exp\Big\{-\gamma\Big[\mathbf{q}_{T-1}^\top\left\{\mathbf{\Lambda}_{T-1}^\top\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\right\} - \mathbf{G}_T\right\}\mathbf{q}_{T-1} + \overline{\mathbf{Q}}_{T-1}^\top\left\{\mathbf{\Lambda}_{T-1}^\top\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\right\} - 2\mathbf{G}_T\right\}\mathbf{q}_{T-1} \\
&\quad - (1-\mathrm{e}^{-\rho})\mathbf{R}_{T-1}^\top\mathbf{q}_{T-1} - \mathbf{H}_{T-1}^\top\mathbf{q}_{T-1} \\
&\quad + W_{T-1} - \mathbf{P}_{T-1}\overline{\mathbf{Q}}_{T-1} + \overline{\mathbf{Q}}_{T-1}^\top\mathbf{G}_T\overline{\mathbf{Q}}_{T-1} + \mathbf{H}_{T-1}^\top\overline{\mathbf{Q}}_{T-1} + (1-\mathrm{e}^{-\rho})\overline{\mathbf{Q}}_{T-1}^\top\mathbf{R}_{T-1}\Big]\Big\} \\
&\quad \times\mathbb{E}\left[\exp\left\{\gamma\left[\mathbf{q}_{T-1}^\top\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\boldsymbol{\kappa}_{T-1} + \mathbf{M}_{T-1}\right\} + \overline{\mathbf{Q}}_{T-1}^\top\left(\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right)\right]\mathbf{v}_{T-1}\right\}\middle|s_{T-1}\right] \\
&\quad \times\mathbb{E}\left[\exp\left\{\gamma\left(\overline{\mathbf{Q}}_{T-1} - \mathbf{q}_{T-1}\right)\boldsymbol{\varepsilon}_{T-1}\right\}\middle|s_{T-1}\right],
\end{aligned}
\tag{A.6}
$$

where $\mathbf{\Pi}_{T-1} := \mathrm{e}^{-\rho}\mathbf{A}_{T-1} + \mathbf{B}_{T-1}$. Using the moment generating function, we obtain

1. First expectation in Eq. (A.6):

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left\{\gamma\left[\mathbf{q}_{T-1}^\top\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\boldsymbol{\kappa}_{T-1} + \mathbf{M}_{T-1}\right\} + \overline{\mathbf{Q}}_{T-1}^\top\left(\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right)\right]\mathbf{v}_{T-1}\right\}\middle|s_{T-1}\right] \\
&= \exp\Big\{\gamma\left[\mathbf{q}_{T-1}^\top\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\boldsymbol{\kappa}_{T-1} + \mathbf{M}_{T-1}\right\} + \overline{\mathbf{Q}}_{T-1}^\top\left(\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right)\right]\left(\mathbf{a}_{T-1}^v + \mathbf{b}_{T-1}^v\mathbf{v}_{T-2}\right) \\
&\quad + \frac{1}{2}\gamma^2\left[\mathbf{q}_{T-1}^\top\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\boldsymbol{\kappa}_{T-1} + \mathbf{M}_{T-1}\right\} + \overline{\mathbf{Q}}_{T-1}^\top\left(\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right)\right]\mathbf{\Sigma}_{T-1}^{\mathbf{v}} \\
&\quad \left[\mathbf{q}_{T-1}^\top\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\boldsymbol{\kappa}_{T-1} + \mathbf{M}_{T-1}\right\} + \overline{\mathbf{Q}}_{T-1}^\top\left(\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right)\right]^\top\Big\};
\end{aligned}
\tag{A.7}
$$

2. Second expectation in Eq. (A.6):

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left\{\gamma\left(\overline{\mathbf{Q}}_{T-1} - \mathbf{q}_{T-1}\right)\boldsymbol{\varepsilon}_{T-1}\right\}\middle|s_{T-1}\right] \\
&= \exp\left\{\gamma\left(\overline{\mathbf{Q}}_{T-1} - \mathbf{q}_{T-1}\right)\boldsymbol{\mu}_{T-1}^{\boldsymbol{\varepsilon}} + \frac{1}{2}\gamma^2\left(\overline{\mathbf{Q}}_{T-1} - \mathbf{q}_{T-1}\right)^\top\mathbf{\Sigma}_{T-1}^{\boldsymbol{\varepsilon}}\left(\overline{\mathbf{Q}}_{T-1} - \mathbf{q}_{T-1}\right)\right\}.
\end{aligned}
\tag{A.8}
$$

Therefore, substituting Eq. (A.7) and (A.8) into Eq. (A.6) and rearranging results in

$$
\begin{aligned}
&V_{T-1}\left[s_{T-1}\right] \\
&= \sup_{q_{T-1}\in\mathbb{R}} -\exp\Big\{-\gamma\Big[-\mathbf{q}_{T-1}^\top\tilde{\mathbf{\Omega}}_{T-1}\mathbf{q}_{T-1} + \left(\overline{\mathbf{Q}}_{T-1}^\top\mathbf{\Theta}_{T-1} + \mathbf{R}_{T-1}^\top\mathbf{\Xi}_{T-1} + \mathbf{v}_{T-2}^\top\mathbf{\Phi}_{T-1} + \mathbf{\Psi}_{T-1}^\top\right)\mathbf{q}_{T-1} \\
&\quad + W_{T-1} - \mathbf{P}_{T-1}^\top\overline{\mathbf{Q}}_{T-1} + \overline{\mathbf{Q}}_{T-1}^\top\left\{\mathbf{G}_T - \frac{1}{2}\gamma\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}\mathbf{\Sigma}_{T-1}^{\mathbf{v}}\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}^\top - \frac{1}{2}\gamma\mathbf{\Sigma}_{T-1}^{\boldsymbol{\varepsilon}}\right\}\overline{\mathbf{Q}}_{T-1} \\
&\quad + \left\{\mathbf{H}_T^\top - \left(\mathbf{a}_{T-1}^{\mathbf{v}}\right)^\top\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}^\top - \left(\boldsymbol{\mu}_{T-1}^{\varepsilon}\right)^\top\right\}\overline{\mathbf{Q}}_{T-1} + (1-\mathrm{e}^{-\rho})\overline{\mathbf{Q}}_{T-1}^\top\mathbf{R}_{T-1} \\
&\quad - \overline{\mathbf{Q}}_{T-1}^\top\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}\mathbf{b}_{T-1}^{\mathbf{v}}\mathbf{v}_{T-2}\Big]\Big\},
\end{aligned}
\tag{A.9}
$$

with the following relations:

$$\begin{aligned}
\tilde{\mathbf{\Omega}}_{T-1} &:= \frac{1}{2}\left(\mathbf{\Omega}_{T-1} + \mathbf{\Omega}_{T-1}^{\top}\right) \\
&= \frac{1}{2}\mathbf{\Lambda}_{T-1}\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\right\} + \frac{1}{2}\left\{\mathbf{\Lambda}_{T-1}\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\right\}\right\}^{\top} - \mathbf{G}_T \\
&\quad + \frac{1}{2}\gamma\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\boldsymbol{\kappa}_{T-1} + \mathbf{M}_T\right\}\mathbf{\Sigma}_{T-1}^{\mathbf{v}}\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\boldsymbol{\kappa}_{T-1} + \mathbf{M}_T\right\}^{\top} + \frac{1}{2}\gamma\mathbf{\Sigma}_{T-1}^{\boldsymbol{\varepsilon}} \ (\in \mathbb{R}^{2\times 2}); \\
\mathbf{\Theta}_{T-1} &:= -\mathbf{\Pi}_{T-1}\mathbf{\Lambda}_{T-1} - 2\mathbf{G}_T - \gamma\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}\mathbf{\Sigma}_{T-1}^{\mathbf{v}}\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\boldsymbol{\kappa}_{T-1} + \mathbf{M}_T\right\}^{\top} \\
&\quad + \gamma\mathbf{\Sigma}_{T-1}^{\boldsymbol{\varepsilon}} \ (\in \mathbb{R}^{2\times 2}); \\
\mathbf{\Xi}_{T-1} &:= -(1 - \mathrm{e}^{-\rho})\mathbf{I}_2 \ (\in \mathbb{R}^{2\times 2}); \\
\mathbf{\Phi}_{T-1} &:= -\left(\mathbf{b}_{T-1}^{v}\right)^{\top}\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\boldsymbol{\kappa}_{T-1} + \mathbf{M}_T\right\}^{\top} \ (\in \mathbb{R}^{2\times 2}); \\
\mathbf{\Psi}_{T-1}^{\top} &:= -\mathbf{H}_T^{\top} - \left(\mathbf{a}_{T-1}^{\mathbf{v}}\right)^{\top}\left\{(\mathbf{I}_2 - \mathbf{\Pi}_{T-1})\boldsymbol{\kappa}_{T-1} + \mathbf{M}_T\right\}^{\top} + \left(\boldsymbol{\mu}_{T-1}^{\boldsymbol{\varepsilon}}\right)^{\top} \ (\in \mathbb{R}^{1\times 2}).
\end{aligned} \tag{A.10}$$

By the assumption 2.1, $\tilde{\mathbf{\Omega}}_{T-1}$ becomes a (symmetric and) positive definite matrix. Finding the optimal execution volume $\mathbf{q}_{T-1}^{*}$ which attains the supremum of Eq. (A.6) is equivalent to finding the one which yields the maximum of the following function $K_{T-1}(\mathbf{q}_{T-1})$ defined as

$$\begin{aligned}
K_{T-1}(\mathbf{q}_{T-1}) &:= -\mathbf{q}_{T-1}^{\top}\tilde{\mathbf{\Omega}}_{T-1}\mathbf{q}_{T-1} + \left(\overline{\mathbf{Q}}_{T-1}^{\top}\mathbf{\Theta}_{T-1} + \mathbf{R}_{T-1}^{\top}\mathbf{\Xi}_{T-1} + \mathbf{v}_{T-2}^{\top}\mathbf{\Phi}_{T-1} + \mathbf{\Psi}_{T-1}^{\top}\right)\mathbf{q}_{T-1} \\
&+ W_{T-1} - \mathbf{P}_{T-1}\overline{\mathbf{Q}}_{T-1} + \overline{\mathbf{Q}}_{T-1}\left\{\mathbf{G}_T - \frac{1}{2}\gamma\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}\mathbf{\Sigma}_{T-1}^{\mathbf{v}}\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}^{\top} - \frac{1}{2}\gamma\mathbf{\Sigma}_{T-1}^{\boldsymbol{\varepsilon}}\right\}\overline{\mathbf{Q}}_{T-1} \\
&+ \left\{\mathbf{H}_T^{\top} - \left(\mathbf{a}_{T-1}^{\mathbf{v}}\right)^{\top}\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}^{\top} - \left(\boldsymbol{\mu}_{T-1}^{\varepsilon}\right)^{\top}\right\}\overline{\mathbf{Q}}_{T-1} + (1 - \mathrm{e}^{-\rho})\overline{\mathbf{Q}}_{T-1}\mathbf{R}_{T-1} \\
&+ \overline{\mathbf{Q}}_{T-1}\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}\mathbf{b}_{T-1}^{\mathbf{v}}\mathbf{v}_{T-2},
\end{aligned} \tag{A.11}$$

since both Eq. (A.6) and Eq. (A.11) are concave functions with respect to $\mathbf{q}_{T-1}$. Thus, by completing the square of K_{T-1}, we obtain the optimal execution volume $\mathbf{q}_{T-1}^{*}$ as

$$\begin{aligned}
\mathbf{q}_{T-1}^{*} &= \tilde{\mathbf{\Omega}}_{T-1}^{-1}\left\{\mathbf{\Theta}_{T-1}^{\top}\overline{\mathbf{Q}}_{T-1} + \mathbf{\Xi}_{T-1}^{\top}\mathbf{R}_{T-1} + \mathbf{\Phi}_{T-1}^{\top}\mathbf{v}_{T-2} + \mathbf{\Psi}_{T-1}\right\} \\
&\quad \left(=: \mathbf{a}_{T-1} + \mathbf{b}_{T-1}\overline{\mathbf{Q}}_{T-1} + \mathbf{c}_{T-1}\mathbf{R}_{T-1} + \mathbf{d}_{T-1}\mathbf{v}_{T-2}\right).
\end{aligned} \tag{A.12}$$

Thus, the optimal value function at time $T-1$ becomes a functional form as follows:

$$\begin{aligned}
&V_{T-1}[s_{T-1}] \\
&= -\exp\Big\{-\gamma\Big[W_{T-1} - \mathbf{P}_{T-1}^{\top}\overline{\mathbf{Q}}_{T-1} + \overline{\mathbf{Q}}_{T-1}^{\top}\mathbf{G}_{T-1}\overline{\mathbf{Q}}_{T-1} + \mathbf{H}_{T-1}^{\top}\overline{\mathbf{Q}}_{T-1} + \overline{\mathbf{Q}}_{T-1}^{\top}\mathbf{I}_{T-1}\mathbf{R}_{T-1} + \mathbf{R}_{T-1}^{\top}\mathbf{J}_{T-1}\mathbf{R}_{T-1} \\
&\quad + \mathbf{L}_{T-1}^{\top}\mathbf{R}_{T-1} + \overline{\mathbf{Q}}_{T-1}^{\top}\mathbf{M}_{T-1}\mathbf{v}_{T-1-1} + \mathbf{R}_{T-1}^{\top}\mathbf{N}_{T-1}\mathbf{v}_{T-2} + \mathbf{v}_{T-2}^{\top}\mathbf{X}_{T-1}\mathbf{v}_{T-2} + \mathbf{Y}_{T-1}^{\top}\mathbf{v}_{T-2} + Z_{T-1}\Big]\Big\},
\end{aligned} \tag{A.13}$$

where

$$\begin{aligned}
&\mathbf{G}_{T-1} := \mathbf{G}_T - \frac{1}{2}\gamma\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}\mathbf{\Sigma}_{T-1}^{\mathbf{v}}\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}^{\top} - \frac{1}{2}\gamma\mathbf{\Sigma}_{T-1}^{\boldsymbol{\varepsilon}} + \frac{1}{4}\mathbf{\Theta}_{T-1}\tilde{\mathbf{\Omega}}_{T-1}^{-1}\mathbf{\Theta}_{T-1}^{\top}; \\
&\mathbf{H}_{T-1}^{\top} := \mathbf{H}_T^{\top} - \left(\mathbf{a}_{T-1}^{\mathbf{v}}\right)^{\top}\left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}^{\top} - \left(\boldsymbol{\mu}_{T-1}^{\varepsilon}\right)^{\top} + \frac{1}{2}\mathbf{\Psi}_{T-1}^{\top}\tilde{\mathbf{\Omega}}_{T-1}^{-1}\mathbf{\Theta}_{T-1}^{\top}; \\
&\mathbf{I}_{T-1} := (1 - \mathrm{e}^{-\rho})\mathbf{I}_2 + \frac{1}{2}\mathbf{\Theta}_{T-1}\tilde{\mathbf{\Omega}}_{T-1}^{-1}\mathbf{\Xi}_{T-1}^{\top}; \quad \mathbf{J}_{T-1} := \frac{1}{4}\mathbf{\Xi}_{T-1}\tilde{\mathbf{\Omega}}_{T-1}^{-1}\mathbf{\Xi}_{T-1}^{\top}; \quad \mathbf{L}_{T-1}^{\top} := \frac{1}{2}\mathbf{\Psi}_{T-1}^{\top}\tilde{\mathbf{\Omega}}_{T-1}^{-1}\mathbf{\Xi}_{T-1}^{\top}; \\
&\mathbf{M}_{T-1} := \left\{\mathbf{\Pi}_{T-1}\boldsymbol{\kappa}_{T-1} - \mathbf{M}_T\right\}\mathbf{b}_{T-1}^{\mathbf{v}} + \frac{1}{2}\mathbf{\Theta}_{T-1}\tilde{\mathbf{\Omega}}_{T-1}^{-1}\mathbf{\Phi}_{T-1}^{\top}; \quad \mathbf{N}_{T-1} := \frac{1}{2}\mathbf{\Xi}_{T-1}\tilde{\mathbf{\Omega}}_{T-1}^{-1}\mathbf{\Phi}_{T-1}^{\top}; \\
&\mathbf{X}_{T-1} := \frac{1}{4}\mathbf{\Phi}_{T-1}\tilde{\mathbf{\Omega}}_{T-1}^{-1}\mathbf{\Phi}_{T-1}^{\top}; \quad \mathbf{Y}_{T-1}^{\top} := \frac{1}{2}\mathbf{\Psi}_{T-1}^{\top}\tilde{\mathbf{\Omega}}_{T-1}^{-1}\mathbf{\Phi}_{T-1}^{\top}; \quad Z_{T-1} := \frac{1}{4}\mathbf{\Psi}_{T-1}^{\top}\tilde{\mathbf{\Omega}}_{T-1}^{-1}\mathbf{\Psi}_{T-1}.
\end{aligned} \tag{A.14}$$

Step 3 For $t \in \{T-2, \ldots, 2\}$, we can assume from the above results that, at time $t+1$, the optimal value function has the following functional form:

$$V_{t+1}\left[s_{t+1}\right] = -\exp\Big\{-\gamma\Big[W_{t+1} - \mathbf{P}_{t+1}^{\top}\overline{\mathbf{Q}}_{t+1} + \overline{\mathbf{Q}}_{t+1}^{\top}\mathbf{G}_{t+1}\overline{\mathbf{Q}}_{t+1} + \mathbf{H}_{t+1}^{\top}\overline{\mathbf{Q}}_{t+1} + \overline{\mathbf{Q}}_{t+1}^{\top}\mathbf{I}_{t+1}\mathbf{R}_{t+1} + \mathbf{R}_{t+1}^{\top}\mathbf{J}_{t+1}\mathbf{R}_{t+1}$$
$$+ \mathbf{L}_{t+1}^{\top}\mathbf{R}_{t+1} + \overline{\mathbf{Q}}_{t+1}^{\top}\mathbf{M}_{t+1}\mathbf{v}_t + \mathbf{R}_{t+1}^{\top}\mathbf{N}_{t+1}\mathbf{v}_t + \mathbf{v}_t^{\top}\mathbf{X}_{t+1}\mathbf{v}_t + \mathbf{Y}_{t+1}^{\top}\mathbf{v}_t + \mathbf{Z}_{t+1}\Big]\Big\}. \quad \text{(A.15)}$$

Then, we can obtain the following calculation by substituting the dynamics of $W_{t+1}, \mathbf{P}_{t+1}, \overline{\mathbf{Q}}_{t+1}, \mathbf{R}_{t+1}, \mathbf{v}_t$ into the equation above:

$$\begin{aligned}
V_t\left[s_t\right] = \sup_{\mathbf{q}_t \in \mathbb{R}^2} & -\exp\Big\{-\gamma\Big[\mathbf{q}_t^{\top}\Big(-\mathbf{\Lambda}_t + \mathbf{\Pi}_t\mathbf{\Lambda}_t + \mathbf{G}_{t+1} - \mathrm{e}^{-\rho}\mathbf{I}_{t+1}\mathbf{A}_t\mathbf{\Lambda}_t + \mathrm{e}^{-2\rho}\mathbf{\Lambda}_t^{\top}\mathbf{A}_t^{\top}\mathbf{J}_{t+1}\mathbf{A}_t\mathbf{\Lambda}_t\Big)\mathbf{q}_t \\
& + \Big[\overline{\mathbf{Q}}_t^{\top}\left(-\mathbf{\Pi}_t\mathbf{\Lambda}_t - 2\mathbf{G}_{t+1} + \mathrm{e}^{-\rho}\mathbf{I}_{t+1}\mathbf{A}_t\mathbf{\Lambda}_t\right) + \mathbf{R}_t^{\top}\left(-(1-\mathrm{e}^{-\rho})\mathbf{I}_2 - \mathrm{e}^{-\rho}\mathbf{I}_{t+1} + 2\mathrm{e}^{-2\rho}\mathbf{J}_{t+1}\mathbf{A}_t\mathbf{\Lambda}_t\right) \\
& + \Big(-\mathbf{H}_{t+1}^{\top} + \mathrm{e}^{-\rho}\mathbf{L}_{t+1}^{\top}\mathbf{A}_t\mathbf{\Lambda}_t\Big)\Big]\mathbf{q}_t + W_t - \mathbf{P}_t^{\top}\overline{\mathbf{Q}}_t + \overline{\mathbf{Q}}_t^{\top}\mathbf{G}_{t+1}\overline{\mathbf{Q}}_t + \mathbf{H}_{t+1}^{\top}\overline{\mathbf{Q}}_t \\
& \overline{\mathbf{Q}}_t^{\top}\left((1-\mathrm{e}^{-\rho})\mathbf{I}_2 + \mathrm{e}^{-\rho}\mathbf{I}_t\right)\mathbf{R}_t + \mathrm{e}^{-2\rho}\mathbf{R}_t^{\top}\mathbf{J}_{t+1}\mathbf{R}_t + \mathrm{e}^{-\rho}\mathbf{L}_{t+1}^{\top}\mathbf{R}_t + Z_{t+1}\Big]\Big\} \\
& \times \mathbb{E}\left[\exp\Big\{-\gamma\Big[\mathbf{v}_t^{\top}\boldsymbol{\xi}_t\mathbf{v}_t + \Big(\mathbf{q}_t^{\top}\boldsymbol{\delta}_t + \overline{\mathbf{Q}}_t\boldsymbol{\eta}_t + \mathbf{R}_t^{\top}\boldsymbol{\theta}_t + \boldsymbol{\phi}_t^{\top}\Big)\mathbf{v}_t\Big]\Big\}\Big|s_t\right] \\
& \times \mathbb{E}\left[\exp\Big\{\gamma\left(\overline{\mathbf{Q}}_t - \mathbf{q}_t\right)^{\top}\boldsymbol{\varepsilon}_t\Big\}\Big|s_t\right], \qquad \text{(A.16)}
\end{aligned}$$

where $\mathbf{\Pi}_t := \mathrm{e}^{-\rho}\mathbf{A}_t + \mathbf{B}_t$ and

$$\begin{aligned}
\boldsymbol{\xi}_t &:= \mathrm{e}^{-2\rho}\boldsymbol{\kappa}_t^{\top}\mathbf{A}_t\mathbf{J}_{t+1}\mathbf{A}_t\boldsymbol{\kappa}_t + \mathrm{e}^{-\rho}\boldsymbol{\kappa}_t\mathbf{A}_t\mathbf{N}_{t+1} + \mathbf{X}_{t+1} \ (\in \mathbb{R}^{2\times 2}); \\
\boldsymbol{\delta}_t &:= -\left(\mathbf{I}_2 - \mathbf{\Pi}_t\right)\boldsymbol{\kappa}_t - \mathrm{e}^{-\rho}\mathbf{I}_{t+1}\mathbf{A}_t\boldsymbol{\kappa}_t + 2\mathrm{e}^{-2\rho}\mathbf{\Lambda}_t\mathbf{A}_t\mathbf{J}_{t+1}\mathbf{A}_t\boldsymbol{\kappa}_t - \mathbf{M}_{t+1} + \mathrm{e}^{-\rho}\mathbf{\Lambda}_t\mathbf{A}_t\mathbf{N}_{t+1} \ (\in \mathbb{R}^{2\times 2}); \\
\boldsymbol{\eta}_t &:= -\mathbf{\Pi}_t\boldsymbol{\kappa}_t + \mathrm{e}^{-\rho}\mathbf{I}_{t+1}\mathbf{A}_t\boldsymbol{\kappa}_t + \mathbf{M}_{t+1} \ (\in \mathbb{R}^{2\times 2}); \\
\boldsymbol{\theta}_t &:= 2\mathrm{e}^{-2\rho}\mathbf{J}_{t+1}\mathbf{A}_t\boldsymbol{\kappa}_t + \mathrm{e}^{-\rho}\mathbf{N}_{t+1} \ (\in \mathbb{R}^{2\times 2}); \\
\boldsymbol{\phi}_t^{\top} &:= \mathrm{e}^{-\rho}\mathbf{L}_{t+1}^{\top}\mathbf{A}_t\boldsymbol{\kappa}_t + \mathbf{Y}_{t+1}^{\top} \ (\in \mathbb{R}^{1\times 2}).
\end{aligned}$$

The direct calculation leads to the following equalities:

1. First expectation in Eq. (A.16):

$$\begin{aligned}
&\mathbb{E}\left[\exp\Big\{-\gamma\Big[\mathbf{v}_t^{\top}\boldsymbol{\xi}_t\mathbf{v}_t + \Big(\mathbf{q}_t^{\top}\boldsymbol{\delta}_t + \overline{\mathbf{Q}}_t\boldsymbol{\eta}_t + \mathbf{R}_t^{\top}\boldsymbol{\theta}_t + \boldsymbol{\phi}_t^{\top}\Big)\mathbf{v}_t\Big]\Big\}\Big|s_t\right] \\
&= \frac{|(\mathbf{\Sigma}_t^*)|}{|\mathbf{\Sigma}_t^{\mathbf{v}}|}\exp\Big\{\frac{1}{2}\Big[\left(\mathbf{a}_t^{\mathbf{v}} + \mathbf{b}_t^{\mathbf{v}}\mathbf{v}_{t-1}\right)^{\top}\Big\{\left(\mathbf{\Sigma}_t^{\mathbf{v}}\right)^{-1}\left(\mathbf{\Sigma}_t^*\right)^{-1}\left(\mathbf{\Sigma}_t^{\mathbf{v}}\right)^{-1} - \left(\mathbf{\Sigma}_t^{\mathbf{v}}\right)^{-1}\Big\}\left(\mathbf{a}_t^{\mathbf{v}} + \mathbf{b}_t^{\mathbf{v}}\mathbf{v}_{t-1}\right) \\
&\quad - 2\gamma\Big[\mathbf{q}_t^{\top}\boldsymbol{\delta}_t + \overline{\mathbf{Q}}_t^{\top}\boldsymbol{\eta}_t + \mathbf{R}_t^{\top}\boldsymbol{\theta}_t + \boldsymbol{\phi}_t^{\top}\Big]\left(\mathbf{\Sigma}_t^*\right)^{-1}\left(\mathbf{\Sigma}_t^{\mathbf{v}}\right)^{-1}\left(\mathbf{a}_t^{\mathbf{v}} + \mathbf{b}_t^{\mathbf{v}}\mathbf{v}_{t-1}\right) \\
&\quad + \gamma^2\Big[\mathbf{q}_t^{\top}\boldsymbol{\delta}_t + \overline{\mathbf{Q}}_t^{\top}\boldsymbol{\eta}_t + \mathbf{R}_t^{\top}\boldsymbol{\theta}_t + \boldsymbol{\phi}_t^{\top}\Big]\left(\mathbf{\Sigma}_t^*\right)^{-1}\Big[\mathbf{q}_t^{\top}\boldsymbol{\delta}_t + \overline{\mathbf{Q}}_t^{\top}\boldsymbol{\eta}_t + \mathbf{R}_t^{\top}\boldsymbol{\theta}_t + \boldsymbol{\phi}_t^{\top}\Big]^{\top}\Big]\Big\}; \qquad \text{(A.17)}
\end{aligned}$$

2. Second expectation in Eq. (A.16):

$$\begin{aligned}
&\mathbb{E}\left[\exp\Big\{\gamma\left(\overline{\mathbf{Q}}_t - \mathbf{q}_t\right)\boldsymbol{\varepsilon}_t\Big\}\Big|s_t\right] \\
&= \exp\Big\{\gamma\left(\overline{\mathbf{Q}}_t - \mathbf{q}_t\right)\boldsymbol{\mu}_t^{\boldsymbol{\varepsilon}} + \frac{1}{2}\gamma^2\left(\overline{\mathbf{Q}}_t - \mathbf{q}_t\right)^{\top}\mathbf{\Sigma}_t^{\boldsymbol{\varepsilon}}\left(\overline{\mathbf{Q}}_t - \mathbf{q}_t\right)\Big\}. \qquad \text{(A.18)}
\end{aligned}$$

To derive Eq. (A.17), the following lemma has been used. Although this lemma is a straightforward result, we here note the result as a lemma for this paper to be self–contained.

Lemma A.1. For an n–dimensional normally distributed random variable $\mathbf{X}$ with mean $\boldsymbol{\mu} \in \mathbb{R}^n$ and variance $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ (where $\boldsymbol{\Sigma}$ is a symmetric positive definite matrix), we have

$$\mathbb{E}\Big[\exp\Big\{\mathbf{s}^\top\mathbf{X} + \mathbf{X}^\top\mathbf{V}\mathbf{X}\Big\}\Big] = \frac{|(\boldsymbol{\Sigma}^*)^{-1}|^{\frac{1}{2}}}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{\frac{1}{2}\Big[(\boldsymbol{\mu}^*)^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}^*)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^* - \boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Big]\right\}, \quad \text{(A.19)}$$

where $\mathbf{s} \in \mathbb{R}^n$, $\mathbf{V} \in \mathbb{R}^{n\times n}$, $\boldsymbol{\mu}^* := \boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{s}$ and $\boldsymbol{\Sigma}^* := \boldsymbol{\Sigma} - 2\mathbf{V}$, under the assumption that $\boldsymbol{\Sigma}^*$ is a non–singular matrix. Remind that when $\mathbf{V} = \mathbf{0}$, the result is consistent with the one obtained from the moment generating function.

Proof. We can assume, without loss of generality, that $\mathbf{V}$ is symmetric, since $\mathbf{X}^\top\mathbf{V}\mathbf{X} \in \mathbb{R}$ we have a symmetric matrix $\tilde{\mathbf{V}}(:= \frac{\mathbf{V}+\mathbf{V}^\top}{2})$ which satisfies $\mathbf{X}^\top\tilde{\mathbf{V}}\mathbf{X} = \mathbf{X}^\top\mathbf{V}\mathbf{X}$ even if $\mathbf{V}$ is not symmetric. Define $\mathbf{x} := (x_1, \dots, x_n)^\top \in \mathbb{R}^n$. Then, direct calculation yields

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{\mathbf{s}^\top\mathbf{X} + \mathbf{X}^\top\mathbf{V}\mathbf{X}\Big\}\Big] \\
&= \int_{\mathbb{R}^n} \exp\Big\{\mathbf{s}^\top\mathbf{x} + \mathbf{x}^\top\mathbf{V}\mathbf{x}\Big\} \frac{1}{(2\pi)^{\frac{n}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} \mathrm{d}\mathbf{x} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \int_{\mathbb{R}^n} \exp\left\{-\frac{1}{2}\Big[\mathbf{x}^\top\left(\boldsymbol{\Sigma}^{-1} - 2\mathbf{V}\right)\mathbf{x} - 2(\boldsymbol{\mu}+\boldsymbol{\Sigma}\mathbf{s})^\top\boldsymbol{\Sigma}^{-1}\mathbf{x} + \boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Big]\right\} \mathrm{d}\mathbf{x}, \qquad \text{(A.20)}
\end{aligned}$$

where $\mathrm{d}\mathbf{x} := \mathrm{d}x_1 \cdots \mathrm{d}x_n$. If we set $\boldsymbol{\Sigma}^* := \boldsymbol{\Sigma}^{-1} - 2\mathbf{V}$ and $\boldsymbol{\mu}^* := \boldsymbol{\mu} + \boldsymbol{\Sigma}\mathbf{s}$, then by the assumption that $\boldsymbol{\Sigma}^* \in \mathbb{R}^{n\times n}$ is a non–singular matrix, Eq. (A.20) results in

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{\mathbf{s}^\top\mathbf{X} + \mathbf{X}^\top\mathbf{V}\mathbf{X}\Big\}\Big] \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \int_{\mathbb{R}^n} \exp\left\{-\frac{1}{2}\Big[\mathbf{x}^\top\boldsymbol{\Sigma}^*\mathbf{x} - 2(\boldsymbol{\mu}^*)^\top\boldsymbol{\Sigma}^{-1}\mathbf{x} + \boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Big]\right\} \mathrm{d}\mathbf{x} \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \int_{\mathbb{R}^n} \exp\left\{-\frac{1}{2}\Big(\mathbf{x}^\top - (\boldsymbol{\Sigma}^*)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Big)\boldsymbol{\Sigma}^*\Big(\mathbf{x}^\top - (\boldsymbol{\Sigma}^*)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Big)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\Big[-(\boldsymbol{\mu}^*)^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}^*)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^* + \boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Big]\right\} \mathrm{d}\mathbf{x} \\
&= \frac{(2\pi)^{\frac{n}{2}}|(\boldsymbol{\Sigma}^*)^{-1}|^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{\frac{1}{2}\Big[(\boldsymbol{\mu}^*)^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}^*)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^* - \boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Big]\right\} \\
&\quad \times \underbrace{\int_{\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}|(\boldsymbol{\Sigma}^*)^{-1}|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\Big(\mathbf{x}^\top - (\boldsymbol{\Sigma}^*)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Big)\Big\{(\boldsymbol{\Sigma}^*)^{-1}\Big\}^{-1}\Big(\mathbf{x}^\top - (\boldsymbol{\Sigma}^*)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Big)\right\} \mathrm{d}\mathbf{x}}_{=1} \\
&= \frac{|(\boldsymbol{\Sigma}^*)^{-1}|^{\frac{1}{2}}}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{\frac{1}{2}\Big[(\boldsymbol{\mu}^*)^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}^*)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^* - \boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Big]\right\}. \qquad \text{(A.21)}
\end{aligned}$$

□

Eq. (A.21) can be rewritten as follows:

$$\begin{aligned}
&\frac{|(\boldsymbol{\Sigma}^*)^{-1}|^{\frac{1}{2}}}{|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left\{\frac{1}{2}\Big[(\boldsymbol{\mu}^*)^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{\Sigma}^*)^{-1}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^* - \boldsymbol{\mu}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\Big]\right\} \\
&= \frac{\sqrt{\left|(\boldsymbol{\Sigma}^*)^{-1}\right|}}{\sqrt{|\boldsymbol{\Sigma}|}} \exp\left\{\frac{1}{2}\Big[\boldsymbol{\mu}^\top\Big\{\boldsymbol{\Sigma}(\boldsymbol{\Sigma}^*)^{-1}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\Big\}\boldsymbol{\mu} + 2\mathbf{s}^\top(\boldsymbol{\Sigma}^*)^{-1}(\boldsymbol{\Sigma})^{-1}\boldsymbol{\mu} + \mathbf{s}^\top(\boldsymbol{\Sigma}^*_t)^{-1}\mathbf{s}\Big]\right\}. \qquad \text{(A.22)}
\end{aligned}$$

Therefore, setting $\mathbf{s}^\top := -\gamma\Big[\mathbf{q}_t^\top\boldsymbol{\delta}_t + \overline{\mathbf{Q}}_t^\top\boldsymbol{\eta}_t + \mathbf{R}_t^\top\boldsymbol{\theta}_t + \boldsymbol{\phi}_t^\top\Big]$ and $\mathbf{V} := -\frac{\gamma}{2}\left(\boldsymbol{\xi}_t + \boldsymbol{\xi}_t^\top\right)$ in Eq. (A.21) yields Eq. (A.17).

Substituting Eq. (A.17) and (A.18) into Eq. (A.16) and rearranging results in

$$
\begin{aligned}
V_t[s_t] = \sup_{\mathbf{q}_t\in\mathbb{R}^2} -\exp\Big\{ -\gamma\Big[&-\mathbf{q}_t^\top\tilde{\boldsymbol{\Omega}}_t\mathbf{q}_t + \Big[\overline{\mathbf{Q}}_t^\top\boldsymbol{\Theta}_t + \mathbf{R}_t^\top\boldsymbol{\Xi}_t + \mathbf{v}_{t-1}^\top\boldsymbol{\Phi}_t + \boldsymbol{\Psi}_t^\top\Big]\mathbf{q}_t + W_t - \mathbf{P}_t^\top\overline{\mathbf{Q}}_t \\
&+ \overline{\mathbf{Q}}_t^\top\Big[\mathbf{G}_{t+1} - \frac{1}{2}\gamma\boldsymbol{\eta}_t\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\eta}_t^\top - \frac{1}{2}\gamma\boldsymbol{\Sigma}_t^\varepsilon\Big]\overline{\mathbf{Q}}_t + \Big[\mathbf{H}_{t+1}^\top + \left(\mathbf{a}_t^\mathbf{v}\right)^\top\left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1}\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\eta}_t^\top \\
&- \gamma\boldsymbol{\phi}_t^\top\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\eta}_t^\top - \left(\boldsymbol{\mu}_t^{\boldsymbol{\varepsilon}}\right)^\top\Big]\overline{\mathbf{Q}}_t + \overline{\mathbf{Q}}_t^\top\Big[(1-\mathrm{e}^{-\rho})\mathbf{I}_2 + \mathrm{e}^{-\rho}\mathbf{I}_t - \gamma\boldsymbol{\eta}_t\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\theta}_t^\top\Big]\mathbf{R}_t \\
&+ \mathbf{R}_t^\top\Big[\mathrm{e}^{-2\rho}\mathbf{J}_{t+1} - \frac{1}{2}\gamma\boldsymbol{\theta}_t\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\theta}_t^\top\Big]\mathbf{R}_t + \Big[\mathrm{e}^{-\rho}\mathbf{L}_{t+1}^\top + \left(\mathbf{a}_t^\mathbf{v}\right)^\top\left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1}\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\theta}_t^\top - \gamma\left(\boldsymbol{\phi}_t\right)^\top\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\theta}_t^\top\Big]\mathbf{R}_t \\
&+ \overline{\mathbf{Q}}_t^\top\boldsymbol{\eta}_t\left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1}\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\mathbf{b}_t^\mathbf{v}\mathbf{v}_{t-1} + \mathbf{R}_t^\top\boldsymbol{\theta}_t\left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1}\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\mathbf{b}_t^\mathbf{v}\mathbf{v}_{t-1} - \frac{1}{2\gamma}\mathbf{v}_{t-1}^\top\left(\mathbf{b}_t^\mathbf{v}\right)^\top\boldsymbol{\Sigma}_t^{**}\mathbf{b}_t^\mathbf{v}\mathbf{v}_{t-1} \\
&+ \Big[-\frac{1}{\gamma}\left(\mathbf{a}_t^\mathbf{v}\right)^\top\boldsymbol{\Sigma}_t^{**}\mathbf{b}_t^\mathbf{v} + \boldsymbol{\phi}_t^\top\left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1}\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\mathbf{b}_t^\mathbf{v}\Big]\mathbf{v}_{t-1} + Z_{t+1} - \frac{1}{2\gamma}\left(\mathbf{a}_t^\mathbf{v}\right)^\top\boldsymbol{\Sigma}_t^{**}\mathbf{a}_t^\mathbf{v} \\
&+ \left(\mathbf{a}_t^\mathbf{v}\right)^\top\left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1}\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\phi}_t - \frac{1}{2\gamma}\boldsymbol{\phi}_t^\top\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\phi}_t + x_t\Big]\Big\}, \qquad \text{(A.23)}
\end{aligned}
$$

where $\boldsymbol{\Sigma}_t^* := \left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1} + \gamma\left(\boldsymbol{\xi}_t + \boldsymbol{\xi}_t^\top\right)$, $\boldsymbol{\Sigma}_t^{**} := \left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1}\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1} - \left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1}$, $x_t := \log\frac{|(\boldsymbol{\Sigma}_t^*)|}{|\boldsymbol{\Sigma}_t^\mathbf{v}|}$ and

$$
\begin{aligned}
\tilde{\boldsymbol{\Omega}}_t &:= \frac{1}{2}\left(\boldsymbol{\Omega}_t + \boldsymbol{\Omega}_t^\top\right) \\
&= \frac{1}{2}\boldsymbol{\Lambda}_t\left(\mathbf{I}_2 - \boldsymbol{\Pi}_t\right) + \frac{1}{2}\left\{\boldsymbol{\Lambda}_t\left(\mathbf{I}_2 - \boldsymbol{\Pi}_t\right)\right\}^\top - \mathbf{G}_{t+1} + \frac{1}{2}\mathrm{e}^{-\rho}\mathbf{I}_{t+1}\mathbf{A}_t\boldsymbol{\Lambda}_t + \frac{1}{2}\left\{\mathrm{e}^{-\rho}\mathbf{I}_{t+1}\mathbf{A}_t\boldsymbol{\Lambda}_t\right\}^\top \\
&\quad - \mathrm{e}^{-2\rho}\boldsymbol{\Lambda}_t\mathbf{A}_t\mathbf{J}_{t+1}\mathbf{A}_t\boldsymbol{\Lambda}_t + \frac{1}{2}\gamma\boldsymbol{\delta}_t\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\delta}_t^\top + \frac{1}{2}\gamma\boldsymbol{\Sigma}_t^{\boldsymbol{\varepsilon}} \ (\in\mathbb{R}^{2\times2}); \\
\boldsymbol{\Theta}_t &:= -\boldsymbol{\Pi}_t\boldsymbol{\Lambda}_t - 2\mathbf{G}_{t+1} + \mathrm{e}^{-\rho}\mathbf{I}_{t+1}\mathbf{A}_t\boldsymbol{\Lambda}_t - \gamma\boldsymbol{\eta}_t\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\delta}_t^\top + \gamma\boldsymbol{\Sigma}_t^{\boldsymbol{\varepsilon}} \ (\in\mathbb{R}^{2\times2}); \\
\boldsymbol{\Xi}_t &:= -(1-\mathrm{e}^{-\rho})\mathbf{I}_2 - \mathrm{e}^{-\rho}\mathbf{I}_{t+1} + 2\mathrm{e}^{-2\rho}\mathbf{J}_{t+1}\mathbf{A}_t\boldsymbol{\Lambda}_t - \gamma\boldsymbol{\theta}_t\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\delta}_t^\top \ (\in\mathbb{R}^{2\times2}); \\
\boldsymbol{\Phi}_t &:= \left(\mathbf{b}_t^\mathbf{v}\right)^\top\left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1}\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\delta}_t^\top \ (\in\mathbb{R}^{2\times2}); \\
\boldsymbol{\Psi}_t^\top &:= -\mathbf{H}_{t+1}^\top + \mathrm{e}^{-\rho}\mathbf{L}_{t+1}^\top\mathbf{A}_t\boldsymbol{\Lambda}_t + \left(\mathbf{a}_t^\mathbf{v}\right)^\top\left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1}\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\delta}_t^\top - \gamma\boldsymbol{\phi}_t^\top\left(\boldsymbol{\Sigma}_t^*\right)^{-1}\boldsymbol{\delta}_t^\top + \left(\boldsymbol{\mu}_t^{\boldsymbol{\varepsilon}}\right)^\top \ (\in\mathbb{R}^{1\times2}). \qquad \text{(A.24)}
\end{aligned}
$$

Thus, under the following assumptions (or regularity conditions):

1. $\boldsymbol{\Sigma}_t^* := \left(\boldsymbol{\Sigma}_t^\mathbf{v}\right)^{-1} + \gamma\left(\boldsymbol{\xi}_t + \boldsymbol{\xi}_t^\top\right)$ is non-sungular;
2. $\tilde{\boldsymbol{\Omega}}_t := \frac{1}{2}\left(\boldsymbol{\Omega}_t + \boldsymbol{\Omega}_t^\top\right)$ is positive (semi)definite,

we can derive the optimal execution volume at time t via the same method used in step 2 and obtain as follows:

$$
\begin{aligned}
\mathbf{q}_t^* &= \tilde{\boldsymbol{\Omega}}_t^{-1}\left\{\boldsymbol{\Theta}_t^\top\overline{\mathbf{Q}}_t + \boldsymbol{\Xi}_t^\top\mathbf{R}_t + \boldsymbol{\Phi}_t^\top\mathbf{v}_{t-1} + \boldsymbol{\Psi}_t\right\} \qquad \text{(A.25)} \\
&\quad \left(=: \mathbf{a}_t + \mathbf{b}_t\overline{\mathbf{Q}}_t + \mathbf{c}_t\mathbf{R}_t + \mathbf{d}_t\mathbf{v}_{t-1}\right).
\end{aligned}
$$

Finally, by substituting this into Eq. (A.23) the optimal value function becomes

$$
\begin{aligned}
V_t[s_t] = -\exp\Big\{ -\gamma\Big[&W_t - \mathbf{P}_t^\top\overline{\mathbf{Q}}_t + \overline{\mathbf{Q}}_t^\top\mathbf{G}_t\overline{\mathbf{Q}}_t + \mathbf{H}_t^\top\overline{\mathbf{Q}}_t + \overline{\mathbf{Q}}_t^\top\mathbf{I}_t\mathbf{R}_t + \mathbf{R}_t^\top\mathbf{J}_t\mathbf{R}_t + \mathbf{L}_t^\top\mathbf{R}_t \\
&+ \overline{\mathbf{Q}}_t^\top\mathbf{M}_t\mathbf{v}_{t-1} + \mathbf{R}_t^\top\mathbf{N}_t\mathbf{v}_{t-1} + \mathbf{v}_{t-1}^\top\mathbf{X}_t\mathbf{v}_{t-1} + \mathbf{Y}_t^\top\mathbf{v}_{t-1} + \mathbf{Z}_t\Big]\Big\}, \qquad \text{(A.26)}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{G}_t &:= \mathbf{G}_{t+1} - \frac{1}{2}\gamma\boldsymbol{\eta}_t \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \boldsymbol{\eta}_t^\top - \frac{1}{2}\gamma\boldsymbol{\Sigma}_t^{\boldsymbol{\varepsilon}} + \frac{1}{4}\boldsymbol{\Theta}_t\tilde{\boldsymbol{\Omega}}_t^{-1}\boldsymbol{\Theta}_t^\top;\\
\mathbf{H}_t^\top &:= \mathbf{H}_{t+1}^\top + \left(\mathbf{a}_t^{\mathbf{v}}\right)^\top \left(\boldsymbol{\Sigma}_t^{\mathbf{v}}\right)^{-1} \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \boldsymbol{\eta}_t^\top - \gamma\boldsymbol{\varphi}_t^\top \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \boldsymbol{\eta}_t^\top - \left(\boldsymbol{\mu}_t^{\boldsymbol{\varepsilon}}\right)^\top + \frac{1}{2}\boldsymbol{\Psi}_t^\top\tilde{\boldsymbol{\Omega}}_t^{-1}\boldsymbol{\Theta}_t^\top;\\
\mathbf{I}_t &:= (1-\mathrm{e}^{-\rho})\mathbf{I}_2 + \mathrm{e}^{-\rho}\mathbf{I}_t - \gamma\boldsymbol{\eta}_t \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \boldsymbol{\theta}_t^\top + \frac{1}{2}\boldsymbol{\Theta}_t\tilde{\boldsymbol{\Omega}}_t^{-1}\boldsymbol{\Xi}_t^\top;\\
\mathbf{J}_t &:= \mathrm{e}^{-2\rho}\mathbf{J}_{t+1} - \frac{1}{2}\gamma\boldsymbol{\theta}_t \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \boldsymbol{\theta}_t^\top + \frac{1}{4}\boldsymbol{\Xi}_t\tilde{\boldsymbol{\Omega}}_t^{-1}\boldsymbol{\Xi}_t^\top;\\
\mathbf{L}_t^\top &:= \mathrm{e}^{-\rho}\mathbf{L}_{t+1}^\top + \left(\mathbf{a}_t^{\mathbf{v}}\right)^\top \left(\boldsymbol{\Sigma}_t^{\mathbf{v}}\right)^{-1} \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \boldsymbol{\theta}_t^\top - \gamma\left(\boldsymbol{\phi}_t\right)^\top \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \boldsymbol{\theta}_t^\top + \frac{1}{2}\boldsymbol{\Psi}_t^\top\tilde{\boldsymbol{\Omega}}_t^{-1}\boldsymbol{\Xi}_t^\top;\\
\mathbf{M}_t &:= \boldsymbol{\eta}_t \left(\boldsymbol{\Sigma}_t^{\mathbf{v}}\right)^{-1} \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \mathbf{b}_t^{\mathbf{v}} + \frac{1}{2}\boldsymbol{\Theta}_t\tilde{\boldsymbol{\Omega}}_t^{-1}\boldsymbol{\Phi}_t^\top;\\
\mathbf{N}_t &:= \boldsymbol{\theta}_t \left(\boldsymbol{\Sigma}_t^{\mathbf{v}}\right)^{-1} \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \mathbf{b}_t^{\mathbf{v}} + \frac{1}{2}\boldsymbol{\Xi}_t\tilde{\boldsymbol{\Omega}}_t^{-1}\boldsymbol{\Phi}_t^\top;\\
\mathbf{X}_t &:= -\frac{1}{2\gamma}\left(\mathbf{b}_t^{\mathbf{v}}\right)^\top \boldsymbol{\Sigma}_t^{**}\mathbf{b}_t^{\mathbf{v}} + \frac{1}{4}\boldsymbol{\Phi}_t\tilde{\boldsymbol{\Omega}}_t^{-1}\boldsymbol{\Phi}_t^\top;\\
\mathbf{Y}_t^\top &:= -\frac{1}{\gamma}\left(\mathbf{a}_t^{\mathbf{v}}\right)^\top \boldsymbol{\Sigma}_t^{**}\mathbf{b}_t^{\mathbf{v}} + \boldsymbol{\phi}_t^\top \left(\boldsymbol{\Sigma}_t^{\mathbf{v}}\right)^{-1} \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \mathbf{b}_t^{\mathbf{v}} + \frac{1}{2}\boldsymbol{\Psi}_t^\top\tilde{\boldsymbol{\Omega}}_t^{-1}\boldsymbol{\Phi}_t^\top;\\
Z_t &:= Z_{t+1} - \frac{1}{2\gamma}\left(\mathbf{a}_t^{\mathbf{v}}\right)^\top \boldsymbol{\Sigma}_t^{**}\mathbf{a}_t^{\mathbf{v}} + \left(\mathbf{a}_t^{\mathbf{v}}\right)^\top \left(\boldsymbol{\Sigma}_t^{\mathbf{v}}\right)^{-1} \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \boldsymbol{\phi}_t - \frac{1}{2\gamma}\boldsymbol{\phi}_t^\top \left(\boldsymbol{\Sigma}_t^*\right)^{-1} \boldsymbol{\phi}_t\\
&\quad + x_t + \frac{1}{4}\boldsymbol{\Psi}_t^\top\tilde{\boldsymbol{\Omega}}_t^{-1}\boldsymbol{\Psi}_t.
\end{aligned}
\tag{A.27}
$$

□

B Proof of Theorem 2.2

According to the above settings, the value function at the maturity becomes

$$V_{T+1}\big[s_{T+1}\big] = -\exp\Big\{-\gamma\Big[W_{T+1} - \big(\mathbf{P}_{T+1} + \boldsymbol{\chi}_{T+1}\overline{\mathbf{Q}}_{T+1}\big)^{\top}\overline{\mathbf{Q}}_{T+1}\Big]\Big\}, \tag{B.1}$$

and thereby, the optimal value function at time T is calculated as follows:

$$\begin{aligned}
V_T\big[s_T\big] &= \sup_{q_T\in\mathbb{R}} \mathbb{E}\Big[V_{T+1}\big[W_{T+1}, \mathbf{P}_{T+1}, \overline{\mathbf{Q}}_{T+1}, \mathbf{R}_{T+1}, \mathbf{v}_T\big]\Big| W_T, \mathbf{P}_T, \overline{\mathbf{Q}}_T, \mathbf{R}_T, \mathbf{v}_{T-1}\Big]\\
&= \sup_{q_T\in\mathbb{R}} \mathbb{E}\Big[-\exp\Big\{-\gamma\Big[W_{T+1} - \big(\mathbf{P}_{T+1} + \boldsymbol{\chi}_{T+1}\overline{\mathbf{Q}}_{T+1}\big)\,\overline{\mathbf{Q}}_{T+1}\Big]\Big\}\Big| s_T\Big]\\
&= \sup_{q_T\in\mathbb{R}} -\exp\Big\{-\gamma\Big[-\mathbf{q}_T^{\top}\tilde{\boldsymbol{\Omega}}_T\mathbf{q}_T + \Big[\overline{\mathbf{Q}}_T^{\top}\boldsymbol{\Theta}_T + \mathbf{R}_T^{\top}\boldsymbol{\Xi}_T + \mathbf{v}_{T-1}^{\top}\boldsymbol{\Phi}_T + \boldsymbol{\Psi}_T^{\top}\Big]\mathbf{q}_T + W_T - \mathbf{P}_T^{\top}\overline{\mathbf{Q}}_t\\
&\quad + \overline{\mathbf{Q}}_T^{\top}\Big[-\Big(\boldsymbol{\chi}_{T+1} + \boldsymbol{\chi}_{T+1}^{\top}\Big) - \frac{1}{2}\gamma\boldsymbol{\Pi}_T\boldsymbol{\kappa}_T\boldsymbol{\Sigma}_T^{\mathbf{v}}\boldsymbol{\kappa}_T\boldsymbol{\Pi}_T - \frac{1}{2}\gamma\boldsymbol{\Sigma}_T^{\boldsymbol{\varepsilon}}\Big]\overline{\mathbf{Q}}_T\\
&\quad + \Big[-\left(\mathbf{a}_T^{\mathbf{v}}\right)^{\top}\boldsymbol{\kappa}_T\boldsymbol{\Pi}_T - \left(\boldsymbol{\mu}_T^{\boldsymbol{\varepsilon}}\right)^{\top}\Big]\overline{\mathbf{Q}}_T + (1-\mathrm{e}^{-\rho})\overline{\mathbf{Q}}_T^{\top}\mathbf{R}_T - \overline{\mathbf{Q}}_T^{\top}\boldsymbol{\Pi}_T\boldsymbol{\kappa}_T\mathbf{b}_T^{\mathbf{v}}\mathbf{v}_{T-1}\Big]\Big\},
\end{aligned} \tag{B.2}$$

where $\boldsymbol{\Pi}_T := \mathrm{e}^{-\rho}\mathbf{A}_T + \mathbf{B}_T$, and

$$\begin{aligned}
\tilde{\boldsymbol{\Omega}}_T &:= \frac{1}{2}\Big(\boldsymbol{\Omega}_T + \boldsymbol{\Omega}_T^{\top}\Big)\\
&= \frac{1}{2}\boldsymbol{\Lambda}_T\left(\mathbf{I}_2 - \boldsymbol{\Pi}_T\right) + \frac{1}{2}\left\{\boldsymbol{\Lambda}_T\left(\mathbf{I}_2 - \boldsymbol{\Pi}_T\right)\right\}^{\top} + \frac{1}{2}\Big(\boldsymbol{\chi}_{T+1} + \boldsymbol{\chi}_{T+1}^{\top}\Big) + \frac{1}{2}\gamma\left(\mathbf{I}_2 - \boldsymbol{\Pi}_T\right)\boldsymbol{\kappa}_T\boldsymbol{\Sigma}_T^{\mathbf{v}}\boldsymbol{\kappa}_T\left(\mathbf{I}_2 - \boldsymbol{\Pi}_T\right)\\
&\quad + \frac{1}{2}\gamma\boldsymbol{\Sigma}_T^{\boldsymbol{\varepsilon}}\ (\in\mathbb{R}^{2\times 2});\\
\boldsymbol{\Theta}_T &:= -\boldsymbol{\Pi}_T\boldsymbol{\Lambda}_T + \Big(\boldsymbol{\chi}_{T+1} + \boldsymbol{\chi}_{T+1}^{\top}\Big) - \gamma\boldsymbol{\Pi}_T\boldsymbol{\kappa}_T\boldsymbol{\Sigma}_T^{\mathbf{v}}\boldsymbol{\kappa}_T\left(\mathbf{I}_2 - \boldsymbol{\Pi}_T\right) + \gamma\boldsymbol{\Sigma}_T^{\boldsymbol{\varepsilon}}\ (\in\mathbb{R}^{2\times 2});\\
\boldsymbol{\Xi}_T &:= -(1-\mathrm{e}^{-\rho})\mathbf{I}_2\ (\in\mathbb{R}^{2\times 2});\\
\boldsymbol{\Phi}_T &:= -\left(\mathbf{b}_T^{\mathbf{v}}\right)^{\top}\boldsymbol{\kappa}_T\left(\mathbf{I}_2 - \boldsymbol{\Pi}_T\right)\ (\in\mathbb{R}^{2\times 2});\\
\boldsymbol{\Psi}_T^{\top} &:= -\left(\mathbf{a}_T^{\mathbf{v}}\right)^{\top}\boldsymbol{\kappa}_T\left(\mathbf{I}_2 - \boldsymbol{\Pi}_T\right) + \boldsymbol{\mu}_T^{\boldsymbol{\varepsilon}}\ (\in\mathbb{R}^{1\times 2}).
\end{aligned} \tag{B.3}$$

We can derive the optimal execution volume satisfying Eq. (B.2) by obtaining the optimal execution volume q_t^* which attains the maximum of

$$\begin{aligned}
K_T\big[\mathbf{q}_T\big] &:= -\mathbf{q}_T^{\top}\tilde{\boldsymbol{\Omega}}_T\mathbf{q}_T + \Big[\overline{\mathbf{Q}}_T^{\top}\boldsymbol{\Theta}_T + \mathbf{R}_T^{\top}\boldsymbol{\Xi}_T + \mathbf{v}_{T-1}^{\top}\boldsymbol{\Phi}_T + \boldsymbol{\Psi}_T^{\top}\Big]\mathbf{q}_T + W_T - \mathbf{P}_T^{\top}\overline{\mathbf{Q}}_T\\
&\quad + \overline{\mathbf{Q}}_T^{\top}\Big[-\Big(\boldsymbol{\chi}_{T+1} + \boldsymbol{\chi}_{T+1}^{\top}\Big) - \frac{1}{2}\gamma\boldsymbol{\Pi}_T\boldsymbol{\kappa}_T\boldsymbol{\Sigma}_T^{\mathbf{v}}\boldsymbol{\kappa}_T\boldsymbol{\Pi}_T - \frac{1}{2}\gamma\boldsymbol{\Sigma}_T^{\boldsymbol{\varepsilon}}\Big]\overline{\mathbf{Q}}_T\\
&\quad + \Big[-\left(\mathbf{a}_T^{\mathbf{v}}\right)^{\top}\boldsymbol{\kappa}_T\boldsymbol{\Pi}_T - \left(\boldsymbol{\mu}_T^{\boldsymbol{\varepsilon}}\right)^{\top}\Big]\overline{\mathbf{Q}}_T + (1-\mathrm{e}^{-\rho})\overline{\mathbf{Q}}_T^{\top}\mathbf{R}_T - \overline{\mathbf{Q}}_T^{\top}\boldsymbol{\Pi}_T\boldsymbol{\kappa}_T\mathbf{b}_T^{\mathbf{v}}\mathbf{v}_{T-1}.
\end{aligned} \tag{B.4}$$

Eq. (B.4) is a quadratic function with a negative definite matrix $\boldsymbol{\Omega}_T + \boldsymbol{\Omega}_T^{\top}$ with respect to $\mathbf{q}_T$, and thereby a concave function with respect to $\mathbf{q}_T$, which leads to the concavity of Eq. (B.2) with respect to $\mathbf{q}_T$. Therefore, by completing the square of $K_T\big[\mathbf{q}_T\big]$ with respect to $\mathbf{q}_T$, we obtain the optimal execution volume at time $t = T$:

$$\begin{aligned}
\mathbf{q}_T^* =: f(s_T) &= \tilde{\boldsymbol{\Omega}}_T^{-1}\Big\{\boldsymbol{\Theta}_T^{\top}\overline{\mathbf{Q}}_T + \boldsymbol{\Xi}_T^{\top}\mathbf{R}_T + \boldsymbol{\Phi}_T^{\top}\mathbf{v}_T + \boldsymbol{\Psi}_T\Big\}\\
&\quad \big(=: \mathbf{a}_{T-1}^* + \mathbf{b}_{T-1}^*\overline{\mathbf{Q}}_{T-1} + \mathbf{c}_{T-1}^*\mathbf{R}_{T-1} + \mathbf{d}_{T-1}^*\mathbf{v}_{T-1}\big),
\end{aligned} \tag{B.5}$$

where

$$\mathbf{a}_T^* := \tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Psi}_T;\quad \mathbf{b}_T^* := \tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Theta}_T^{\top};\quad \mathbf{c}_T^* := \tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Xi}_T^{\top};\quad \mathbf{d}_T^* := \tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Phi}_T^{\top},$$

and by substituting this into Eq. (B.2) the optimal value function becomes

$$V_T[s_T] = -\exp\Big\{ -\gamma\Big[W_T - \mathbf{P}_T^\top\overline{\mathbf{Q}}_T + \overline{\mathbf{Q}}_T^\top\mathbf{G}_T\overline{\mathbf{Q}}_T + \mathbf{H}_T^\top\overline{\mathbf{Q}}_T + \overline{\mathbf{Q}}_T^\top\mathbf{I}_T\mathbf{R}_T + \mathbf{R}_T^\top\mathbf{J}_T\mathbf{R}_T + \mathbf{L}_T^\top\mathbf{R}_T + \overline{\mathbf{Q}}_T^\top\mathbf{M}_T\mathbf{v}_{T-1} + \mathbf{R}_T^\top\mathbf{N}_T\mathbf{v}_{T-1} + \mathbf{v}_{T-1}^\top\mathbf{X}_T\mathbf{v}_{T-1} + \mathbf{Y}_T^\top\mathbf{v}_{T-1} + \mathbf{Z}_T\Big]\Big\}, \quad \text{(B.6)}$$

where

$$\begin{aligned}
\mathbf{G}_T^* &:= -\frac{1}{2}\left(\boldsymbol{\chi}_{T+1} + \boldsymbol{\chi}_{T+1}^\top\right) - \frac{1}{2}\gamma\boldsymbol{\Pi}_T\boldsymbol{\kappa}_T\boldsymbol{\Sigma}_T^{\mathbf{v}}\boldsymbol{\kappa}_T\boldsymbol{\Pi}_T - \frac{1}{2}\gamma\boldsymbol{\Sigma}_T^{\boldsymbol{\varepsilon}} + \frac{1}{4}\boldsymbol{\Theta}_T\tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Theta}_T^\top;\\
\mathbf{H}_T^{*\top} &:= -\left(\mathbf{a}_T^{\mathbf{v}}\right)^\top\boldsymbol{\kappa}_T\boldsymbol{\Pi}_T - \left(\boldsymbol{\mu}_T^{\varepsilon}\right)^\top + \frac{1}{2}\boldsymbol{\Psi}_{T-1}^\top\tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Theta}_{T-1}^\top;\\
\mathbf{I}_T^* &:= (1-\mathrm{e}^{-\rho})\mathbf{I}_2 + \frac{1}{2}\boldsymbol{\Theta}_T\tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Xi}_T^\top; \quad \mathbf{J}_T^* := \frac{1}{4}\boldsymbol{\Xi}_T\tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Xi}_T^\top; \quad \mathbf{L}_T^{*\top} := \frac{1}{2}\boldsymbol{\Psi}_T^\top\tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Xi}_T^\top;\\
\mathbf{M}_T^* &:= -\boldsymbol{\Pi}_T\boldsymbol{\kappa}_T\mathbf{b}_T^{\mathbf{v}} + \frac{1}{2}\boldsymbol{\Theta}_T\tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Phi}_T^\top; \quad \mathbf{N}_T^* := \frac{1}{2}\boldsymbol{\Xi}_T\tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Phi}_T^\top;\\
\mathbf{X}_T^* &:= \frac{1}{4}\boldsymbol{\Phi}_T\tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Phi}_T^\top; \quad \mathbf{Y}_T^{*\top} := \frac{1}{2}\boldsymbol{\Psi}_T^\top\tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Phi}_T^\top; \quad Z_T := \frac{1}{4}\boldsymbol{\Psi}_T^\top\tilde{\boldsymbol{\Omega}}_T^{-1}\boldsymbol{\Psi}_T.
\end{aligned} \quad \text{(B.7)}$$

For $t \in \{T-1,\dots,1\}$, we can recursively derive the optimal execution volume and optimal value function at each time by a similar derivation which we use to obtain the optimal execution volume in the last subsection for time $t \in \{T-2,\dots,1\}$. □

Graduate School of Economics
Osaka University, Osaka, 560–0043, Japan
&
Center for Mathematical Modeling and Data Science
Osaka University, Osaka, 560–8531, Japan
E-mail address: ohnishi@econ.osaka-u.ac.jp

Masamitsu OHNISHI

Graduate School of Economics
Osaka University, Osaka, 560–0043, Japan
&
Graduate School of Management
Tokyo Metropolitan University, Tokyo, 100–0005, Japan
E-mail address: shimosshi1q94@gmail.com

Makoto SHIMOSHIMIZU