An inverse problem for stationary Kirschhoff plate equations

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Abstract

We consider an inverse problem for stationary Kirschhoff plate equations. It is proved that a single pair of surface Cauchy data $(u, \Delta u)$ uniquely determine an inclusion where the deflection and bending displacement of a plate vanish.

1 Introduction and main result

This note is concerned with an inverse problem arising from plate bending problems modelled by the Kirchhoff theory of plates in elasticity. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$ (i.e., C^2), and let $D \subset \Omega$ be an open subset of Ω such that $\Omega\backslash\overline{D}$ is connected. In the stationary case, we consider an isotropic, homogeneous plate in the region $\Omega\backslash\overline{D}$ under pure bending governed by (which is also known as stationary Euler-Bernoulli equation)

$$\begin{cases} \Delta^2 u = \omega^2 u & \text{in} \quad \Omega\backslash\overline{D}, \\ u = \Delta u = 0 & \text{on} \quad \partial D, \\ u = f, \quad \Delta u = g & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $f \in H^{3/2}(\partial\Omega)$, $g \in H^{-1/2}(\partial\Omega)$. In (1), u and Δu represent the deflection and the bending displacement of the plate, respectively. The frequency $\omega > 0$ is assumed to be such that the above Kirchhoff plate problem admits a unique solution

$$u \in X := \{u : u \in H^2(\Omega\backslash\overline{D}), \quad u = \Delta u = 0 \quad \text{on} \quad \partial D\}.$$

In this paper we are interested in the inverse problem of recovering ∂D from knowledge of a single pair $(f, g) \in H^{3/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$.

Theorem 1.1. *Suppose that D is a polygon and $|f| > 0$ on Ω. Then the interior boundary ∂D can be uniquely determined by the observation data (f, g).*

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2 Lemma

Define $x' := (x_1, -x_2)$ for $x = (x_1, x_2) \in \mathbb{R}^2$.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded symmetric domain with respect to the x_1-axis and write $\Gamma := \Omega \cap \{x_2 = 0\}$. Suppose that $u \in H^2(\Omega)$ is a solution to*

$$\Delta u = v \quad in \quad \Omega, \qquad u = 0 \quad on \quad \Gamma,$$

where $v \in L^2(\Omega)$ satisfies the symmetric relation $v(x) = -v(x')$ for all $x \in \Omega$. Then we have the same symmetry for u, that is, $u(x) = -u(x')$ for all $x \in \Omega$.

Proof. Set $\Omega^\pm := \Omega \cap \{x : x_2 \gtrless 0\}$. We extend $u|_{\Omega^+}$ from Ω^+ to Ω as follows

$$w(x) = \begin{cases} u(x) & \text{if} \quad x \in \Omega^+, \\ -u(x') & \text{if} \quad x \in \Omega^-. \end{cases}$$

Since $u = 0$ on Γ and $v(x) = -v(x')$, it is easy to verify $\Delta w = v$ in Ω and $w = u$, $\partial_\nu w = \partial_\nu v$ on Γ. By Holmgren's uniqueness theorem, we obtain $w = u$ in Ω, implying that $u(x) = -u(x')$ for $x \in \Omega^-$. By the symmetry of the domain Ω, we obtain $u(x) = -u(x')$ for all $x \in \Omega$. $\square$

Lemma 2.2. *Let Ω and Γ be given as in Lemma 2.1. Suppose that $u \in H^2(\Omega)$ is a solution to*

$$\Delta^2 u = 0 \quad in \quad \Omega, \qquad u = \Delta u = 0 \quad on \quad \Gamma.$$

Then (i) $u(x) = -u(x')$ for all $x \in \Omega$. (ii) If $u = \Delta u = 0$ on a line segment $L \subset \Omega$, then the same relations hold on $L' := \{x' : x \in L\}$.

Proof. Setting $v = \Delta u \in L^2(\Omega)$, we see $\Delta v = 0$ in Ω and $v = 0$ on Γ. By reflection principle for harmonic functions, we get the symmetric relation $v(x) = -v(x')$ for $x \in \Omega$. Since $u = 0$ on Γ, applying Lemma 2.1 gives the relation in the first assertion. The second assertion follows directly from the first one. $\square$

Remark 2.1. *Lemma 2.1 applies to the following system:*

$$\Delta^2 u - \omega^2 u = 0 \quad in \quad \Omega, \qquad u = \Delta u = 0 \quad on \quad \Gamma,$$

where $\omega > 0$. In fact, the above boundary value problem can be equivalently formulated as the system

$$\begin{aligned} \Delta U &= AU, \quad U = \begin{pmatrix} u \\ \Delta u \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ \omega^2 & 0 \end{pmatrix}, \\ U &= 0 \quad on \quad \Gamma. \end{aligned}$$

Applying the proof of Lemma 2.1, one can show that $U(x) = -U(x')$ for $x \in \Omega$.

3 Proof of Theorem 1.1

Suppose that D_1 and D_2 are two polygons contained inside Ω, and let (f_j, g_j) be the boundary observations on $\partial\Omega$ that correspond to solutions u_j to (1) with $D = D_j$. Assuming that $(f_1, g_1) = (f_2, g_2)$ and $|f_j| > 0$ $(j = 1, 2)$, we need to prove that $D_1 = D_2$. To prove Theorem 1.1, we employ path and reflection arguments first developed in [1] for the Helmholtz equation and later modified in [4, 10]. We shall carry out the proof following the arguments in [4, Section 3.1] but modified to be applicable for the equation (1).

Proof of Theorem 1.1 Suppose on the contrary that $D_1 \neq D_2$, we shall derive a contraction by two steps.

Step 1. Existence of a nodal set. Set $v_j = \Delta u_j$ and $U_j = (u_j, v_j)$. Then

$$\Delta U_j = A U_j \quad \text{in} \quad G, \quad U_j = (f_j, g_j)^\top \quad \text{on} \quad \partial\Omega,$$

where G denotes the connected component of $(\Omega\backslash\overline{D}_1) \cap (\Omega\backslash\overline{D}_2)$ such that $\partial\Omega \subset \partial G$. The coincidence of the observation data $f_1 = f_2$ and $g_1 = g_2$ on $\partial\Omega$ together with the assumption of ω gives arise to $U_1 = U_2 := U$ in G. This in turn implies

$$u_1 = u_2 \quad \text{in} \quad G. \tag{2}$$

The nodal set of u_j, which we denote by Σ_j, is defined as the set of line segments in $\overline{\Omega}\backslash\overline{D}_j$ on which both u_j and Δu_j vanish. Since $\Omega\backslash\overline{D}_j$ is connected, we obtain $\partial G\backslash\partial\Omega \nsubseteq D_1 \cap D_2$. Hence, without loss of generality we assume that

$$S := (\partial D_1 \backslash \partial D_2) \cap \partial G \neq \emptyset.$$

Since both D_1 and D_2 are polygons, we can always find a line segment L lying on S. By (2), this implies that

$$u_2 = u_1 = 0, \qquad \Delta u_2 = \Delta u_1 = 0 \qquad \text{on} \quad L \subset \Omega\backslash\overline{D}_2$$

and thus $\Sigma_2 \neq \emptyset$.

Step 2. Derive a contradiction by path and reflection arguments. Since u_2 is analytic, $\Omega\backslash\overline{D}_2$ is connected and $|f_2| > 0$ on $\partial\Omega$, the set Σ_2 must be bounded. Otherwise, Σ_2 must intersect with $\partial\Omega$ at some point O, leading to contraction with $|f_2(O)| > 0$. On the other hand, the two end points of any nodal line segment of Σ_2 must lie on ∂D_2. Choose a point $x_0 \in L \subset \partial G$ and a continuous and injective path $\gamma(t)$ $(t \geq 0)$ connecting x_0 and some point $y \in \partial\Omega$. Without loss of generality, we suppose that $\gamma(0) = x_0$ and $\gamma(T) = y$ for some $T > 0$. Denote by $\mathcal{M}$ the set of intersection points of γ with all nodal sets of u_2, i.e.,

$$\mathcal{M} := \{x : x \in \{\gamma(t) : t \in [0, T]\} \cap \Sigma_2\}.$$

The set $\mathcal{M}$ is not empty, since at least $x_0 = \gamma(0) \in \mathcal{M}$. Obviously, $y = \gamma(T) \notin \mathcal{M}$.

Observe that the set $\mathcal{M}$ is bounded. Moreover, it is closed, hence compact; see the arguments in the proof of [10, Lemma 2]. Thus, there exists $T^* \in (0, T)$ such that

$\gamma(t^*) \in \mathcal{M}$ and $\{\gamma(t) : t \in (T^*, T)\} \cap \mathcal{M} = \emptyset$. Let $L^* \subset \Sigma_2$ be the finite nodal line segment possing through x^*. We now apply the reflection principle of Lemma 2.2 (ii) to prove the existence of a new nodal line segment of u_2 which intersects Ω.

By coordinate rotation we can assume without loss of generality that L^* lies on the x_1-axis. Note that the two end points of L^* must lie on ∂D_2. Choose $x^+ = \gamma(T^* + \epsilon)$ for $\epsilon > 0$ sufficiently small and $x^- := (x^+)'$. Let $\Sigma^\pm$ be the connected component of $\Omega \backslash (L^* \cup D_2)$ containing $x^\pm$, and denote by $E^\pm$ the connected component of $\Sigma^\pm \cap (\Sigma^\mp)'$ containing $x^\pm$.

Setting $E = E^+ \cup L^* \cup E^-$. Then E is a connected open set with the boundary $\partial E \subset \partial D_2 \cup (\partial D_2)' \cup \partial\Omega$. Applying the reflection principle for bi-harmonic functions (see Remark 2.1), we get $u_2 = \Delta u_2 = 0$ on ∂E^+, because the same conditions hold on both L^* and $(E^+)'$. By the assumption $|f_2| > 0$ on $\partial\Omega$, we see $\partial E^+ \cap \partial\Omega = \emptyset$, implying that $E \subset \Omega$ is a bounded open set containing $\gamma(T^*)$. Recalling the definition of $\gamma(t)$, we conclude that $\gamma(t)$ must intersect ∂E at some $t' > T^*$. Therefore, there must exist a new nodal line segment passing through $\gamma(t')$. This is a contradiction to the definition of T^* and L^*. This contradiction implies $D_1 = D_2$.

Remark 3.1. *(i) The positivity assumption* $|f| > 0$ *on* $\partial\Omega$ *can be replaced by either the distance assumption* $\mathrm{diam}(D) < \mathrm{dist}(D, \Omega)$ *or the irrational condition of each corner of D; see [6, 11].*

(ii) The proof of Theorem 1.1 implies that u must be "singular" (that is, non-analytic) at corner points. This excludes the possibility of analytical extension in a corner domain and is important in designing inversion algorithms with a single measurement data; see e.g. the enclosure method [7], the range test approach [8, 9] as well as [12, Chapter 5] and the data-driven scheme [5].

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