The uniqueness problem of Rayleigh wave in Kelvin viscoelastic half-space

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1 Introduction

Waves that propagate in elastic solid can be divided into two main categories: body waves and surface waves. Two and only two types of body waves in an unbounded solid, namely longitudinal wave (P-wave) and shear waves (SV-wave and SH-wave), can be propagated independently. Rayleigh wave is one kind of surface waves which propagates along the free boundary and decays exponentially away from the boundary. Rayleigh wave is essentially the formation of interference on the surface of the medium of P-wave and SV-wave. It was firstly introduced as a solution of the free boundary problem for an elastic half space by Lord Rayleigh (1885) [1], who summed it up as the simultaneous solution of the equations of P-wave and SV-wave. Yet to this day, the method of the characteristic equation of surface wave velocity he deduced is still the important way to seek for Rayleigh wave [2].

There remains a dispute about the number of Rayleigh waves in viscoelastic medium until now. Rayleigh wave in viscoelastic media was early and deeply studied by Scholte [3], who proofed that Rayleigh wave also existed on the surface of half-space viscoelastic media, but he was not sure whether the Rayleigh wave he calculated was the only one valid. In 1960, from the mathematical aspect of the effective characteristic root, Bland [4] raised the question of the existence and uniqueness of Rayleigh wave in viscoelastic media "It has not yet been shown that for any viscoelastic material there is one and only one such root" [4, p.75]. In response to this question, after a series of studies based on linear viscoelastic models, Currie et al. [5,6,7] came to a conclusion in 1977 that there was more than one possible Rayleigh wave in the viscoelastic half-space surface. In most cases, there existed two while in some special material there would even be three possible Rayleigh waves [6]. What's more, they predicted that the velocity of viscoelastic surface wave would sometimes be higher than that of the body wave, and there would even be retrograde wave [5,6].

After that, Bland's question seemed to be solved. But, on one hand, there is no report about the practical examples that the velocity of surface wave will surpass the body wave velocity nor the retrograde propagation phenomenon. On the other hand, in 2001 Romeo [8,9] promoted Nkemzi's elastic Rayleigh wave equation [10] to linear viscoelastic situation, thus put forward an opposite viewpoint. He took advantage of complex modulus to analyze the root number of the characteristic equation of viscoelastic half-space Rayleigh wave. Although his derivation process contained multiple variable substitutions, which appeared complicated and obscure, he deemed that there is only one complex root valid, while the other two are invalid. This implied that there is only one truly valid Rayleigh wave in viscoelastic half-space surface.

Obviously, Romeo's opinion denied the conclusion by Currie et al., but it hasn't been widely accepted yet. Except that Ivanov and Savova's [11] study supports Romeo's opinion only in the sense of wavelength fixed condition, studies in the common sense of frequency fixed condition [12,13] and more researchers' studies [14,15,16] are in favor of the conclusion of Currie et al. The possibility of the existence of a retrograde Rayleigh surface wave is even suggested [16], and there maybe third types of Rayleigh waves in the case of some special combinations of material parameters [16]. What is worth noting is that the study in 2014 by Chirita et al [16]. is carried out based on Kelvin viscoelastic model, which is simple but very commonly adopted in practical engineering. It is interesting but a little unbelievable that the process of derivation, settings of variable and multiple variable substitutions by Chirita et al. in the study [16] were similar to Romeo's method, but in the conclusion, Romeo's opinion is completely negated. The obscure of mathematical derivation process in Chirita et al.'s article brings some difficulty to understanding for us, but the authors' work arouses our interest and makes us want to find out the truth.

This paper is to propose a brief way of handling this essential problem within half-space Kelvin viscoelastic medium. Starting from the dynamic equations of transverse wave and longitudinal wave based on Kelvin viscoelastic model, we derive the characteristic equation of Rayleigh surface wave in Kelvin viscoelastic half-space in a much simpler way and in terms of complex wavenumber. We then propose a relatively convenient and concise method which is able to directly analyze the validity of characteristic roots in a physical sense. Finally, the uniqueness problem of Rayleigh wave is discussed and we reach the conclusion that there is only one Rayleigh wave in Kelvin viscoelastic half-space surface, confirming Romeo's [8,9] conclusion under the assumption of Kelvin model. Meanwhile, we point out the error of Chirita et al [16] have made in handling the result, negating their viewpoint that there is not only one Rayleigh wave in Kelvin viscoelastic half-space surface.

2 Basic equations in viscoelastic medium

According to the theory of Continuum Mechanics, the dynamic equilibrium equation neglecting body force is expressed as

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sigma_{ij,j} \tag{2.1}$$

where σ_{ij} is second order symmetric stress tensor, ρ is the density of the medium, and u_i is the displacement vector, $i, j = 1,2,3$ denote the axis.

For a linear isotropic viscoelastic medium, the constitutive equation can be uniformly expressed as

$$\sigma_{ij} = \lambda(t)\delta_{ij} * \mathrm{d}u_{k,k} + \mu(t) * \left[\mathrm{d}u_{i,j} + \mathrm{d}u_{j,i}\right] \tag{2.2}$$

where $\lambda(t), \mu(t)$ are the Lame coefficients of viscoelastic material which depend on the time, * denotes convolution operator, δ_{ij} is the Kronecker symbol. By submitting the relation (2.2) into (2.1), we obtain the displacement vector satisfying the dynamic equilibrium equation

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \left[\lambda(t)+\mu(t)\right] * \mathrm{d}u_{j,ji} + \mu(t) * \mathrm{d}u_{i,jj} \tag{2.3}$$

As for constitutive relations of linear viscoelastic medium, there are exactly many mathematical models. In this paper we adopt Kelvin model, which is a simplified model more efficient for describing the attenuation of elastic waves. It is also called Voigt model or Kelvin-Meyer-Voigt model [17, p.99] and it is commonly used in practical engineering, especially in the field of seismic exploration.

Under the assumption of the Kelvin model, the above equation of displacement vector can then be expressed as

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \left[(\lambda+\mu)+(\lambda'+\mu')\frac{\partial}{\partial t}\right]u_{j,ji} + \left(\mu+\mu'\frac{\partial}{\partial t}\right)u_{i,jj} \tag{2.4}$$

Where, λ and μ are the elastic partition of Lame coefficients, while λ' and μ' describe two variables of the viscous partition which is independent on time, whose effect is corresponding to the λ and μ in describing elastic medium.

Applying Helmholtz decomposition theorem on displacement vector field

$$u_i = \Phi_{,i} + e_{ijk}\Psi_{k,j}, \quad \Psi_{k,k} = 0$$

Where Φ is the scalar potential function, Ψ_k is the vector potential function, $k = 1,2,3$ denote the axis. Substitute it in (2.4), change the differential order, the longitudinal wave equation and the transverse wave equation can be obtained in expression of potential functions

$$\begin{aligned} \rho\frac{\partial^2 \Phi}{\partial t^2} &= (\lambda+2\mu)\Phi_{,jj} + (\lambda'+2\mu')\left(\frac{\partial \Phi}{\partial t}\right)_{,jj} \\ \rho\frac{\partial^2 \Psi_k}{\partial t^2} &= \mu\Psi_{k,jj} + \mu'\left(\frac{\partial \Psi_k}{\partial t}\right)_{,jj} \end{aligned} \tag{2.5}$$

The above equation(2.5) shows that, in Kelvin viscoelastic medium, there are also two kinds of independent wave modes like in the elastic medium, namely, the P-wave and the S-wave. But we should pay attention to the addition of first order partial derivative term for time in their wave equations.

3 The characteristic equation for viscoelastic medium Rayleigh wave

As shown in Fig. 1, the lower half space is characterized by Kelvin viscoelastic medium, whose elastic coefficients are λ and μ. Its density is ρ and the viscoelastic coefficients are λ' and μ'. The origin O of the axis is located on the free surface of the half space, with the axis Ox on the free surface and the vertically downward axis Oz pointing to the inner medium part. Assume the propagation direction of the wave pointing x along the plane xz, so the displacement component u_x, u_z are independent on y and $u_y = 0$ so that it could be considered as a plane problem. The vibrations of particles happen only in plane xz and the potential function of the displacement vector in space is independent on y.

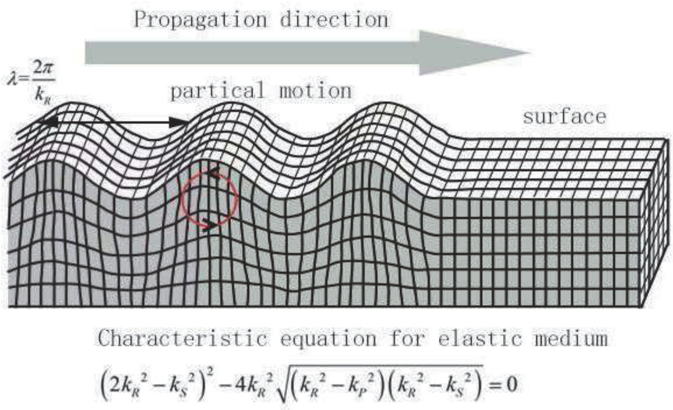

Fig. 1 Diagram of Rayleigh wave (modified from picture on wiki pedia)

Since the formation of Rayleigh wave propagating on the free surface is due to the interference and superposition of the transverse wave and longitunidal wave, thus the problem comes down to establishing the simultaneous solution of the P-wave equation and the S-wave equation, meanwhile finding the solution that the free surface boundary condition equation is satisfied. Here as Fig 1. shows, to be specifically, the wave equation (2.5) can be expressed as

$$\begin{aligned}
\rho\frac{\partial^2\Phi}{\partial t^2} &= (\lambda+2\mu)\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}\right)+(\lambda'+2\mu')\frac{\partial}{\partial t}\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}\right)\\
\rho\frac{\partial^2\Psi_k}{\partial t^2} &= \mu\left(\frac{\partial^2\Psi_k}{\partial x^2}+\frac{\partial^2\Psi_k}{\partial z^2}\right)+\mu'\frac{\partial}{\partial t}\left(\frac{\partial^2\Psi_k}{\partial x^2}+\frac{\partial^2\Psi_k}{\partial z^2}\right)
\end{aligned} \tag{3.1}$$

Where the scalar potential function is $\Phi = \Phi(x, z, t)$, the vector potential function Ψ_k is taken as $\Psi_y(x, z, t)$. From equation (3.1) it is easy to find that Φ and Ψ_y satisfy the same form of equation, so they should have the same form of expression. To begin with we try to find the solution in the form bellow

$$\begin{aligned} \Phi(x, z, t) &= ae^{-\alpha z}\,\mathrm{e}^{i(Kx-\omega t)} \\ \Psi_y(x, z, t) &= be^{-\beta z}\,\mathrm{e}^{i(Kx-\omega t)} \end{aligned} \tag{3.2}$$

Where a and b are both real non-zero constants; α and β are positive real constants just like the condition in elastic medium, expressing the attenuations on the direction of axis z; ω denotes the circular frequency of the motivation source. What is different from that in the elastic condition is K, which denotes the complex circular wavenumber, whose imaginary part reflects the attenuation effect of viscoelastic medium on the propagation along the free surface. α and β above act as coefficients attenuating exponentially with the increase of depth in medium (both in elastic and viscoelastic condition), which are firstly introduced by Rayleigh and commonly accepted. So that the undetermined coefficient K, namely, the complex wavenumber is the key factor to determine the character of Rayleigh surface wave in Kelvin viscoelastic medium. In order to determine K, the boundary stress condition must be used:

$$\begin{aligned} \sigma_{zx}\,|_{z=0} &= 0 \\ \sigma_{zz}\,|_{z=0} &= 0 \end{aligned} \tag{3.3}$$

Taking use of constitutive equation of Kelvin viscoelastic medium, the stress σ_{zx}, σ_{zz} are expressed by displacement u_x, u_z. Finally, substitute the potential functions above with it

$$\begin{aligned} u_x &= \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi_y}{\partial z} \\ u_z &= \frac{\partial \Phi}{\partial z} + \frac{\partial \Psi_y}{\partial x} \end{aligned} \tag{3.4}$$

Then

$$\begin{aligned} \sigma_{zx} &= \left(\mu + \mu' \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \Psi_y}{\partial x^2} - \frac{\partial^2 \Psi_y}{\partial z^2} + 2\frac{\partial^2 \Phi}{\partial x \partial z}\right) \\ \sigma_{zz} &= \left[(\lambda + 2\mu) + (\lambda' + 2\mu')\frac{\partial}{\partial t}\right]\frac{\partial^2 \Phi}{\partial z^2} + \left(\lambda + \lambda' \frac{\partial}{\partial t}\right)\frac{\partial^2 \Phi}{\partial x^2} + 2\left(\mu + \mu' \frac{\partial}{\partial t}\right)\frac{\partial^2 \Psi_y}{\partial x \partial z} \end{aligned} \tag{3.5}$$

Substitute (3.5) with (3.2) and simplify it, on the free surface $z = 0$ it satisfies

$$\sigma_{zx}\big|_{z=0}=\left\{-2i\alpha K\left(\mu-i\omega\mu'\right)\cdot a-\left[\mu\left(K^2+\beta^2\right)-i\omega\mu'\left(K^2+\beta^2\right)\right]\cdot b\right\}\mathrm{e}^{i(Kx-\omega t)}$$
$$\sigma_{zz}\big|_{z=0}=\left\{\left\{\left[\left(\lambda+2\mu\right)\alpha^2-\lambda K^2\right]-i\omega\left[\left(\lambda'+2\mu'\right)\alpha^2-\lambda' K^2\right]\right\}\cdot a-2i\beta K\left(\mu-i\omega\mu'\right)\cdot b\right\}\mathrm{e}^{i(Kx-\omega t)} \tag{3.6}$$

Substitute (3.3) with (3.6), we get the homogeneous linear equations for a and b

$$-2i\alpha K\left(\mu-i\omega\mu'\right)\cdot a-\left[\mu\left(K^2+\beta^2\right)-i\omega\mu'\left(K^2+\beta^2\right)\right]\cdot b=0$$
$$\left\{\left[\left(\lambda+2\mu\right)\alpha^2-\lambda K^2\right]-i\omega\left[\left(\lambda'+2\mu'\right)\alpha^2-\lambda' K^2\right]\right\}\cdot a-2i\beta K\left(\mu-i\omega\mu'\right)\cdot b=0 \tag{3.7}$$

If there is non-zero root for (3.7), the coefficient determinant should be zero. Rearrange it

$$(i\omega\mu'-\mu)(\beta^2+K^2)\left\{\left[(\lambda+2\mu)\alpha^2-\lambda K^2\right]-i\omega\left[(\lambda'+2\mu')\alpha^2-\lambda' K^2\right]\right\}-4(i\mu+\omega\mu')^2\alpha\beta K^2=0$$
(3.8)

Equation(3.8)above is the characteristic equation to determine the complex wavenumber K of Rayleigh surface wave. It is seen that for this problem, coefficient K is not only related to the material modulus(λ, μ, λ', μ'),but also related to the circular frequency ω. Meanwhile it has mutual inference with (α, β)

What is special is that, in purely elastic condition, $\lambda'=\mu'=0$, the imaginary part of complex wavenumber K is zero. In this condition, K is exactly the circular wavenumber k_R in elastic Rayleigh wave. What has been known is that $\alpha^2={k_R}^2-k_P^2, \beta^2={k_R}^2-k_S^2$, here k_P, k_S are respectively the circular wavenumbers of elastic longitudinal wave and transverse wave. The equation of Kelvin viscoelastic Rayleigh wave (3.8) retrogrades to

$$(2{k_R}^2-k_S^2)^2-4{k_R}^2\sqrt{({k_R}^2-k_P^2)({k_R}^2-k_S^2)}=0 \tag{3.9}$$

Equation (3.9) above is well-known equation for Rayleigh wave in elastic medium.

Apparently, for particular case, the values of complex wavenumber K can be obtained by solving characteristic equation of Kelvin viscoelastic Rayleigh wave(3.8).But commonly it is complicated to find the analytic solution of complex wavenumber K. We try to get simpler form of equation (3.8) to analyze the propagation character of Rayleigh wave, especially the number of Rayleigh wave in viscoelastic medium. We introduce two more complex moduli

$$\Lambda=\lambda-i\omega\lambda',\quad \mathrm{M}=\mu-i\omega\mu'$$

It is easily found that the newly introduced complex moduli Λ, M are composed of two parts. The real parts of them are the Lame elastic coefficient λ, μ ,while the imaginary parts act as the viscous part λ' , μ' of Lame coefficient times circular frequency ω , In the same time we can keep the dimension consistent. So, equation (3.8) is rearranged into following form

$$\Lambda K^4 + [\Lambda\beta^2 - (\Lambda + 2\mathrm{M})\alpha^2 + 4\mathrm{M}\alpha\beta]K^2 - \alpha^2\beta^2(\Lambda + 2\mathrm{M}) = 0 \qquad (3.10)$$

In the deduction above, we follow the use of α, β as undetermined coefficients, but the meaning of them has changed. We try to set relations bellow

$$\begin{aligned} \alpha^2 &= K^2 - K_P^2 \\ \beta^2 &= K^2 - K_S^2 \end{aligned} \qquad (3.11)$$

Where $K_P^2 = \dfrac{\rho\omega^2}{\Lambda + 2\mathrm{M}}$, $K_S^2 = \dfrac{\rho\omega^2}{M}$. K_P, K_S respectively express the wavenumbers of longitudinal wave and transverse wave in viscoelastic medium. Like the condition in the Rayleigh wavenumber K, it is also complex. Substitute (3.10) with (3.11), rearrange it

$$(2K^2 - K_S^2)^2 - 4K^2\sqrt{(K^2 - K_P^2)(K^2 - K_S^2)} = 0 \qquad (3.12)$$

Thus, we have derived the Rayleigh wave equation (3.12) in viscoelastic medium, though the same in the form, but the related coefficients are the solution for the characteristic equation of viscoelastic Rayleigh wave.

Based on equation (3.12) above we could solve the complex wavenumber K of surface wave. Now K is not only related to the material coefficients of material $\lambda, \mu, \lambda', \mu'$,but also related to circular frequency ω .What's more it has relationship with complex numbers α, β . Rearrange the equation above, we can get six order equation about K

$$16(K_S^2 - K_P^2)K^6 + (16K_P^2K_S^2 - 24K_S^4)K^4 + 8K_S^6K^2 - K_S^8 = 0 \qquad (3.13)$$

Beacause equation (3.13) includes only even order terms, there are three pairs of complex roots with inter conjugate. In the analysis bellow, we only focus on three complex roots whose imaginary parts are positive.

4. The uniqueness problem of Rayleigh wave

Equation (3.13) can be simplified into

$$16\left(1 - \frac{K_P^2}{K_S^2}\right)R^6 + \left(16\frac{K_P^2}{K_S^2} - 24\right)R^4 + 8R^2 - 1 = 0 \qquad (4.1)$$

Where $R=\dfrac{K}{K_S}$ is the ratio between the wavenumber of Rayleigh wave to that of transverse wave, let $Z=\dfrac{\Lambda}{\mathrm{M}}$ be the ratio of complex modulus, then equation (4.1) is further simplified into

$$16\left(Z+1\right)R^6-8(3Z+4)R^4+8(Z+2)R^2-(Z+2)=0 \tag{4.2}$$

If we pay attention to the real part and imaginary part of complex modulus ratio Z, namely let $Z=p+qi$, where $p=\dfrac{\lambda\mu+\left(\omega\lambda'\right)\left(\omega\mu'\right)}{\mu^2+\left(\omega\mu'\right)^2}, q=\dfrac{\lambda\left(\omega\mu'\right)-\mu\left(\omega\lambda'\right)}{\mu^2+\left(\omega\mu'\right)^2}$, p,q reflect the ratio relationship between Lame coefficients. So it should satisfy $p>0$. Based on the values of the real part p and imaginary part q of Z, the solution R to equation(4.2) can be solved. We can use it to analyze the number of Rayleigh wave.

Rayleigh wave is generated by interference between longitudinal wave and transverse wave, therefore in a physical sense, the velocity (phase velocity) of Rayleigh wave should not surpass the velocity of transverse wave. Because the phase velocity of Rayleigh wave can be decided by $v=\dfrac{\omega}{\mathrm{Re}\left(K\right)}$, the phase velocity of transverse wave is decided by $v_S=\dfrac{\omega}{\mathrm{Re}\left(K_S\right)}$. The Rayleigh wave velocity is lower than that of transverse wave, that is

$$\mathrm{Re}\left(K\right)>\mathrm{Re}\left(K_S\right) \tag{4.3}$$

If we take K,K_S,R in polar coordinate system, from $K_S\left(\rho_S,\theta_S\right),R\left(\rho_R,\theta_R\right)$ we can obtain $K\left(\rho_S\rho_R,\theta_S+\theta_R\right)$. Then, According to equation(4.3), we have

$$\rho_S\rho_R\cos\left(\theta_S+\theta_R\right)>\rho_S\cos\left(\theta_S\right)$$

Simplified into

$$\left(\rho_R\cos\theta_R-1\right)\cos\theta_S>\rho_R\sin\theta_R\sin\theta_S \tag{4.4}$$

Judging from $K_S^2=\dfrac{\rho\omega^2}{\mathrm{M}}$, M's augment $\theta_M\in\left[0,-\dfrac{\pi}{2}\right)$, K_S's augment $\theta_S\in\left[0,\dfrac{\pi}{4}\right)$,take consideration of equation(4.3), it can be known that when $\theta_R>0$, $\rho_R\cos\theta_R>1$;when $\theta_R<0$, $\rho_R\cos\theta_R>1+\rho_R\sin\theta_R$. Because when **Z** is conjugated, the complex root R in correspond should also be conjugated, so the complex root R should satisfy

$$\mathrm{Re}(R) > 1 - \left|\mathrm{Im}(R)\right| \tag{4.5}$$

Namely the root should be located to the left side of the purple line in Fig 2(a)

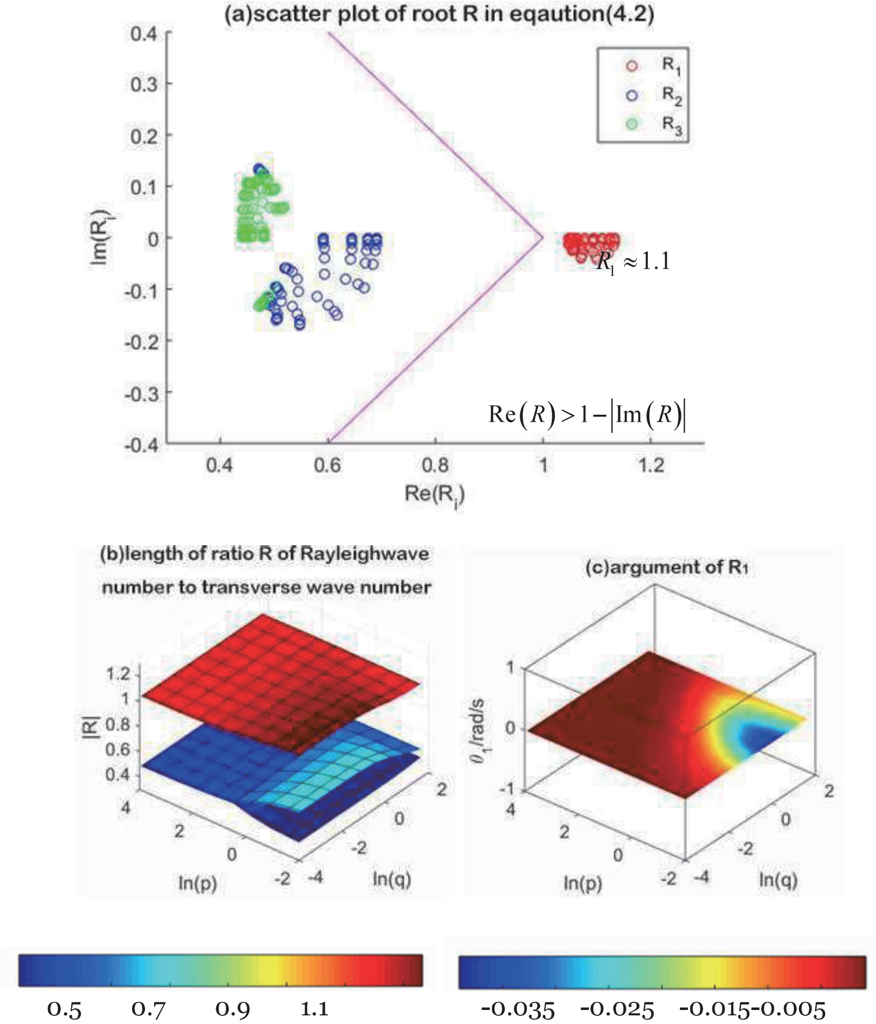

Fig. 2 relation between characteristic root of equation(4.2)and coefficient p,q （take $p \in [0.01,10000], q \in [0.0001,100]$ ） (a) scatter plot of characteristic root；(b) the relation between variation of characteristic root and p,q ； (c)the relation between argument R_1 and p,q .

From the result we can know that there is only one root R_1 that satisfies the Rayleigh wave equation to relation $\mathrm{Re}(K) > \mathrm{Re}(K_S)$. Namely, there is only one possible Rayleigh wave in correspond to the characteristic root R_1 correspond. The characteristic roots R_2, R_3 are against the physical nature that the velocity of Rayleigh wave should be lower than the transverse wave, so they are all invalid. What's more, since the argument R_1 is rather small, it could be known that $K \approx 1.1K_S$ .Namely, similar to the condition of elastic medium, the phase velocity of Rayleigh wave is about 0.9 times that of transverse wave.

Then we analyze the study of Currie et al[5,6,7] in 1977 that there is not only one valid Rayleigh wave in linear viscoelastic half-space medium as well as the study of Chirita et al[16] in 2014 on the method of solving several solutions of the Rayleigh wave in Kelvin viscoelastic condition. When they were studying the characteristic roots of Rayleigh wave, they also required that the characteristic roots should satisfy the charactrer of attenuation and the propagation direction, but none of them have taken consideration of the relative value between the velocity of Rayleigh wave and transverse wave. Take the example of the article of Chirita et al[16], the table below shows their several calculation examples. We find that there is only one root close to R_1 in this paper whose corresponding Rayleigh wave velocity is lower than that of transverse wave, the ratio modulus of the other two roots to the velocity of transverse wave are larger than 2. Namely, the velocities of Rayleigh wave are far higher than that of transverse wave, which fits the condition of R_2, R_3 in this article. So, in physical sense, there is only one valid Rayleigh wave in their calculation examples. So it could be considered that the wrong result concluded in the reference of Chirita et al[16] is due to their error in handling the results.

Table 2 The values of v, r_1 and r_2 for the set of the Rayleigh wave solutions for some specified viscoelastic materials characterized by $\lambda_0 = \mu_0$, C_1 and C_2

C_1	C_2	v/c_2	r_1	r_2
0.5	1.0	$0.860958 - 0.423783i$	$0.862209 + 0.116357i$	$0.346807 - 0.0847438i$
1.0	0.8	$\boxed{0.492535 - 1.66342i}$	$1.16655 + 1.35667i$	$0.392526 + 2.17912i$
		$0.864529 - 0.326975i$	$0.840434 + 0.0303701i$	$0.39616 - 0.0314553i$
1.75	1.0	$0.38025 - 1.97115i$	$0.0141401 - 0.513189i$	$0.0090599 - 1.69235i$
modulus length is greater than 2		$0.743741 - 1.60562i$	$0.0234002 + 0.265052i$	$0.0215112 + 1.5048i$
		$0.816203 - 0.423236i$	$0.848123 - 0.00121024i$	$0.3926790 + 0.00120344i$

Fig. 3 Some analysis of numerical example from Chirita et al.

5 Conclusion

We derive the characteristic equation of Rayleigh surface wave in Kelvin viscoelastic half-space in a much simpler way and in terms of complex wavenumber. We then propose a new and simple method which is able to directly analyze the validity of characteristic roots in a physical sense. Finally, we reach the conclusion that there is only one Rayleigh wave in Kelvin viscoelastic half-space surface, confirming Romeo's [8,9] conclusion under the assumption of Kelvin model. Meanwhile, we point out the error of Chirita et al[16] have made in handling the result, negating their viewpoint that there is not only one Rayleigh wave in Kelvin viscoelastic half-space surface.

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