On polynomial solutions of the Lamé and Stokes systems

Hiroya ITO*

The University of Electro-Communications

1 Introduction

The Laplacian Δ is one of the most important differential operators in Mathematics. Solutions of the Laplace equation $\Delta u = 0$ are called harmonic functions, which play significant roles in many subjects of mathematical research fields. It is well known that harmonic polynomials in n variables are well classified. Moreover, the restriction of nonzero elements of $\mathcal{H}_m$ to the unit sphere $\mathbb{S}^{n-1}$, called spherical harmonics of degree m, become eigenfunctions of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1}$ with the common eigenvalue $m(m+n-2)$, and this restriction is a linear isomorphism between $\mathcal{H}_m$ and the space of spherical harmonics of degree m. Furthermore, we can think that all the harmonic polynomials in $\mathbb{R}^n$ (or all the spherical harmonics) generate most function spaces on $\mathbb{S}^{n-1}$ (see Theorem 1 below). We shall study that such beautiful theory can be partially generalized to vector-valued elliptic systems.

Fixing n variables $x_1, \ldots, x_n$ with $n \geq 2$, for each $m \in \mathbb{N}_0$, we denote by $\mathcal{P}_m$ the vector space of all the homogeneous polynomials of degree m in $x = (x_1, \ldots, x_n)$, and by $\mathcal{H}_m$ its subspace consisting of those in $\mathcal{P}_m$ which are harmonic. Moreover, $\mathring{\mathcal{H}}_m$ denotes the vector space of all the functions on $\mathbb{S}^{n-1}$ obtained by restricting each element of $\mathcal{H}_m$ to $\mathbb{S}^{n-1}$; each element of $\mathring{\mathcal{H}}_m$ is called a spherical harmonic of degree m:

$$\mathcal{H}_m = \{u \in \mathcal{P}_m \mid \Delta u = 0\}, \quad \mathring{\mathcal{H}}_m = \{u|_{\mathbb{S}^{n-1}} \mid u \in \mathcal{H}_m\}.$$

Then, the dimension d_m of $\mathcal{P}_m$ is given by $d_m = \binom{m+n-1}{n-1}$ and the restriction map $\mathcal{H}_m \ni u \mapsto u|_{\mathbb{S}^{n-1}} \in \mathring{\mathcal{H}}_m$ is, due to the homogeneity of elements of $\mathcal{H}_m$, a linear isomorphism: $\mathcal{H}_m \cong \mathring{\mathcal{H}}_m$. Fundamental properties of spherical harmonics on $\mathbb{S}^{n-1}$ are described in the following theorem (see, e.g., Chapter 2 of Shimakura [4], Chapter 3 of Simon [5], Nomura [2]).

Theorem 1. *The space $\mathring{\mathcal{H}}_m$ ($m \in \mathbb{N}_0$) has the following properties.*

(i) *The dimension of $\mathring{\mathcal{H}}_m$ is given by $\dim \mathring{\mathcal{H}}_m = d_m - d_{m-2}$, where $d_{-1} = d_{-2} = 0$.*

(ii) *$L^2(\mathbb{S}^{n-1}) = \bigoplus_{m=0}^{\infty} \mathring{\mathcal{H}}_m$ in the sense that*

$$\mathring{\mathcal{H}}_\ell \perp \mathring{\mathcal{H}}_m \ (\ell \neq m) \ \text{ in } \ L^2(\mathbb{S}^{n-1}) \quad \text{and} \quad \overline{\operatorname{span}(\bigcup_{m=0}^{\infty} \mathring{\mathcal{H}}_m)}^{L^2} = L^2(\mathbb{S}^{n-1}).$$

In the present paper, we consider the homogeneous equation of the Lamé system

$$\mathcal{L}\boldsymbol{u} := \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla(\operatorname{div}\boldsymbol{u}) = \boldsymbol{0} \quad \text{in } \mathbb{R}^n \tag{1}$$

*e-mail: `ito-hiroya@uec.ac.jp`

for n-vector valued functions (vector fields) $\boldsymbol{u}$, where λ and μ are elasticity constants. We study the structure of the restriction of polynomial solutions of (1) to $\mathbb{S}^{n-1}$ (or analogues of spherical harmonics for (1)). Here, the operator $\mathcal{L}$ of (1) appears in linear theory of isotropic elasticity and the constants λ and μ are assumed to satisfy

$$\mu(\lambda+2\mu)>0, \qquad \gamma := \frac{\lambda+\mu}{\lambda+3\mu} \in (-1,1). \tag{2}$$

The symbol of $\mathcal{L}$ is

$$L(\xi) = \mu|\xi|^2 I + (\lambda+\mu)\boldsymbol{\xi}\otimes\boldsymbol{\xi}, \tag{3}$$

whose eigenvalues are given by $\mu|\xi|^2$ (multiplicity $n-1$) and $(\lambda+2\mu)|\xi|^2$ (simple), where we write ξ in boldface in order to clarify that $\boldsymbol{\xi}$ is a column vector. Assumption (2) implies that $\mathcal{L}=L(\partial)$ is a strongly elliptic system. In a similar way we also deal with polynomial solutions of the homogeneous equations of the Stokes system.

2 Orthogonally invariant partial differential operators for vector fields

Let $P(\partial)$ be a partial differential operator with constant coefficients for scalar fields u on $\mathbb{R}^n$. It is well-known that $P(\partial)$ is invariant under the special orthogonal group $\mathrm{SO}(n)$ if and only if $P(\partial)$ is in the form $P(\partial)u = f(\Delta)u$ for some polynomial $f(t)$. How about the case $P(\partial)$ is for vector fields $\boldsymbol{u}$ on $\mathbb{R}^n$? The following theorem In the case $P(\partial)$ for scalar functions u, it is well-known that $P(\partial)$ is invariant under $\mathrm{SO}(n)$ if and only if it is in the form $P(\partial)u = f(\Delta)u$. The following theorem shows, in a sense, the necessity of considering the Lame system.

Theorem 2. *A partial differential operator $P(\partial)$ with constant coefficients for vector fields $\boldsymbol{u}$ on $\mathbb{R}^n$ is invariant under the orthogonal group $\mathrm{O}(n)$ if and only if $P(\partial)$ is in the form*

$$P(\partial)\boldsymbol{u} = f(\Delta)\boldsymbol{u} + g(\Delta)\,\nabla(\operatorname{div}\boldsymbol{u}).$$

for some polynomials $f(t)$ and $g(t)$.

Even if we restrict $\mathrm{O}(n)$ to $\mathrm{SO}(n)$ in Theorem 2, then the conclusion is valid for $n \geq 4$, but not for $n = 2, 3$, in which $P(\partial)\boldsymbol{u}$ may contain additional terms, for example $h(\Delta)\operatorname{rot}\boldsymbol{u}$ if $n = 3$.

3 L-harmonic vector fields and L-harmonics

Denote by $\boldsymbol{\mathcal{P}}_m$ the vector space of all n-vector homogeneous polynomials in $x = (x_1, \ldots, x_n)$ of degree m. We define subspaces $\boldsymbol{\mathcal{H}}_m$ and $\boldsymbol{\mathcal{H}}_m^L$ of $\boldsymbol{\mathcal{P}}_m$ by

$$\boldsymbol{\mathcal{H}}_m = \{\boldsymbol{u}\in\boldsymbol{\mathcal{P}}_m \mid \Delta\boldsymbol{u} = \boldsymbol{0}\}, \quad \boldsymbol{\mathcal{H}}_m^L = \{\boldsymbol{u}\in\boldsymbol{\mathcal{P}}_m \mid \mathcal{L}\boldsymbol{u} = \boldsymbol{0}\}.$$

Elements of $\boldsymbol{\mathcal{H}}_m^L$ are called *L-harmonic polynomials* of degree m.

Vector functions on $\mathbb{S}^{n-1}$ obtained by restricting L-harmonic polynomials are called *spherical L-harmonics*. We represent the vector spaces of such vector functions (vector fields) as

$$\mathring{\boldsymbol{\mathcal{H}}}_m = \{\boldsymbol{u}|_{\mathbb{S}^{n-1}} \mid \boldsymbol{u}\in\boldsymbol{\mathcal{H}}_m\}, \quad \mathring{\boldsymbol{\mathcal{H}}}_m^L = \{\boldsymbol{u}|_{\mathbb{S}^{n-1}} \mid \boldsymbol{u}\in\boldsymbol{\mathcal{H}}_m^L\}.$$

Corresponding to Theorem 1 for spherical harmonics, the following theorem for spherical L-harmonics has been established through joint research with Prof. Honda and Prof. Jimbo.

Theorem 3 ([1]). *The space* $\mathring{\boldsymbol{\mathcal{H}}}_m^L$ $(m \in \mathbb{N}_0)$ *has the following properties.*

(i) *The dimension of* $\mathring{\boldsymbol{\mathcal{H}}}_m^L$ *is given by* $\dim \mathring{\boldsymbol{\mathcal{H}}}_m^L = \dim \boldsymbol{\mathcal{H}}_m^L = n(d_m - d_{m-2})$, *where* $d_{-1} = d_{-2} = 0$.

(ii) *For each* $m \in \mathbb{N}$, *the sum* $\mathring{\boldsymbol{\mathcal{H}}}_0^L + \mathring{\boldsymbol{\mathcal{H}}}_1^L + \cdots + \mathring{\boldsymbol{\mathcal{H}}}_m^L$ *is a direct sum.*

(iii) *The linear span of* $\bigcup_{m=0}^{\infty} \mathring{\boldsymbol{\mathcal{H}}}_m^L$ *is dense in* $\boldsymbol{L}^2(\mathbb{S}^{n-1})$ *with the* L^2*-norm.*

4 Case $n = 2$

In this section we consider the case $n = 2$. The spaces $\boldsymbol{\mathcal{H}}_m$, $\boldsymbol{\mathcal{H}}_m^L$, $\mathring{\boldsymbol{\mathcal{H}}}_m$, $\mathring{\boldsymbol{\mathcal{H}}}_m^L$ are defined not only for nonnegative integer m but also negative integer m. For example, $\boldsymbol{u} \in \mathring{\boldsymbol{\mathcal{H}}}_m^L$ for $m < 0$ implies that $\boldsymbol{u}$ is a vector field solution of (1) in $\mathbb{R}^2 \setminus \{0\}$ which is homogeneous in $x = (x_1, x_2)$ of degree m.

Let $\boldsymbol{u} = (u_1, u_2)$ be a real vector field solution of $\boldsymbol{\mathcal{L}}\boldsymbol{u} = \boldsymbol{0}$ in $\mathbb{R}^2 \setminus \{0\}$. Then the complex function $U(z) := u_1(x_1, x_2) + iu_2(x_1, x_2)$ $(z := x_1 + ix_2)$ satisfies

$$\frac{\partial}{\partial \overline{z}}\Big(\frac{\partial U}{\partial z} + \gamma \overline{\frac{\partial U}{\partial z}}\Big) = 0 \quad \text{in } \mathbb{C} \setminus \{0\},$$

which is solved as

$$U = \varphi(z) - \gamma z \overline{\varphi'(z)} + \overline{\psi(z)} + 2c \log|z| - \gamma \overline{c} \Big(\frac{z}{|z|}\Big)^2$$

where $\varphi(z), \psi(z)$ are holomorphic functions in $\mathbb{C} \setminus \{0\}$, and $c \in \mathbb{C}$ is a constant ([3]). Using this fact, we have the following assertions.

Theorem 4. *The spaces* $\boldsymbol{\mathcal{H}}_m^L$ *and* $\mathring{\boldsymbol{\mathcal{H}}}_m^L$ *have the following bases.*

(i) $\boldsymbol{\mathcal{H}}_0^L = \mathbb{R}^2$. *For* $m \neq 0$, *the space* $\boldsymbol{\mathcal{H}}_m^L$ *has a basis*

$$\left\{ \begin{bmatrix} \mathrm{Re}[z^m - \gamma m z \overline{z^{m-1}}] \\ \mathrm{Im}[z^m - \gamma m z \overline{z^{m-1}}] \end{bmatrix}, \begin{bmatrix} -\mathrm{Im}[z^m + \gamma m z \overline{z^{m-1}}] \\ \mathrm{Re}[z^m + \gamma m z \overline{z^{m-1}}] \end{bmatrix}, \begin{bmatrix} \mathrm{Re}[z^m] \\ -\mathrm{Im}[z^m] \end{bmatrix}, \begin{bmatrix} \mathrm{Im}[z^m] \\ \mathrm{Re}[z^m] \end{bmatrix} \right\}.$$

(ii) $\mathring{\boldsymbol{\mathcal{H}}}_0^L = \mathbb{R}^2$. *For* $m \neq 0$, *the space* $\mathring{\boldsymbol{\mathcal{H}}}_m^L$ *has a basis*

$$\left\{ \begin{bmatrix} \cos m\theta - \gamma m \cos{(m-2)\theta} \\ \sin m\theta + \gamma m \sin{(m-2)\theta} \end{bmatrix}, \begin{bmatrix} -\sin m\theta + \gamma m \sin{(m-2)\theta} \\ \cos m\theta + \gamma m \cos{(m-2)\theta} \end{bmatrix}, \begin{bmatrix} \cos m\theta \\ -\sin m\theta \end{bmatrix}, \begin{bmatrix} \sin m\theta \\ \cos m\theta \end{bmatrix} \right\}.$$

Corollary 5.

(i) $\mathring{\boldsymbol{\mathcal{H}}}_0^L + \mathring{\boldsymbol{\mathcal{H}}}_1^L + \cdots + \mathring{\boldsymbol{\mathcal{H}}}_m^L = \mathring{\boldsymbol{\mathcal{H}}}_0 + \mathring{\boldsymbol{\mathcal{H}}}_1 + \cdots + \mathring{\boldsymbol{\mathcal{H}}}_m$ *for* $m \geq 1$.

(ii) *The sum* $\mathring{\boldsymbol{\mathcal{H}}}_0^L + \mathring{\boldsymbol{\mathcal{H}}}_{-1}^L + \cdots + \mathring{\boldsymbol{\mathcal{H}}}_{-m}^L$ $(m \geq 1)$ *is a direct sum, and satisfies*

$$\mathring{\boldsymbol{\mathcal{H}}}_0 + \mathring{\boldsymbol{\mathcal{H}}}_1 + \cdots + \mathring{\boldsymbol{\mathcal{H}}}_{m-2} \subset \mathring{\boldsymbol{\mathcal{H}}}_0^L + \mathring{\boldsymbol{\mathcal{H}}}_{-1}^L + \cdots + \mathring{\boldsymbol{\mathcal{H}}}_{-m}^L \subset \mathring{\boldsymbol{\mathcal{H}}}_0 + \mathring{\boldsymbol{\mathcal{H}}}_1 + \cdots + \mathring{\boldsymbol{\mathcal{H}}}_{m+2} \quad \text{for } m \geq 2.$$

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