Spectral analysis on the elastic Neumann–Poincaré operator *

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Abstract

The elastic Neumann–Poincaré (eNP) operator is a boundary integral operator that appears naturally when we solve classical boundary value problems for the Lamé system using layer potentials, and there is rapidly growing interest in its spectral properties recently in relation to cloaking by anomalous localized resonance (CALR). In this workshop, the speaker reported two results on the spectrum of the eNP operator. The first one is the polynomial compactness of the three-dimensional eNP operator on a $C^{1,\alpha}$ surface for $\alpha > 0$, which describes a distribution of eigenvalues. The second one is on the essential spectrum of the two-dimensional eNP operator on a curve which is smooth except at a corner.

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1 Introduction

The elastic Neumann–Poincaré (eNP) operator is a boundary integral operator that appears naturally when we solve classical boundary value problems for the Lamé system using layer potentials. Recently, there is rapidly growing interest in the spectral properties of the eNP operator in relation to cloaking by anomalous localized resonance (CALR). Anomalous localized resonance occurs at the accumulation point of eigenvalues, which motivates us to investigate the spectral structure of the eNP operator.

The Lamé system, a system of equations of linear elasticity, is described by

$$\mathcal{L}_{\lambda,\mu} u := \mu \Delta u + (\lambda + \mu) \nabla \nabla \cdot u = f,$$

where $u = (u_1, \ldots, u_d)$ $(d = 2, 3)$ is the displacement, (λ, μ) are the Lamé constants, and f is the body force. In what follows, we assume that the pair of constants (λ, μ) satisfies the strong convexity condition:

$$\mu > 0, \quad d\lambda + 2\mu > 0.$$

The Lamé operator has the divergence form as follows. Let $\mathbb{C} = (C_{ijkl})_{i,j,k,l=1}^d$ be the isotropic elasticity tensor corresponding to (λ, μ), namely,

$$C_{ijkl} := \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$$

Here, δ_{ij} denotes the Kronecker delta. Also, let $\widehat{\nabla} u$ be the symmetric gradient of a vector-valued function u, namely,

$$\widehat{\nabla} u := \frac{1}{2} \left(\nabla u + (\nabla u)^T \right),$$

where $(\nabla u)^T$ is the transpose of the matrix ∇u. Then, the Lamé system is also described as

$$\mathcal{L}_{\lambda,\mu} u = \nabla \cdot (\mathbb{C} \widehat{\nabla} u) = f.$$

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Let us describe CALR. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ ($d = 2, 3$). Let $a(x)$ be a function in $\mathbb{R}^d \backslash \partial\Omega$ such that

$$a(x) = \begin{cases} k, & x \in \Omega, \\ 1, & x \in \mathbb{R}^d \backslash \overline{\Omega}, \end{cases}$$

where k is a complex number. We assume that $k = k_0 + i\delta$ in (1.1) for $\delta > 0$, where

$$k_0 := -\frac{\lambda + 3\mu}{\lambda + \mu}.$$

We consider the following transmission problem:

$$\begin{cases} \nabla \cdot (a(x)\mathbb{C}\widehat{\nabla} u) = f \text{ in } \mathbb{R}^d, \\ |u(x)| = O(|x|^{1-d}) \text{ as } |x| \to \infty, \end{cases} \tag{1.1}$$

where f is a function compactly supported in $\mathbb{R}^d \backslash \overline{\Omega}$ and satisfies

$$\int_{\mathbb{R}^d} f \, dx = 0.$$

For the transmission problem (1.1), let us define an energy E of the function u by

$$E(u) := \int_\Omega \widehat{\nabla} u : \overline{\mathbb{C}\widehat{\nabla} u} \, dx = \int_\Omega \lambda |\nabla \cdot u|^2 + (d-1)\mu |\widehat{\nabla} u|^2 \, dx.$$

Here, $A : B = \sum_{ij} a_{ij} b_{ij}$ for two matrices $A = (a_{ij})$ and $B = (b_{ij})$. Let u_δ be the solution to the problem (1.1). Then, CALR is characterized by two conditions:

1. $\limsup_{\delta \downarrow 0} \delta E(u_\delta) = \infty$,
2. There exist positive constants C and R such that $|u_\delta(x)| < C$ for all $|x| > R$.

Let $v_\delta := u_\delta / \sqrt{\delta E(u_\delta)}$. Then, we have $\delta E(v_\delta) = 1$ and $|v_\delta(x)| \to 0$ as $\delta \downarrow 0$ for $|x| > R$. In other words, we cannot observe the displacement outside a ball.

In the two dimension case, Ando et al. showed that CALR occurs when the domain Ω is a ball or an ellipse, and when the support of f locates in a suitable area [2]. Also, in the three dimension case, Deng et al. showed that CALR occurs when the domain Ω is a ball and when the support of f locates in a suitable area [9]. Moreover, they showed the same result for the elastic wave case [10]. All of the above results are shown by the spectral analysis of the eNP operator, or by the layer potential technique.

Motivated by these previous works, we are going to work on the spectral analysis of the eNP operator.

2 Potential Theory

Let us introduce two operators originating from the potential theory. Let $\mathbf{\Gamma}(x) = (\Gamma(x))_{i,j=1}^d$ be the fundamental solution to the Lamé system associated with the Lamé constants (λ, μ), namely,

$$\Gamma_{ij}(x) := \begin{cases} \dfrac{\alpha_1}{2\pi} \delta_{ij} \log |x| - \dfrac{\alpha_2}{2\pi} \dfrac{x_i x_j}{|x|^2}, & d = 2, \\ -\dfrac{\alpha_1}{4\pi} \dfrac{\delta_{ij}}{|x|} - \dfrac{\alpha_2}{4\pi} \dfrac{x_i x_j}{|x|^3}, & d = 3, \end{cases} \quad |x| \neq 0,$$

where

$$\alpha_1 := \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{\lambda + 2\mu}\right), \quad \alpha_2 := \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\lambda + 2\mu}\right).$$

Also, let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ $(d = 2, 3)$. Then, the single layer potential $\mathbf{S}$ is defined by

$$\mathbf{S}[\varphi](x) := \int_{\partial\Omega} \mathbf{\Gamma}(x-y)\varphi(y)\, d\sigma_y, \qquad \text{a.e. } x \in \mathbb{R}^d.$$

For a vector-valued function u, the conormal derivative $\partial_\nu u$ corresponding to the Lamé system is defined by

$$\partial_\nu u := (\mathbb{C}\widehat{\nabla} u)n = \lambda(\nabla \cdot u)n + 2\mu(\widehat{\nabla} u)n,$$

where n is the outward unit normal to $\partial\Omega$. Then, the eNP operator $\mathbf{K}^*$ is defined by

$$\mathbf{K}^*[\varphi](x) := \text{p.v.} \int_{\partial\Omega} \partial_{\nu_x} \mathbf{\Gamma}(x-y)\varphi(y)\, d\sigma_y, \qquad \text{a.e. } x \in \partial\Omega.$$

Here, we consider the conormal derivative $\partial_{\nu_x}\mathbf{\Gamma}(x-y)$ of the matrix columnwise and p.v. stands for the Cauchy principal value.

We will show some properties of these operators. For details, see [1].

Proposition 2.1 (Jump Formula)**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. For $\varphi \in L^2(\partial\Omega)^d$, we have*

$$\partial_\nu \mathbf{S}[\varphi]|_\pm = \left(\pm\frac{1}{2}I + \mathbf{K}^*\right)[\varphi].$$

where the subscript $+$ and $-$ in the left hand side implies the limit from outside and inside Ω, respectively.

Proposition 2.2. *For any $\varphi \in L^2(\partial\Omega)^d$, we have $(\mathbf{S}[\varphi], \varphi)_{L^2(\partial\Omega)^d} \le 0$. Moreover, if $\mathbf{S}[\varphi] = 0$, then $\varphi = 0$.*

Remark 2.1. *To be precise, we need a slight modification in Proposition 2.2 when d $= 2$. For the detail, see [2].*

From Proposition 2.2, we can see that the sesquilinear form

$$(\varphi, \psi)_* := (-\mathbf{S}[\varphi], \psi)_{L^2(\partial\Omega)^d}$$

is indeed an inner product on $L^2(\partial\Omega)^d$. Let $\|\cdot\|_* := (\cdot,\cdot)_*^{1/2}$. Since $\mathbf{S} : L^2(\partial\Omega)^d \to L^2(\partial\Omega)^d$ is bounded, we have

$$\|\varphi\|_* \le \|\mathbf{S}\| \|\varphi\|_{L^2(\partial\Omega)^d}$$

for all $\varphi \in L^2(\partial\Omega)^d$.

Let $\mathcal{H}$ be the completion of the space $L^2(\partial\Omega)^d$ with respect to the norm $\|\cdot\|_*$. Then, we can show that $\mathcal{H} = H^{-1/2}(\partial\Omega)^d$ equipped with the inner product

$$\langle \varphi, \psi \rangle_* := -\langle \varphi, \mathbf{S}[\psi] \rangle_{H^{-1/2}, H^{1/2}}$$

for all $\varphi, \psi \in H^{-1/2}(\partial\Omega)^d$ by regarding the single layer potential $\mathbf{S}$ as the operator $\mathbf{S} : H^{-1/2}(\partial\Omega)^d \to H^{1/2}(\partial\Omega)^d$.

By introducing the inner product $\langle \varphi, \psi \rangle_*$, the eNP operator $\mathbf{K}^*$ becomes a self-adjoint operator on $\mathcal{H}$, which follows from Plemelj's symmetrization principle

$$\mathbf{S}\mathbf{K}^* = \mathbf{K}\mathbf{S},$$

where $\mathbf{K}$ is the L^2-adjoint of the eNP operator $\mathbf{K}^*$. Thus, the spectrum $\sigma(\mathbf{K}^*)$ of the eNP operator on $\mathcal{H}$ should be real. By further discussion on solvability of a transmission problem, we can show that

$$\sigma(\mathbf{K}^*) \subset (-1/2, 1/2).$$

In what follows, we investigate the spectral structure of the eNP operator $\mathbf{K}^*$ on $\mathcal{H}$ more deeply.

3 Polynomial Compactness

So far, we obtain two results on the spectral structure of the eNP operator $\mathbf{K}^*$ on the space $\mathcal{H}$.

The first one is polynomial compactness of the three-dimensional eNP operator. This result was obtained by a joint work with Hyeonbae Kang (Inha University, Korea).

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ with the $C^{1,\alpha}$-smooth boundary for some $\alpha > 0$. Let $\mathbf{K}^*$ be the eNP operator on $\partial\Omega$ corresponding to the pair of Lamé constants (λ, μ). Let $p_3(t) := t(t + \kappa_0)(t - \kappa_0)$, where κ_0 is given by*

$$\kappa_0 := \frac{k_0 + 1}{2(k_0 - 1)} = \frac{\mu}{2(\lambda + 2\mu)}.$$

Then, $p_3(\mathbf{K}^*)$ is compact on $H^{-1/2}(\partial\Omega)^3$. Moreover, $\mathbf{K}^*(\mathbf{K}^* + \kappa_0 I)$, $\mathbf{K}^*(\mathbf{K}^* - \kappa_0 I)$ and $(\mathbf{K}^*)^2 - \kappa_0^2 I$ are not compact on $H^{-1/2}(\partial\Omega)^3$.

From Theorem 3.1 and the spectral mapping theorem, we obtain the following result on an asymptotic behavior of eigenvalues.

Corollary 3.1. *The spectrum of $\mathbf{K}^*$ on $H^{-1/2}(\partial\Omega)^3$ consists of three non-empty sequences of eigenvalues which converge to 0, κ_0 and $-\kappa_0$, respectively.*

In the two-dimensional case, the counterpart of Theorem 3.1 is proved by replacing the polynomial p_3 by $p_2(t) := (t + \kappa_0)(t - \kappa_0)$ [2]. Also, Theorem 3.1 was once proved by assuming C^∞-smoothness on the boundary $\partial\Omega$ [3]. Theorem 3.1 is an extension of these two results.

The key idea to a proof is to localize the eNP operator and to approximate it by surface Riesz transforms, which is also the main idea in [3].

Let $G(u) = (g_{ij})_{i,j=1,2}$ be a positive-definite symmetric matrix valued function on $\mathbb{R}^2$ such that $G(u) = I$ (the identity matrix) for u outside a compact set. We assume that G is C^α-smooth for some $\alpha > 0$. In fact, G is a metric tensor corresponding to a $C^{1,\alpha}$-smooth boundary $\partial\Omega$ of a certain bounded domain Ω in $\mathbb{R}^3$ (see (3.8) and (3.9)). Let

$$L(u, u - v) = \langle u - v, G(u)(u - v)\rangle^{-3/2}. \tag{3.1}$$

The surface Riesz transform is defined by

$$R_j^g[f](u) = \frac{1}{2\pi}\text{p.v.}\int_{\mathbb{R}^2} (u_j - v_j)L(u, u - v)f(v)dv, \quad j = 1, 2. \tag{3.2}$$

Here, p.v. stands for the Cauchy principal value and u_j is the j-th component of the point u. The operator R_j^g is a singular integral operator of non-convolution type and bounded on $L^2(\mathbb{R}^2)$.

The surface Riesz transforms R_j^g satisfies the following relations, where $A \equiv B$ for two operators A and B bounded on $L^2(\mathbb{R}^2)$ (or $L^2(\mathbb{R}^2)^3$) means that $A - B$ is compact on $L^2(U)$ (resp. $L^2(U)^3$) for any bounded open set $U \subset \mathbb{R}^2$.

Theorem 3.2. *Let R_j^g, $j = 1, 2$, be surface Riesz transforms defined by the metric tensor G. Suppose that G is C^α-smooth for some $\alpha > 0$. Then, following identities hold:*

$$R_1^g R_2^g - R_2^g R_1^g \equiv 0 \tag{3.3}$$

and

$$\det(G)(g_{11}(R_1^g)^2 + 2g_{12}R_1^g R_2^g + g_{22}(R_2^g)^2) \equiv -I. \tag{3.4}$$

Remark 3.1. *Theorem 3.2 could be shown by the pseudodifferential calculus with rough coefficients (for example, see [15]).*

In order to show Theorem 3.2, we introduce an auxiliary operator R_{ij}. In what follows, we use the notation:

$$r_j(u, v) := v_j L(u, v), \quad j = 1, 2, \tag{3.5}$$

where $L(u,v)$ is defined by (3.1). Observe that

$$R_i^g R_j^g[f](u) = \lim_{\delta_1,\delta_2\downarrow 0} \frac{1}{4\pi^2} \int_{|u-v|>\delta_1} r_i(u,u-v) \int_{|v-w|>\delta_2} r_j(v,v-w) f(w)\, dw\, dv$$

for a.e. u, where the limit exists either in the point-wise sense or L^2-sense. Define the operator R_{ij} by

$$R_{ij}[f](u) = \lim_{\delta_1,\delta_2\downarrow 0} \frac{1}{4\pi^2} \int_{|u-v|>\delta_1} r_i(u,u-v) \int_{|v-w|>\delta_2} r_j(u,v-w) f(w)\, dw\, dv$$

for a.e. u. We emphasize that the difference between $R_i^g R_j^g[f](u)$ and $R_{ij}[f](u)$ lies in the r_j appeared in the formulas: the first one is $r_j(v,v-w)$ while the second one is $r_j(u,v-w)$.

The following proposition is the key ingredient in proving Theorem 3.2.

Proposition 3.1. *If the metric tensor $G(u)$ is C^α for some $\alpha > 0$, then*

$$R_i^g R_j^g \equiv R_{ij}. \tag{3.6}$$

for $i, j = 1, 2$.

It is shown in [3] that the eNP operator can expressed in terms of surface Riesz transforms. We review it and prove Theorem 3.1 using Theorem 3.2.

Let

$$\mathbf{K}_1(x,y) = \frac{n_x(x-y)^T - (x-y)n_x^T}{2\pi|x-y|^3},$$

where n_x is the outward unit normal at x, and let

$$\mathbf{T}[f](x) := \text{p.v.} \int_{\partial\Omega} \mathbf{K}_1(x,y) f(y)\, d\sigma(y), \quad x \in \partial\Omega.$$

It is proved in [2, 3] that

$$\mathbf{K}^* \equiv k_0 \mathbf{T}. \tag{3.7}$$

Here (3.7) means that the difference $\mathbf{K}^* - k_0\mathbf{T}$ is compact on $L^2(\partial\Omega)^3$. We emphasize that $\mathbf{T}$ is a singular integral operator and bounded on $L^2(\partial\Omega)^3$ (see [6]).

Denoting $n_x = (n_1(x), n_2(x), n_3(x))^T$, we have

$$\mathbf{K}_1(x,y) = \frac{1}{2\pi|x-y|^3} \begin{bmatrix} 0 & K_{12}(x,y) & K_{13}(x,y) \\ -K_{12}(x,y) & 0 & K_{23}(x,y) \\ -K_{13}(x,y) & -K_{23}(x,y) & 0 \end{bmatrix},$$

where

$$\begin{aligned} K_{12}(x,y) &= n_1(x)(x_2-y_2) - n_2(x)(x_1-y_1), \\ K_{13}(x,y) &= n_1(x)(x_3-y_3) - n_3(x)(x_1-y_1), \\ K_{23}(x,y) &= n_2(x)(x_3-y_3) - n_3(x)(x_2-y_2). \end{aligned}$$

Let

$$T_{ij}[f](x) := \text{p.v.} \int_{\partial\Omega} \frac{K_{ij}(x,y)}{2\pi|x-y|^3} f(y) d\sigma(y),$$

so that

$$\mathbf{T} = \begin{bmatrix} 0 & T_{12} & T_{13} \\ -T_{12} & 0 & T_{23} \\ -T_{13} & -T_{23} & 0 \end{bmatrix}.$$

Let U be a coordinate chart in $\partial\Omega$ so that there is an open set D in $\mathbb{R}^2$ and a parametrization $\Phi : D \to U$, namely,

$$x = \Phi(u) = (\varphi_1(u), \varphi_2(u), \varphi_3(u)), \quad x \in U, \quad u \in D.$$

Then the metric tensor of the surface, denoted by $G(u) = (g_{ij}(u))_{i,j=1}^2$, is given by

$$dx_1^2 + dx_2^2 + dx_3^2 = g_{11}du_1^2 + 2g_{12}du_1du_2 + g_{22}du_2^2,$$

where

$$g_{11} = |\partial_1\Phi|^2, \quad g_{12} = g_{21} = \partial_1\Phi \cdot \partial_2\Phi, \quad g_{22} = |\partial_2\Phi|^2. \tag{3.8}$$

Here and afterwards, ∂_j denotes the j-th partial derivative. In short, we have

$$G(u) = D\Phi(u)^T D\Phi(u), \tag{3.9}$$

where $D\Phi$ is the 3×2 Jacobian matrix of Φ. We then extend $G(u)$ to $\mathbb{R}^2$ in such a way that $G(u) = I$ for u outside a compact set. With this metric tensor, the surface Riesz transform is defined by (3.2).

Choose open sets U_j ($j = 1, 2$) in $\partial\Omega$ so that $\overline{U_1} \subset U_2$ and $\overline{U_2} \subset U$. Let χ_j ($j = 1, 2$) be $C^{1,\alpha}$-smooth functions such that $\chi_1 = 1$ in U_1, $\mathrm{supp}(\chi_1) \subset U_2$, $\chi_2 = 1$ in U_2, and $\mathrm{supp}(\chi_2) \subset U$. We denote by M_j the multiplication operator by χ_j, i.e.,

$$M_j[f](x) = \chi_j(x)f(x),$$

and by $\widetilde{M_j}$ the multiplication operator by $\chi_j(\Phi(u))$ for $j = 1, 2$. Let Φ^* be a pull back operator, namely,

$$\Phi^*[f](u) := f(\Phi(u))|\partial_1\Phi \times \partial_2\Phi(u)|.$$

For ease of notation, we set

$$m_{11} := (g_{11}\partial_2\varphi_3 - g_{12}\partial_1\varphi_3), \tag{3.10}$$
$$m_{12} := (g_{21}\partial_2\varphi_3 - g_{22}\partial_1\varphi_3), \tag{3.11}$$
$$m_{21} := -(g_{11}\partial_2\varphi_2 - g_{12}\partial_1\varphi_2), \tag{3.12}$$
$$m_{22} := -(g_{21}\partial_2\varphi_2 - g_{22}\partial_1\varphi_2), \tag{3.13}$$
$$m_{31} := (g_{11}\partial_2\varphi_1 - g_{12}\partial_1\varphi_1), \tag{3.14}$$
$$m_{32} := (g_{21}\partial_2\varphi_1 - g_{22}\partial_1\varphi_1), \tag{3.15}$$

and denote by M_{ij} the multiplication operator by m_{ij}. We emphasize that m_{ij} are C^α.

Let

$$X_{12} := \widetilde{M_2}(M_{11}R_1^g + M_{12}R_2^g)\widetilde{M_1},$$
$$X_{13} := \widetilde{M_2}(M_{21}R_1^g + M_{22}R_2^g)\widetilde{M_1},$$
$$X_{23} := \widetilde{M_2}(M_{31}R_1^g + M_{32}R_2^g)\widetilde{M_1},$$

and let

$$\mathbf{R} := \begin{bmatrix} 0 & X_{12} & X_{13} \\ -X_{12} & 0 & X_{23} \\ -X_{13} & -X_{23} & 0 \end{bmatrix}.$$

Then it is proved in [3] that the following relation holds:

$$\Phi^* M_2 \mathbf{T} M_1 \equiv \mathbf{R}\Phi^*.$$

Note that the crux of the matter in Theorem 3.1 is that

$$p_3(\mathbf{K}^*) = \mathbf{K}^*((\mathbf{K}^*)^2 - k_0^2\mathbf{I}) \equiv 0. \tag{3.16}$$

In view of (3.7) this fact follows once we have

$$\mathbf{T}^3 - \mathbf{T} \equiv 0,$$

which in turn follows from the following proposition:

Proposition 3.2. *It holds that*

$$\mathbf{R}^3 - \widetilde{M_1}\mathbf{R} \equiv 0. \tag{3.17}$$

Since Theorem 3.2 proves Proposition 3.2, we see that the operator $p_3(\mathbf{K}^*)$ is compact on $L^2(\partial\Omega)^3$.

In order to push the polynomial compactness of $\mathbf{K}^*$ on $L^2(\partial\Omega)^3$ up to that on $\mathcal{H}$, we borrow a result on function spaces with two norms [14]. Let us review the idea briefly. Let L be a pre-Hilbert space with an inner product $(\cdot,\cdot)$. We introduce two norms on L; $\|\cdot\|_B$ and $\|\cdot\|_H := (\cdot,\cdot)^{1/2}$. Denote L with $\|\cdot\|_B$ and $\|\cdot\|_H$ by L_B and L_H, respectively. We assume that L_B is a Banach space and that there exists a constant C such that $\|f\|_H \le C\|f\|_B$ for all $f \in L$. Let A be a self-adjoint (or a Hermittian) operator on L, that is,

$$(Af, g) = (f, Ag)$$

for all f, $g \in L$. Furthermore, let $\mathcal{H}$ be the completion of L_H with respect to $\|\cdot\|_H$ and $\tilde{A}$ be the extension of A to $\mathcal{H}$. Under these settings, we have the following proposition.

Proposition 3.3. *Let A be a self-adjoint operator on L and suppose that A is compact on L_B. Then, $\tilde{A}$ is compact on $\mathcal{H}$.*

Theorem 3.1 follows from Proposition 3.3 by setting $L_B = L^2(\partial\Omega)^3$, $\mathcal{H} = H^{-1/2}(\partial\Omega)^3$ and $A = p_3(\mathbf{K}^*)$.

4 Essential Spectrum

The second one is a characterization of the essential spectrum of the eNP operator when Ω is a planar domain with a corner. This result was obtained by a joint work with Eric Bonnetier (Université Grenoble–Alpes), Charles Dapogny (Université Grenoble–Alpes) and Hyeonbae Kang.

We consider a bounded domain Ω in $\mathbb{R}^2$ such that $\partial\Omega$ is smooth except at a single corner point, of angle α, $0 < \alpha < 2\pi$, $\alpha \neq \pi$ (our results extend quite straightforwardly to the case of domains showing multiple corners). By translation and rotation, we may assume that the corner is located at the origin, and that for some $R_0 > 0$,

$$\Omega \cap B_{R_0} = \{x = (r\cos\theta, r\sin\theta) | 0 \le r < R_0,\ 0 < \theta < \alpha\}, \tag{4.1}$$

where B_ρ denotes the ball of radius $\rho > 0$ centered at the origin. Here and in the sequel, an arbitrary point $x \in \mathbb{R}^2$ is indifferently identified with its Cartesian coordinates (x_1, x_2) or its polar coordinates (r, θ) centered at the origin.

We define a set $\Sigma(k_0, \alpha)$ by

$$\Sigma(k_0, \alpha) := \overline{\{p \in (0, 1-k_0) | \tilde{d}(p,\xi) = 0 \text{ for some } \xi > 0\}}, \tag{4.2}$$

where

$$\begin{aligned} \tilde{d}(p,\xi) &= 16 d_+(p,\xi) d_-(p,\xi), \\ d_\pm(p,\xi) &= f_{1,\pm}(p,\xi) f_{2,\pm}(p,\xi) + g(p,\xi), \\ f_{1,\pm}(p,\xi) &= \sinh(\alpha\xi)(p-1) \pm \xi\sin\alpha, \\ f_{2,\pm}(p,\xi) &= \sinh((2\pi-\alpha)\xi)(p+k_0) \pm \xi\sin\alpha, \\ g(p,\xi) &= p(p-1+k_0)\sinh^2((\pi-\alpha)\xi). \end{aligned}$$

Then, we have the following relation.

Theorem 4.1. *Let $0 < \alpha < 2\pi$. Then, we have*

$$\sigma_{ess}(\mathbf{K}^*) \supset \frac{1}{2} - \frac{1}{1-k_0}\Sigma(k_0, \alpha).$$

Remark 4.1. *In the workshop, the speaker presented that the equality holds, namely*

$$\sigma_{ess}(\mathbf{K}^*) = \frac{1}{2} - \frac{1}{1-k_0}\Sigma(k_0, \alpha).$$

However, we found a mistake in our proof. So, we excluded the opposite inclusion in the report.

A proof of Theorem 4.1 is based on the idea of Bonnetier and Zhang [4], where they discussed the essential spectrum of the NP operator for the Laplace equation. It is worth mentioning that Perfekt and Putinar also discussed it with the complex analysis [17]. However, we do not think that their idea works in the case of Lamé system because it is not invariant under a conformal mapping.

Let us briefly review some relative works on the essential spectrum of $\mathbf{K}^*$ on different function spaces from the energy space $H^{-1/2}(\partial\Omega)^2$. Mitrea [16] investigated the essential spectrum of $\mathbf{K}^*$ on $L^p(\partial\Omega)^2$ for $1 < p < \infty$ by applying the Mellin transform to the eNP operator directly, which originates from Cotabel and Stephan [7]. Also, based on the idea of [7], Diomeda and Lisena [11] discussed the well-posedness of a transmission problem on the Sobolev space $H^{3/2}(\mathbb{R}^2)$. Compared with these works, we discuss the essential spectrum of $\mathbf{K}^*$ on $H^{-1/2}(\partial\Omega)^2$, which is related to the well-posedness of the transmission problem on $H^1(\mathbb{R}^2)$.

The definition of the essential spectrum is as the following.

Definition 4.1. *Let $T : H \to H$ be a bounded self-adjoint operator on a Hilbert space H. An element $\beta \in \mathbb{R}$ belongs to the essential spectrum $\sigma_{\text{ess}}(T)$ of T if $(\beta\mathrm{I} - T)$ fails to be Fredholm.*

From the physical point of view, the essential spectrum is of particular interest since it is a resonance effect characterized by values of the properties inside the inclusion associated to 'generalized eigenfunctions' which are highly concentrated; see for instance [12] in the context of electrostatics.

From Theorem 4.1 and by analyzing structures of the set $\Sigma(k_0, \alpha)$, we obtain the following three theorems on bounds of the essential spectrum of the eNP operator $\mathbf{K}^*$. Here, we use the notation κ_0 appearing in Theorem 3.1.

Theorem 4.2. *Let $0 < \alpha < 2\pi$. If*

$$-k_0 > 1 + \max\left\{\frac{|\sin\alpha|}{\alpha}, \frac{|\sin\alpha|}{2\pi-\alpha}\right\}, \tag{4.3}$$

then,

$$\left[-\kappa_0 - \frac{|\sin\alpha|}{(2\pi-\alpha)(1-k_0)}, -\kappa_0\right] \cup \left[\kappa_0, \kappa_0 + \frac{|\sin\alpha|}{\alpha(1-k_0)}\right] \subset \sigma_{\text{ess}}(\mathbf{K}^*).$$

Theorem 4.3. *For $0 < \alpha < 4\pi/3$ ($\alpha \neq \pi$), there exists a positive number δ_1 depending only on k_0 and α such that*

$$[\kappa_0 - \delta_1, \kappa_0] \subset \sigma_{\text{ess}}(\mathbf{K}^*).$$

Theorem 4.4. *For $2\pi/3 < \alpha < 2\pi$ ($\alpha \neq \pi$), there exists a positive number δ_2 depending only on k_0 and α such that*

$$[-\kappa_0, -\kappa_0 + \delta_2] \subset \sigma_{\text{ess}}(\mathbf{K}^*).$$

Roughly speaking, Theorem 4.2 states that, provided the relation (4.3) holds, there are two disjoint closed intervals containing $-\kappa_0$ and κ_0 as endpoints belonging to $\sigma_{\text{ess}}(\mathbf{K}^*)$. The next two theorems, Theorem 4.3 and Theorem 4.4, hold without the assumption (4.3); together with Theorem 4.2, they show that there are two closed intervals inside $\sigma_{\text{ess}}(\mathbf{K}^*)$ containing $-\kappa_0$ and κ_0 as interior points. Notice that, strictly speaking, the essential spectrum could very well lie only on one side of either κ_0 or $-\kappa_0$.

Our proof relies on construction of a singular Weyl sequence.

Proposition 4.1. *Let A be a self-adjoint operator on a separable Hilbert space H. One real value κ belongs to $\sigma_{\mathrm{ess}}(A)$ if and only if there exists a singular Weyl sequence, that is, a sequence of vectors $(\varphi_n) \subset H$ such that*

$$\begin{cases} \|\varphi_n\| = 1, \\ (\kappa I - A)[\varphi_n] \to 0 \text{ strongly in } H, \\ \varphi_n \rightharpoonup 0 \text{ weakly in } H. \end{cases}$$

Let O be a bounded smooth domain in $\mathbb{R}^2$ such that $\overline{\Omega} \subset O$, and let us consider the following transmission problem.

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u = f & \text{in } O \backslash \overline{\Omega}, \\ \mathcal{L}_{\lambda,\mu} u = f & \text{in } \Omega, \\ u|_+ = u|_- & \text{on } \partial\Omega, \\ \partial_\nu u|_+ = k \partial_\nu u|_- & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial O. \end{cases} \tag{4.4}$$

The space $H_0^1(O)^2$ of functions in $H^1(O)^2$ with null trace on ∂O is equipped with the inner product $(\cdot,\cdot)_e$ and the associated norm $||\cdot||_e$ defined by:

$$(u,v)_e := \int_O \mathbb{C}\widehat{\nabla} u : \overline{\widehat{\nabla} v}\, dx = \int_O \left(\lambda(\nabla\cdot u)(\overline{\nabla \cdot v}) + 2\mu \widehat{\nabla} u : \overline{\widehat{\nabla} v} \right) dx,$$

where $A : B := \sum_{i,j=1}^2 a_{ij} b_{ij}$ is the Frobenius inner product of two matrices $A = (a_{ij})_{i,j=1}^2$ and $B = (b_{ij})_{i,j=1}^2$, and where the function $\overline{u}$ represents the complex conjugate of the function u. Let us recall that $(\cdot,\cdot)_e$ is indeed an inner product on $H_0^1(O)^2$ and that the norm $\|\cdot\|_e$ is a norm equivalent to the classical norm

$$\|u\|^2_{H_0^1(\Omega)^2} := \int_\Omega \nabla u : \overline{\nabla u}\, dx,$$

as follows from the classical Korn inequality and the strong convexity assumption; see for instance [5] Chap. 6 and the references therein about this point.

We now come to the definition of the Poincaré variational operator $T_\Omega : H_0^1(O)^2 \to H_0^1(O)^2$ for the Lamé system: for $u \in H_0^1(\Omega)^2$, $T_\Omega u$ is the unique element in $H_0^1(O)^2$ such that

$$(T_\Omega u, v)_e = \int_\Omega \left(\lambda(\nabla\cdot u)(\overline{\nabla \cdot v}) + 2\mu \widehat{\nabla} u : \overline{\widehat{\nabla} v} \right) dx$$

for all $v \in H_0^1(O)^2$. The existence of $T_\Omega u$ is guaranteed by the Riesz representation theorem. It readily stems from its definition that T_Ω is self-adjoint on $H_0^1(O)^2$ (with respect to the inner product $(\cdot,\cdot)_e$). Also, T_Ω is a positive operator with operator norm $\|T_\Omega\| = 1$.

The Poincaré variational operator has the following spectral structure.

Lemma 4.1. *The spectrum $\sigma(T_\Omega)$ of T_Ω is contained in the interval $[0,1]$. Moreover, the space $H_0^1(\Omega)^2$ has the following orthogonal decomposition:*

$$H_0^1(\Omega)^2 = \mathrm{Ker}(T_\Omega) \oplus \mathrm{Ker}(I - T_\Omega) \oplus \mathcal{H},$$

where:

- $\mathrm{Ker}(T_\Omega) = \left\{ u \in H_0^1(\Omega)^2,\ u|_D \in \Psi \right\}$;
- $\mathrm{Ker}(I - T_\Omega) = \left\{ u \in H_0^1(\Omega)^2,\ u = 0 \text{ on } \Omega \backslash \overline{D} \right\} \approx H_0^1(D)^2$;
- *$\mathcal{H}$ is defined by:*

$$\mathcal{H} = \left\{ u \in H_0^1(\Omega)^2 \mid \mathcal{L}_{\lambda,\mu} u = 0 \text{ in } D \cup (\Omega \backslash \overline{D}),\ \int_{\partial D} \partial_\nu u^+ \cdot \psi_j \, ds = 0,\ j = 1,2,3 \right\}. \tag{4.5}$$

There is a relation between the essential spectrum of T_Ω and that of the eNP operator $\mathbf{K}^*$.

Proposition 4.2.

$$\sigma_{\text{ess}}(T_\Omega)\backslash\{0,1\} = \frac{1}{2} - \sigma_{\text{ess}}(\mathbf{K}^*).$$

In what follows, we discuss the essential spectrum of the Poincaré variational operator instead of that of the eNP operator.

We consider the following transmission problem

$$\nabla \cdot (a(x)\mathbb{C}\widehat{\nabla} u) = 0 \text{ in } \mathbb{R}^2, \tag{4.6}$$

where

$$a(x) = \begin{cases} k(<0), & 0<\theta<\alpha, \\ 1, & \alpha<\theta<2\pi \end{cases}$$

with the polar coordinates $x = (r\cos\theta, r\sin\theta)$. We also write the displacement vector using the polar coordinates as

$$u = u_r(r,\theta)e_r + u_\theta(r,\theta)e_\theta,$$

where $e_r = (\cos\theta, \sin\theta)$ and $e_\theta = (-\sin\theta, \cos\theta)$, and we seek a formal solution of the form

$$u_r(r,\theta) = r^\eta \varphi_r(\theta), \quad u_\theta(r,\theta) = r^\eta \varphi_\theta(\theta), \tag{4.7}$$

with a complex number η.

The pair $(\varphi_r, \varphi_\theta)$ in (4.7) should satisfy

$$\begin{cases} \mu\varphi_r'' + ((\lambda+\mu)\eta - (\lambda+3\mu))\varphi_\theta' + (\lambda+2\mu)(\eta^2-1)\varphi_r = 0, \\ (\lambda+2\mu)\varphi_\theta'' + ((\lambda+\mu)\eta + (\lambda+3\mu))\varphi_r' + \mu(\eta^2-1)\varphi_\theta = 0 \end{cases} \tag{4.8}$$

in $(0,\alpha)\cup(\alpha,2\pi)$ with the transmission conditions

$$\begin{cases} \varphi_r(0) = \varphi_r(2\pi), \\ \varphi_\theta(0) = \varphi_\theta(2\pi), \\ k\{\mu\varphi_r'(0) + \mu(\eta-1)\varphi_\theta(0)\} = \mu\varphi_r'(2\pi) + \mu(\eta-1)\varphi_\theta(2\pi), \\ k\{(\lambda+2\mu)\varphi_\theta'(0) + (\lambda\eta+\lambda+2\mu)\varphi_r(0)\} = (\lambda+2\mu)\varphi_\theta'(2\pi) + (\lambda\eta+\lambda+2\mu)\varphi_r(2\pi), \\ \varphi_r(\alpha_-) = \varphi_r(\alpha_+), \\ \varphi_\theta(\alpha_-) = \varphi_\theta(\alpha_+), \\ k\{\mu\varphi_r'(\alpha_-) + \mu(\eta-1)\varphi_\theta(\alpha_-)\} = \mu\varphi_r'(\alpha_+) + \mu(\eta-1)\varphi_\theta(\alpha_+), \\ k\{(\lambda+2\mu)\varphi_\theta'(\alpha_-) + (\lambda\eta+\lambda+2\mu)\varphi_r(\alpha_-)\} = (\lambda+2\mu)\varphi_\theta'(\alpha_+) + (\lambda\eta+\lambda+2\mu)\varphi_r(\alpha_+), \end{cases} \tag{4.9}$$

where $\alpha_\pm = \lim_{\epsilon\downarrow 0} \alpha \pm \epsilon$.

We rewrite the system (4.8) as a system of first order ordinary differential equations. To achieve this, let $\varphi_1 = \varphi_r$, $\varphi_2 = \varphi_\theta$, $\varphi_3 = \varphi_r'$, $\varphi_4 = \varphi_\theta'$; then, we have:

$$\frac{d}{d\theta}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{bmatrix} = A\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\end{bmatrix}, \tag{4.10}$$

where A is the matrix with constant entries:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{(\lambda+2\mu)(1-\eta^2)}{\mu} & 0 & 0 & \dfrac{(\lambda+3\mu)-(\lambda+\mu)\eta}{\mu} \\ 0 & \dfrac{\mu(1-\eta^2)}{\lambda+2\mu} & -\dfrac{(\lambda+\mu)\eta+(\lambda+3\mu)}{\lambda+2\mu} & 0 \end{bmatrix}$$

The characteristic polynomial of this matrix reads: for all $\gamma \in \mathbb{C}$,

$$\det(\gamma I - A) = \gamma^4 + 2(1+\eta^2)\gamma^2 + (\eta^2-1)^2.$$

Another calculation reveals that A has four distinct eigenvalues, which are given by;

$$\gamma_1 := (1+\eta)i, \quad \gamma_2 := -(1+\eta)i, \quad \gamma_3 := (1-\eta)i, \quad \gamma_4 := -(1-\eta)i,$$

and the corresponding eigenvectors are respectively given by

$$\begin{bmatrix} 1 \\ i \\ (1+\eta)i \\ -(1+\eta) \end{bmatrix}, \begin{bmatrix} 1 \\ -i \\ -(1+\eta)i \\ -(1+\eta) \end{bmatrix}, \begin{bmatrix} \eta+k_0 \\ -(\eta-k_0)i \\ (1-\eta)(\eta+k_0)i \\ (1-\eta)(\eta-k_0) \end{bmatrix}, \text{ and } \begin{bmatrix} \eta+k_0 \\ (\eta-k_0)i \\ -(1-\eta)(\eta+k_0)i \\ (1-\eta)(\eta-k_0) \end{bmatrix}.$$

It is worth noticing that all four eigenvalues γ_j, $j = 1, ..., 4$ are independent of the Lamé constants (λ, μ), while the associated eigenvectors actually depends on them through the parameter k_0. Now defining the change of basis matrix

$$P = \begin{bmatrix} 1 & 1 & \eta+k_0 & \eta+k_0 \\ i & -i & -(\eta-k_0)i & (\eta-k_0)i \\ (1+\eta)i & -(1+\eta)i & (1-\eta)(\eta+k_0)i & -(1-\eta)(\eta+k_0)i \\ -(1+\eta) & -(1+\eta) & (1-\eta)(\eta-k_0) & (1-\eta)(\eta-k_0) \end{bmatrix},$$

it follows that A rewrites

$$A = P \,\mathrm{diag}[\gamma_1, \gamma_2, \gamma_3, \gamma_4]\, P^{-1},$$

and so the general solution to (4.10) is given by

$$(\varphi_1, \varphi_2, \varphi_3, \varphi_4)^T = PD(\theta)a, \tag{4.11}$$

where a is an arbitrary constant vector in $\mathbb{C}^4$, and

$$D(\theta) := \begin{bmatrix} e^{\gamma_1\theta} & 0 & 0 & 0 \\ 0 & e^{\gamma_2\theta} & 0 & 0 \\ 0 & 0 & e^{\gamma_3\theta} & 0 \\ 0 & 0 & 0 & e^{\gamma_4\theta} \end{bmatrix}.$$

Let a_D and a_Ω be constant vectors in (4.11) in intervals $(0, \alpha)$ and $(\alpha, 2\pi)$, respectively. With the above notations, the transmission conditions (4.9) can be written as

$$\begin{bmatrix} I(k)MPD(0) & -MPD(2\pi) \\ I(k)MPD(\alpha) & -MPD(\alpha) \end{bmatrix} \begin{bmatrix} a_D \\ a_\Omega \end{bmatrix} = 0, \tag{4.12}$$

where

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \mu(\eta-1) & \mu & 0 \\ \lambda\eta+\lambda+2\mu & 0 & 0 & \lambda+2\mu \end{bmatrix}, \quad I(k) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{bmatrix}.$$

In order for (4.12) to have a non-trivial solution, the matrix in the above system has to be degenerate, that is:

$$\det \begin{bmatrix} I(k)MPD(0) & -MPD(2\pi) \\ I(k)MPD(\alpha) & -MPD(\alpha) \end{bmatrix} = 0. \tag{4.13}$$

Our task is now to find η and k satisfying this equation for given values of the Lamé constants λ, μ and the aperture angle α. To achieve this goal, assuming that η is neither 0 nor 1, P and M are invertible. Therefore, we may factorize (4.13) as:

$$\det \begin{bmatrix} I(k)MPD(0) & -MPD(2\pi) \\ I(k)MPD(\alpha) & -MPD(\alpha) \end{bmatrix}$$

$$
\begin{aligned}
&= \det \begin{bmatrix} MP & 0 \\ 0 & MP \end{bmatrix} \begin{bmatrix} P^{-1}M^{-1}I(k)MP & -D(2\pi) \\ P^{-1}M^{-1}I(k)MPD(\alpha) & -D(\alpha) \end{bmatrix} \\
&= p^{-4}(\det MP)^2 \det\left(D(\alpha - 2\pi)B(p,\eta) - B(p,\eta)D(\alpha)\right),
\end{aligned}
$$

where we have introduced the reduced variable:

$$
p = \frac{k_0 - 1}{k - 1} \tag{4.14}
$$

and the 4×4 complex-valued matrix:

$$
B(p,\eta) = pP^{-1}M^{-1}I(k)MP.
$$

Notice that since $k < 0$ and $k_0 < 0$, the reduced variable p satisfies:

$$
p \in (0, 1 - k_0). \tag{4.15}
$$

Note also that the entries of $B(p,\eta)$ depend on the Lamé coefficients (λ, μ), although the dependence is not made explicit in the notations. The condition that (4.13) is now reduced to the following equation:

$$
d(p,\eta) := \det(D(\alpha - 2\pi)B(p,\eta) - B(p,\eta)D(\alpha)) = 0. \tag{4.16}
$$

Through straightforward but tedious calculations, one can see that

$$
B(p,\eta) = \begin{bmatrix} k_0 + p & \eta & k_0 + \eta^2 & \eta(k_0 + 1) \\ \eta & k_0 + p & \eta(k_0 + 1) & k_0 + \eta^2 \\ -1 & 0 & p - 1 & -\eta \\ 0 & -1 & -\eta & p - 1 \end{bmatrix}.
$$

Expanding (4.16), we obtain:

$$
\begin{aligned}
d(p,\eta) = {} & 4(\cos(2\pi\eta) - 1)^2(p-1)^2(p+k_0)^2 \\
& + 4\eta^2(1 - \cos(2\alpha))(\cos((4\pi - 2\alpha)\eta) - 1)(p+k_0)^2 \\
& + 4\eta^2(1 - \cos(2\alpha))(\cos(2\alpha\eta) - 1)(p-1)^2 \\
& + 8k_0(\cos(2\pi\eta) - 1)(\cos((2\pi - 2\alpha)\eta) - 1)(p-1)(p+k_0) \\
& - 8\eta^2(1 - \cos(2\alpha))(\cos(2\alpha\eta) - 1)(p-1)(p+k_0) \\
& + 4\{k_0(\cos((2\pi - 2\alpha)\eta) - 1) - (1 - \cos(2\alpha))\eta^2\}^2.
\end{aligned} \tag{4.17}
$$

We should emphasize that the same formula as (4.17) was also derived by Diomeda and Lisena [11]. It is also worth mentioning that the function $d(p,\eta)$ is even with respect to η, namely,

$$
d(p,-\eta) = d(p,\eta).
$$

Let us presently seek the pure imaginary zeros $\eta = i\xi$ ($\xi \in \mathbb{R}$) of the determinant $d(p,\eta)$. For simplicity, we let $\widetilde{d}(p,\xi) := d(p,i\xi)$. It then follows from (4.17) that:

$$
\begin{aligned}
\widetilde{d}(p,\xi) = {} & 4(\cosh(2\pi\xi) - 1)^2(p-1)^2(p+k_0)^2 \\
& - 4\xi^2(1 - \cos(2\alpha))(\cosh((4\pi - 2\alpha)\xi) - 1)(p+k_0)^2 \\
& - 4\xi^2(1 - \cos(2\alpha))(\cosh(2\alpha\xi) - 1)(p-1)^2 \\
& + 8k_0(\cosh(2\pi\xi) - 1)(\cosh((2\pi - 2\alpha)\xi) - 1)(p-1)(p+k_0) \\
& + 8\xi^2(1 - \cos(2\alpha))(\cosh((2\pi - 2\alpha)\xi) - 1)(p-1)(p+k_0) \\
& + 4\{k_0(\cosh((2\pi - 2\alpha)\xi) - 1) + (1 - \cos(2\alpha))\xi^2\}^2.
\end{aligned}
$$

Elementary computations based on the elementary identity

$$\cosh s - \cosh t = 2 \sinh \frac{s+t}{2} \sinh \frac{s-t}{2}, \ \ s,t \in \mathbb{C},$$

show that $\widetilde{d}(p,\xi)$ admits the following factorization:

$$\widetilde{d}(p,\xi) = 16\, d_+(p,\xi)\, d_-(p,\xi), \ \text{ where } d_\pm(p,\xi) := f_{1,\pm}(p,\xi)\, f_{2,\pm}(p,\xi) + g(p,\xi), \tag{4.18}$$

and

$$f_{1,\pm}(p,\xi) := \sinh(\alpha\xi)(p-1) \pm \xi \sin\alpha, \tag{4.19}$$
$$f_{2,\pm}(p,\xi) := \sinh((2\pi-\alpha)\xi)(p+k_0) \pm \xi \sin\alpha, \tag{4.20}$$
$$g(p,\xi) := p(p-1+k_0)\sinh^2((\pi-\alpha)\xi). \tag{4.21}$$

Let us remark that, owing to the estimate (4.15) for p, one has $p(p-1+k_0) < 0$.

Summarizing the above argument, the condition $p \in \Sigma(k_0,\alpha)$ holds if and only if there is a formal solution to the transmission problem (4.6) of the form (4.7) with the pure imaginary exponent $\eta = i\xi$.

Now we are ready to prove the following inclusion.

Proposition 4.3. *For any $k_0 < -1$ and $0 < \alpha < 2\pi$, $\alpha \neq \pi$, one has:*

$$\frac{1}{1-k_0}\Sigma(k_0,\alpha) \subset \sigma_{\text{ess}}(T_\Omega).$$

Since $\sigma_{\text{ess}}(T_\Omega)$ is closed, it suffices to show that all of the interior points in $(1/(1-k_0))\Sigma(k_0,\alpha)$ belong to $\sigma_{\text{ess}}(T_\Omega)$. For such a number p, there exists a nontrivial solution to the transmission problem (4.6) of the form (4.7) with the pure imaginary exponent $\eta = i\xi$.

Straight computations show that if u and v are of the form $u = u_r(r,\theta)e_r + u_\theta(r,\theta)e_\theta$ and $v = v_r(r,\theta)e_r + v_\theta(r,\theta)e_\theta$, then

$$\nabla \cdot u = \partial_r u_r + r^{-1}u_r + r^{-1}\partial_\theta u_\theta, \tag{4.22}$$

and

$$\begin{aligned}(\widehat{\nabla} u) : (\widehat{\nabla} v) = {} & \partial_r u_r \partial_r v_r + r^{-2}\left(u_r + \partial_\theta u_\theta\right)\left(v_r + \partial_\theta v_\theta\right) \\ & + \frac{1}{2}\left(\partial_r u_\theta + r^{-1}\partial_\theta u_r - r^{-1}u_\theta\right)\left(\partial_r v_\theta + r^{-1}\partial_\theta v_r - r^{-1}v_\theta\right),\end{aligned} \tag{4.23}$$

where ∂_r and ∂_θ denotes the partial derivative with respect to r and θ, respectively. Thus, we see that the solution u given in (4.7) with $\eta = i\xi$ and $\xi > 0$ satisfies:

$$|\nabla \cdot u|^2 \leq Cr^{-2} \quad \text{and} \quad |\widehat{\nabla} u|^2 \leq Cr^{-2} \text{ in } B_{R_0}. \tag{4.24}$$

Let $\chi_1 : \mathbb{R}^+ \to [0,1]$ be smooth a cut-off function such that:

$$\chi_1(s) = 0, \ s \leq 1; \quad \chi_1(s) = 1, \ s \geq 2; \quad |\chi_1'(s)| \leq C;$$

choosing $r_0 < R_0/2$, let $\chi_2 : \mathbb{R}^+ \to [0,1]$ be another smooth cut-off function such that:

$$\chi_2(s) = 0, \ s \geq 2r_0; \quad \chi_2(s) = 1, \ s \leq r_0; \quad |\chi_2'(s)| \leq C.$$

In the above equation, and throughout the proof, $C > 0$ is a positive constant which may change from one line to the next, but is in any case independent of ϵ. We set $\chi_1^\epsilon(r) := \chi_1(r/\epsilon)$ for $\epsilon > 0$, and define

$$u_\epsilon(x) = s_\epsilon \chi_1^\epsilon(r)\chi_2(r)u(x), \quad x \in B_{R_0}, \tag{4.25}$$

where the constant s_ϵ is chosen so that $\|u_\epsilon\|_e = 1$. We can show that (u_ϵ) is the desired singular Weyl sequence, which proves Proposition 4.3.

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