INVERSE SOURCE PROBLEM FOR KLEIN–GORDON EQUATION IN DE SITTER SPACE-TIME

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Abstract. De Sitter space-time is a solution of the vacuum Einstein equation with a positive cosmological constant in Euclidean space. We prove a uniqueness theorem to determine a time-independent source term for the Klein–Gordon equation in de Sitter space-time up to a neighborhood of an open subset when the mass in the equation has a particular value. We establish a new method based on the Duhamel's principle and theory of distributions with compact supports to deal with inverse problems for such equations having time-dependent coefficients.

1. Introduction and Main Theorem

Let $n \in \mathbb{N}$ and $T, H, m > 0$ be constants. In this paper, we consider the initial value problem

(1.1)
$$\begin{cases} P_m u := \partial_t^2 u - e^{-2Ht}\Delta u + nH\partial_t u + m^2 u = S\left(\int_0^t e^{-H\tau} d\tau\right) f(x) & \text{in } (0,T)\times\mathbb{R}^n, \\ u(0,\cdot) = \partial_t u(0,\cdot) = 0 & \text{on } \mathbb{R}^n, \end{cases}$$

where f and S are smooth functions. Let $(t, x) = (t, x^1, \ldots, x^n) \in \mathbb{R}^{1+n}$ be the Cartesian coordinate. If we use de Sitter space-time in Euclidean space $(\mathbb{R}^{1+n}, g)$ defined by

$$g = -dt \otimes dt + e^{2Ht} \sum_{j=1}^{n} dx^j \otimes dx^j,$$

which satisfies the Einstein equation with a positive cosmological constant, then our differential operator P_m can be written as

$$P_m = \Box_g + m^2,$$

where $\Box_g$ denotes the d'Alembertian with respect to the metric g. Namely, (1.1) is the initial value problem of the Klein–Gordon equation in de Sitter space-time. This type of equations is dealt by Yagdjian and Galstian [9], [10]. The semi-linear case is also studied by Yagdjian [8] and Nakamura [5].

Let $\Omega \subset \mathbb{R}^n$ be an open subset. We consider the inverse source problem to determine the source term f up to a neighborhood of Ω from a data of a solution u to (1.1) in $\Omega \times (0, T)$. This inverse problem relates to the inverse source problem for the equations having time-dependent coefficients. This type of inverse problems are studied by Jiang, Liu and Yamamoto [3] by the Bukhgeim–Klibanov method [1]. They prove the local Hölder stability under the assumption that there exists

Date: April 30, 2020.

the Carleman estimate for a suitable weight function. However, the Bukhgeim–Klibanov method in no longer useful since our goal is to prove uniqueness up to an outer neighborhood of Ω and the choice of a weight function is unclear. We must generate a new method to solve this difficulty.

To the best knowledge of the author, there are few publications on inverse source problems in the case that the principal part depends on time. Given the situation, we consider a new method to deal with inverse source problems based on the Duhamel's principle and theory of distributions with compact supports.
We denote the class of tempered distributions by $\mathcal{S}'$ and distributions with compact supports by $\mathcal{E}'$.

Theorem 1.1. *Let $H, T > 0$ be constants and $m = \frac{\sqrt{n^2-1}}{2}H$. Let $\Omega \subset \mathbb{R}^n$ be an open subset, $f \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'$ and $S \in C^\infty([0, H^{-1}))$ with $S(0) \neq 0$. Assume $u \in C^\infty([0,T); \mathcal{S}')$ is a solution to the initial value problem* (1.1)*. If $u = 0$ in $[0,T) \times \Omega$, then $f = 0$ up to Ω_T, where Ω_T is defined by*

$$\Omega_T := \left\{ x \in \mathbb{R}^n \mid \operatorname{dist}(x, \Omega) < \frac{1 - e^{-HT}}{H} \right\}.$$

Remark 1. We should remark on the well-posedness of (1.1). Under the assumptions in Theorem 1.1, it is well known that there exits a unique solution $u \in C^\infty([0,T); \mathcal{S}')$. Furthermore, the solution u satisfies

$$\forall t \in [0,T),\ u(t, \cdot) \in C^\infty(\mathbb{R}^n),$$

which is followed by the argument in the proof of Theorem 1.1 by means of the Fourier transform in $\mathcal{S}'$ and the fact that a convolution $*$ is a map

$$* : \mathcal{E}' \times C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n),$$

(e.g., [7, Theorem 27.2]). Thus, the condition $u = 0$ in $[0,T) \times \Omega$ makes sense.

2. Proof of Theorem 1.1

Proof. Let $m = \frac{\sqrt{n^2-1}}{2}H$. We remark that by defining

$$R(\sigma) := \frac{S(\sigma)}{(1 - H\sigma)^{\frac{n+3}{2}}},\ \sigma \in [0, H^{-1})$$

as a smooth function with $R(0) \neq 0$, we can write

$$S\left(\int_0^t e^{-H\tau} d\tau\right) = e^{-\frac{n+3}{2}Ht} R\left(\int_0^t e^{-H\tau} d\tau\right).$$

For the solution $u \in C^\infty([0,T); \mathcal{S}')$, by Lemma 3.1, there exists $v \in C^\infty([0,T); \mathcal{S}')$ such that v is a unique solution to the initial value problem

$$\begin{cases} P_m v = 0 & \text{in } [0,T) \times \mathbb{R}^n, \\ v(0, \cdot) = 0,\ \partial_t v(0, \cdot) = f & \text{on } \mathbb{R}^n. \end{cases}$$

Furthermore, u and v satisfy

$$u(t,x) = e^{-\frac{n-1}{2}Ht} \int_0^t e^{\frac{n-3}{2}Hs} R\left(\int_s^t e^{-H\tau} d\tau\right) v(s,x) ds.$$

The Fourier transform with respect to the space variable x in $\mathcal{S}'$ yields

$$\begin{cases} \partial_t^2 \hat{v} + e^{-2Ht}|\xi|^2\hat{v} + nH\partial_t\hat{v} + m^2\hat{v} = 0 & \text{in } [0,T)\times\mathbb{R}^n, \\ \hat{v}(0,\cdot) = 0,\ \partial_t\hat{v}(0,\cdot) = \hat{f} & \text{on } \mathbb{R}^n, \end{cases}$$

where $\hat{v}(t,\xi)$ represents the Fourier transform of v in $\mathcal{S}'$. The solution is written by

$$\hat{v}(t,\xi) = e^{-\frac{n-1}{2}Ht}\frac{\sin\left(|\xi|\int_0^t e^{-Hs}ds\right)}{|\xi|}\hat{f}(\xi).$$

Define a linear operator $\mathcal{A}^j : C^\infty([0,T);\mathcal{S}') \to C^\infty([0,T);\mathcal{S}')$ for $j\in\mathbb{N}$ as

$$\mathcal{A}^j\hat{w} := i\xi^j e^{-\frac{n-1}{2}Ht}\int_0^t e^{\frac{n-3}{2}H\tau}\int_0^\tau e^{-Hs}ds\hat{w}(\tau,\xi)d\tau.$$

Applying $\mathcal{A}^j$ to the solution $\hat{v}$ yields

$$\begin{aligned} \mathcal{A}^j\hat{v} &= i\xi^j e^{-\frac{n-1}{2}Ht}\int_0^t e^{\frac{n-3}{2}H\tau}\int_0^\tau e^{-Hs}ds\hat{v}(\tau,\xi)d\tau \\ &= i\xi^j e^{-\frac{n-1}{2}Ht}\left(\left[-\int_0^\tau e^{-Hs}ds\frac{\cos\left(|\xi|\int_0^\tau e^{-Hs}ds\right)}{|\xi|^2}\right]_0^t + \int_0^t e^{-H\tau}\frac{\cos\left(|\xi|\int_0^\tau e^{-Hs}ds\right)}{|\xi|^2}d\tau\right)\hat{f}(\xi) \\ &= i\xi^j e^{-\frac{n-1}{2}Ht}\left(-\int_0^t e^{-Hs}ds\frac{\cos\left(|\xi|\int_0^t e^{-Hs}ds\right)}{|\xi|^2} + \frac{\sin\left(|\xi|\int_0^t e^{-Hs}ds\right)}{|\xi|^3}\right)\hat{f}(\xi). \end{aligned}$$

Moreover, define a linear operator $\mathcal{L}^j : C^\infty([0,T);\mathcal{S}') \to C^\infty([0,T);\mathcal{S}')$ with the help of $\mathcal{A}^j$ as

$$\mathcal{L}^j\hat{w} := \mathcal{A}^j\hat{w} + i\frac{\partial}{\partial\xi^j}\hat{w}.$$

Applying $\mathcal{L}^j$ to the solution $\hat{v}$ yields

(2.1)

$$\begin{aligned} \mathcal{L}^j\hat{v} &= \mathcal{A}^j\hat{v} + i\frac{\partial}{\partial\xi^j}\hat{v} \\ &= \mathcal{A}^j\hat{v} + ie^{-\frac{n-1}{2}Ht}\frac{\xi^j|\xi|\int_0^t e^{-Hs}ds\cos\left(|\xi|\int_0^t e^{-Hs}ds\right) - \xi^j\sin\left(|\xi|\int_0^t e^{-Hs}ds\right)}{|\xi|^3}\hat{f} \\ &\quad + e^{-\frac{n-1}{2}Ht}\frac{\sin\left(|\xi|\int_0^t e^{-Hs}ds\right)}{|\xi|}i\frac{\partial}{\partial\xi^j}\hat{f}(\xi) \\ &= e^{-\frac{n-1}{2}Ht}\frac{\sin\left(|\xi|\int_0^t e^{-Hs}ds\right)}{|\xi|}i\frac{\partial}{\partial\xi^j}\hat{f}(\xi) \\ &= e^{-\frac{n-1}{2}Ht}\frac{\sin\left(|\xi|\int_0^t e^{-Hs}ds\right)}{|\xi|}\widehat{x^j f}(\xi). \end{aligned}$$

On the other hand, we define a linear operator $A^j : C^\infty([0,T);\mathcal{S}') \to C^\infty([0,T);\mathcal{S}')$ and $L^j : C^\infty([0,T);\mathcal{S}') \to C^\infty([0,T);\mathcal{S}')$ for $j\in\mathbb{N}$ as

$$A^j w := \frac{\partial}{\partial x^j}\left(e^{-\frac{n-1}{2}Ht}\int_0^t e^{\frac{n-3}{2}H\tau}\int_0^\tau e^{-Hs}dsw(\tau,x)d\tau\right)$$

and

$$L^j w := A^j w + x^j w.$$

We then can easily see that $\widehat{L^j w} = \mathcal{L}^j \hat{w}$ for all $j \in \mathbb{N}$. Let $p \in \mathcal{P}$ be an arbitrary polynomial, where $\mathcal{P}$ denotes a set of all polynomials with variable x. We then have from (2.1)

$$\widehat{p(L)v}(t,\xi) = e^{-\frac{n-1}{2}Ht} \frac{\sin\left(|\xi| \int_0^t e^{-Hs} ds\right)}{|\xi|} \widehat{pf}(\xi),$$

where $L := (L^1, \cdots, L^n)$. The Paley–Wiener theorem shows that $e^{-\frac{n-1}{2}Ht} \frac{\sin\left(|\xi| \int_0^t e^{-Hs} ds\right)}{|\xi|}$ is the Fourier transform of some distribution $s(t) \in \mathcal{E}'$ contained in the ball of radius $\int_0^t e^{-Hs} ds$. More precisely, it is well known that when n is odd number

$$\operatorname{supp} s(t) = \left\{ x \in \mathbb{R}^n \mid |x| = \int_0^t e^{-Hs} ds \right\},$$

and when n is even number,

$$\operatorname{supp} s(t) = \left\{ x \in \mathbb{R}^n \mid |x| \le \int_0^t e^{-Hs} ds \right\}.$$

The inverse Fourier transform in $\mathcal{S}'$ yields

$$p(L)v(t,x) = (s(t) * pf)(x).$$

Our assumption requires

$$(s(t) * pf)(x) = 0 \text{ in } [0,T) \times \Omega \tag{2.2}$$

by the final claim of Lemma 3.1. Let $V(t) \subset \mathbb{R}^n$ be an arbitrary open neighborhood of $\operatorname{supp} s(t)$. Since the polynomials $\mathcal{P}$ form a dense linear subspace of $C^\infty(V(t))$ (e.g., [7, Chapter 15]), we have

$$\forall \varphi \in C_0^\infty(V(t)),\ \exists \{p_k\}_{k=1}^\infty \subset \mathcal{P},\ p_k \xrightarrow{k\to\infty} \varphi$$

in the sense of $C^\infty(\mathbb{R}^n)$. By continuity of multiplication of an element in $C^\infty(\mathbb{R}^n)$ and of convolution as a linear map of $C^\infty(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n)$ (e.g., [7, Theorem 27.3]), it follows from (2.2) that

$$(s(t) * \varphi f)(x) = 0 \text{ in } [0,T) \times \Omega$$

holds for all $\varphi \in C_0^\infty(V(t))$. Finally, it means from the definition of convolution with elements of $\mathcal{E}'$ and $C^\infty(\mathbb{R}^n)$ (e.g., [7, Definition 27.1]) that

$$\langle s(t), \varphi f(x - \cdot) \rangle = 0 \text{ in } [0,T) \times \Omega$$

holds for all $\varphi \in C_0^\infty(V(t))$. By Lemma 3.2, we conclude that for all $(t,x) \in [0,T) \times \Omega$ and all $y \in \operatorname{supp} s(t)$, $f(x-y) = 0$ holds, i.e.,

$$f = 0 \text{ in } \bigcup_{t \in [0,T)} \Omega - \operatorname{supp} s(t).$$

$\square$

3. Appendix

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open subset, $H > 0, m = \frac{\sqrt{n^2-1}}{2}H$ be constants, $f \in C^\infty(\mathbb{R}^n) \cap \mathcal{S}'$ and $R \in C^\infty([0, H^{-1}))$. Assume $u \in C^\infty([0,T);\mathcal{S}')$ is a solution to*

$$\begin{cases} P_m u = e^{-\frac{n+3}{2}Ht} R\left(\int_0^t e^{-H\tau} d\tau\right) f(x) & \text{in } [0,T) \times \mathbb{R}^n, \\ u(0,\cdot) = \partial_t u(0,\cdot) = 0 & \text{on } \mathbb{R}^n. \end{cases}$$

There then exists $v \in C^\infty([0,T);\mathcal{S}')$ such that v is a unique solution to

$$\begin{cases} P_m v = 0 & \text{in } [0,T) \times \mathbb{R}^n, \\ v(0,\cdot) = 0, \ \partial_t v(0,\cdot) = f & \text{on } \mathbb{R}^n, \end{cases}$$

and u and v satisfy the following relation

$$u(t,x) = e^{-\frac{n-1}{2}Ht} \int_0^t e^{\frac{n-3}{2}Hs} R\left(\int_s^t e^{-H\tau} d\tau\right) v(s,x) ds. \tag{3.1}$$

Moreover, if $R(0) \neq 0$ and $u = 0$ in $[0,T) \times \Omega$, then $v = 0$ in $[0,T) \times \Omega$.

Proof. From the argument by means of the Fourier transform in the proof of Theorem 1.1, the existence of such v is proved. It suffices to prove the relation (3.1). For the solution v, we set

$$w(t,x) := e^{-\frac{n-1}{2}Ht} \int_0^t e^{\frac{n-3}{2}Hs} R\left(\int_s^t e^{-H\tau} d\tau\right) v(s,x) ds.$$

It then is obvious that $w \in C^\infty([0,T);\mathcal{S}')$ by the regularity of v. What we have to prove is w satisfies the initial value problem. Some calculations yield

$$\begin{aligned} \partial_t^2 w = &\left(-\frac{n+1}{2} H e^{-Ht} R(0) + e^{-2Ht} R'(0)\right) v + e^{-Ht} R(0) \partial_t v \\ &+ \frac{(n-1)^2}{4} H^2 e^{-\frac{n-1}{2}Ht} \int_0^t e^{\frac{n-3}{2}Hs} R\left(\int_s^t e^{-H\tau} d\tau\right) v(s,\cdot) ds \\ &- nHe^{-\frac{n+1}{2}Ht} \int_0^t e^{\frac{n-3}{2}Hs} R'\left(\int_s^t e^{-H\tau} d\tau\right) v(s,\cdot) ds \\ &+ e^{-\frac{n+3}{2}Ht} \int_0^t e^{\frac{n-3}{2}Hs} R''\left(\int_s^t e^{-H\tau} d\tau\right) v(s,\cdot) ds, \end{aligned}$$

$$
\begin{aligned}
e^{-2Ht}\Delta w &= e^{-\frac{n+3}{2}Ht}\int_0^t e^{\frac{n-3}{2}Hs}R\left(\int_s^t e^{-H\tau}d\tau\right)\Delta v(s,\cdot)ds \\
&= e^{-\frac{n+3}{2}Ht}\int_0^t e^{\frac{n+1}{2}Hs}R\left(\int_s^t e^{-H\tau}d\tau\right)(\partial_s^2 v+3H\partial_s v+m^2 v)(s,\cdot)ds \\
&= e^{-Ht}R(0)\partial_t v - e^{-\frac{n+3}{2}Ht}R\left(\int_0^t e^{-H\tau}d\tau\right)f(x) \\
&\quad + \left(\frac{n-1}{2}He^{-Ht}R(0)+e^{-2Ht}R'(0)\right)v \\
&\quad + e^{-\frac{n+3}{2}Ht}\int_0^t \partial_s^2\left(e^{\frac{n+1}{2}Hs}R\left(\int_s^t e^{-H\tau}d\tau\right)\right)v(s,\cdot)ds \\
&\quad - nHe^{-\frac{n+3}{2}Ht}\int_0^t \partial_s\left(e^{2Hs}R\left(\int_s^t e^{-H\tau}d\tau\right)\right)v(s,\cdot)ds \\
&\quad + \frac{n^2-1}{4}H^2e^{-\frac{n+3}{2}Ht}\int_0^t e^{\frac{n+1}{2}Hs}R\left(\int_s^t e^{-H\tau}d\tau\right)v(s,\cdot)ds \\
&= e^{-Ht}R(0)\partial_t v - e^{-\frac{n+3}{2}Ht}R\left(\int_0^t e^{-H\tau}d\tau\right)f(x) \\
&\quad + \left(\frac{n-1}{2}He^{-Ht}R(0)+e^{-2Ht}R'(0)\right)v \\
&\quad + e^{-\frac{n+3}{2}Ht}\int_0^t e^{\frac{n-3}{2}Hs}R''\left(\int_s^t e^{-H\tau}d\tau\right)v(s,\cdot)ds,
\end{aligned}
$$

$$
\begin{aligned}
nH\partial_t w = &-\frac{n(n-1)}{2}H^2e^{-\frac{n-1}{2}Ht}\int_0^t e^{\frac{n-3}{2}Hs}R\left(\int_s^t e^{-H\tau}d\tau\right)v(s,\cdot)ds \\
&+ nHe^{-Ht}R(0)v + nHe^{-\frac{n+1}{2}Ht}\int_0^t e^{\frac{n-3}{2}Hs}R'\left(\int_s^t e^{-H\tau}d\tau\right)v(s,\cdot)ds
\end{aligned}
$$

and

$$
m^2 w = \frac{n^2-1}{4}H^2e^{-\frac{n-1}{2}Ht}\int_0^t e^{\frac{n-3}{2}Hs}R\left(\int_s^t e^{-H\tau}d\tau\right)v(s,\cdot)ds.
$$

We then have

$$
\begin{aligned}
P_m w &= \partial_t^2 w - e^{-2Ht}\Delta w + nH\partial_t w + m^2 w \\
&= e^{-\frac{n+3}{2}Ht}R\left(\int_0^t e^{-H\tau}d\tau\right)f(x)
\end{aligned}
$$

and

$$
w(0,\cdot)=\partial_t w(0,\cdot)=0.
$$

By the uniqueness of the initial value problem, we can conclude $u=w$. The final claim follows by differentiating (3.1) with respect to t and applying the Gronwall's inequality to the formula. □

Lemma 3.2. *Let $n\in\mathbb{N}$, $f\in C^\infty(\mathbb{R}^n)$ and $\Phi\in\mathcal{E}'$ with a compact support* $\operatorname{supp}\Phi$*. If for an arbitrary open neighborhood V of* $\operatorname{supp}\Phi$*,*

$$
\langle\Phi,\varphi f\rangle=0,\ \forall\varphi\in C_0^\infty(V)
$$

holds, then $f=0$ on $\operatorname{supp}\Phi$*.*

Proof. Suppose there exists $x_0 \in \operatorname{supp}\Phi$, $f(x_0) \neq 0$. Without loss of generality, we can assume $f(x_0) > 0$. Then there exists $r > 0$ such that $f > 0$ on $B_r(x_0)$, which is an open ball centered at x_0 of radius r. For a neighborhood V of $\operatorname{supp}\Phi$ containing $\overline{B_r(x_0)}$ and any $\chi \in C_0^\infty(B_r(x_0))$, define $\varphi := \frac{\chi}{f} \in C_0^\infty(B_r(x_0)) \subset C_0^\infty(V)$. By our assumption

$$\langle \Phi, \varphi f\rangle = \langle \Phi, \chi\rangle = 0$$

holds for all $\chi \in C_0^\infty(B_r(x_0))$, that is, $\Phi = 0$ on $B_r(x_0)$. However since $x_0 \in \operatorname{supp}\Phi$ it follows that $x_0 \notin U$ for all open subset $U \subset \mathbb{R}^n$ on which $\Phi = 0$. By the contradiction argument, we prove our lemma.

□

Acknowledgement

The author would like to thank Professor Masahiro Yamamoto (The University of Tokyo) for many valuable discussions and comments. This work was supported by JSPS and RFBR under the Japan-Russia Research Cooperative Program (project No. J19-721)

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