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Construction of Non-$\times\mu$-Indivisible TKND-AVKF-Fields

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Abstract

In an author's joint work with Hoshi and Mochizuki, we introduced the notion of TKND-AVKF-field [concerning the divisible subgroups of the groups of rational points of semi-abelian varieties] and obtained an anabelian Grothendieck Conjecture-type result for higher dimensional configuration spaces associated to hyperbolic curves over TKND-AVKF-fields. On the other hand, every concrete example of TKND-AVKF-field that appears in this joint work is a $\times\mu$-indivisible field [i.e., a field such that any divisible element of the multiplicative group of the field is a root of unity]. In the present paper, we construct new examples of TKND-AVKF-fields that are not $\times\mu$-indivisible.

Contents

Introduction

Throughout the present paper, we shall use the following notations and conventions: The notation $\mathbb{Z}$ will be used to denote the additive group of integers. The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. We shall refer to a finite extension field of $\mathbb{Q}$ as a *number field*. If p is a prime number, then the notation $\mathbb{Z}_p$ (respectively, $\mathbb{Q}_p$) will be used to

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denote the p-adic completion of $\mathbb{Z}$ (respectively, $\mathbb{Q}$). For any field F of characteristic 0, field extension $F \subseteq E$, abelian variety A over F, positive integer n, and prime number l, we shall write $\overline{F}$ for the algebraic closure [determined up to isomorphisms] of F; $G_F \stackrel{\text{def}}{=} \text{Gal}(\overline{F}/F)$; $F^\times \stackrel{\text{def}}{=} F \setminus \{0\}$; $\mu_n(F) \subseteq F^\times$ for the subgroup of n-th roots of unity $\in F$; $\zeta_n \in \overline{F}$ a primitive n-th root of unity;

$$\mu(F) \stackrel{\text{def}}{=} \bigcup_{m \geq 1} \mu_m(F), \quad F^{\times l^\infty} \stackrel{\text{def}}{=} \bigcap_{m \geq 1} (F^\times)^{l^m}, \quad F^{\times \infty} \stackrel{\text{def}}{=} \bigcap_{m \geq 1} (F^\times)^m,$$

where m ranges over the positive integers; F^{ab} ($\subseteq \overline{F}$) for the maximal abelian extension field of F; F_{div} ($\subseteq \overline{F}$) for the field obtained by adjoining the divisible elements of the multiplicative groups of finite extension fields of F to $\mathbb{Q}$; $A(E)$ for the group of E-valued points of A; $A(E)_{\text{tor}} \subseteq A(E)$ for the subgroup of torsion points; $A[l] \subseteq A(\overline{F})$ for the subgroup of l-torsion points; $T_l A$ for the l-adic Tate module associated to A.

Let us recall the notions of $\times\mu$-indivisible field and TKND-AVKF-field for our purpose [cf. Definition 1.1, (i), (ii), (iii), (iv), (v), below]. Let F be a field of characteristic 0. Then we shall say that F is

- $\times\mu$-*indivisible* if $F^{\times\infty} \subseteq \mu(F)$;
- *stably* $\times\mu$-*indivisible* if, for every finite extension field E of F, it holds that E is $\times\mu$-indivisible;
- *TKND* [i.e., "torally Kummer-nondegenerate"] if $F_{\text{div}} \subseteq \overline{F}$ is an infinite field extension;
- *AVKF* [i.e., "abelian variety Kummer-faithful"] if, for each abelian variety A over a finite extension field E of F, any divisible element $\in A(E)$ is trivial;
- *TKND-AVKF* if F is both TKND and AVKF.

For instance, every subfield of the maximal cyclotomic extension of a number field is TKND-AVKF [cf. [10], Theorem 3.1, and its proof; [10], Remark 3.4.1]. In [2], we proved a certain anabelian Grothendieck Conjecture-type result for higher dimensional configuration spaces associated to hyperbolic curves over TKND-AVKF-fields. Therefore, from the viewpoint of anabelian geometry, it would be important to investigate examples of TKND-AVKF-fields that have not appeared in the literatures yet. On the other hand, we note that every TKND-AVKF-field that appears in [2] is stably $\times\mu$-indivisible. In the present paper, we construct new examples of TKND-AVKF fields that are not $\times\mu$-indivisible [cf. Corollary 2.5]:

Theorem A. *Let p be a prime number; K a number field. Write L ($\subseteq \overline{\mathbb{Q}}$) for the field obtained by adjoining all roots of p to K [so L contains all roots of unity, and $K \subseteq L$ is a nonabelian metabelian Galois extension]. Then L is not $\times\mu$-indivisible, and every subfield of L is TKND-AVKF.*

The key ingredient of the proof of Theorem A is the finiteness theorem of torsion points of abelian varieties [cf. Theorem 2.1] as follows:

Theorem B. *We maintain the notation of Theorem A. Let A be an abelian variety over L. Then, for each finite field extension $L \subseteq M$ $(\subseteq \overline{\mathbb{Q}})$, it holds that $A(M)_{\mathrm{tor}}$ is finite.*

We apply Ribet's theorem concerning the finiteness of torsion points of abelian varieties valued in the maximal cyclotomic extension of a number field [cf. [3], Appendix, Theorem 1], together with Kubo-Taguchi's lemma [cf. [4], Lemma 2.2, (i)], to prove Theorem B.

Finally, we also give an example of a stably $\times\mu$-indivisible field that is not AVKF [cf. Proposition 2.6]:

Proposition C. $\mathbb{Q}(\zeta_4)^{\mathrm{ab}}$ $(\subseteq \overline{\mathbb{Q}})$ *is a stably $\times\mu$-indivisible field that is not AVKF.*

Thus, one may conclude from Theorem A and Proposition C that the notion of AVKF-field is neither stronger nor weaker than the notion of stably $\times\mu$-indivisible field [cf. Remark 2.6.1; [2], Introduction].

1 Basic definitions

In the present section, we recall the definitions of TKND-AVKF-fields and stably $\times\mu$-indivisible fields:

Definition 1.1 ([2], Definition 6.1, (iii); [2], Definition 6.6, (i), (ii), (iii); [10], Definition 3.3, (iv), (v))**.** Let F be a field of characteristic 0; p a prime number.

(i) We shall say that F is p-$\times\mu$ (respectively, $\times\mu$)-*indivisible* if

$$F^{\times p^{\infty}} \subseteq \mu(F) \quad \text{(respectively, } F^{\times\infty} \subseteq \mu(F)).$$

(ii) We shall say that F is *stably* p-$\times\mu$ (respectively, *stably* $\times\mu$)-*indivisible* if, for every finite extension field E of F, it holds that E is p-$\times\mu$ (respectively, $\times\mu$)-indivisible.

(iii) If F satisfies the following condition, then we shall say that F is an *AVKF-field* [i.e., "abelian variety Kummer-faithful field"]:

> Let A be an abelian variety over a finite extension field E of F. Then any divisible element $\in A(E)$ is trivial.

If F is an AVKF-field, then we shall say that F is *AVKF*.

(iv) If $F_{\mathrm{div}} \subseteq \overline{F}$ is an infinite field extension, then we shall say that F is a *TKND-field* [i.e., "torally Kummer-nondegenerate field"]. If F is a TKND-field, then we shall say that F is *TKND*.

(v) If F is both a TKND-field and an AVKF-field, then we shall say that F is a *TKND-AVKF-field*.

Remark 1.1.1. We maintain the notation of Definition 1.1. Then it follows immediately from the various definitions involved that, if F is p-$\times\mu$-indivisible (respectively, $\times\mu$-indivisible; stably p-$\times\mu$-indivisible; stably $\times\mu$-indivisible; AVKF), then every subfield of F is also p-$\times\mu$-indivisible (respectively, $\times\mu$-indivisible; stably p-$\times\mu$-indivisible; stably $\times\mu$-indivisible; AVKF). On the other hand, a similar assertion for TKND does not hold. Indeed, suppose that F is a finitely generated transcendental extension field of an algebraically closed field M [of characteristic 0]. Then it follows immediately from a similar argument to the argument applied in [6], Remark 1.5.4, (i), together with the various definitions involved, that

$$M_{\mathrm{div}} = M = F_{\mathrm{div}} \subsetneq F \subsetneq \overline{F}.$$

Thus, since $M \subseteq F$ is an infinite field extension, we conclude that F is TKND, and M is not TKND.

Remark 1.1.2. We maintain the notation of Definition 1.1. Then it follows immediately from the various definitions involved that $E^{\times\infty} \subseteq E^{\times p^{\infty}}$. Thus, if F is p-$\times\mu$-indivisible (respectively, stably p-$\times\mu$-indivisible), then F is $\times\mu$-indivisible (respectively, stably $\times\mu$-indivisible). On the other hand, if F is stably $\times\mu$-indivisible, then since $\mathbb{Q}^{\mathrm{ab}} \subseteq \overline{\mathbb{Q}}$ is an infinite field extension, it holds that F is TKND [cf. [2], Remark 6.6.2].

Remark 1.1.3. It follows immediately from the various definitions involved that the algebraically closed fields and real closed fields are trivial examples of non-TKND-fields. However, at the time of writing of the present paper, the author does not know to what extent non-TKND-fields exist.

Proposition 1.2. *Let F be an abelian extension field of a number field; p a prime number. Then F is stably p-$\times\mu$-indivisible. In particular, F is stably $\times\mu$-indivisible [cf. Remark 1.1.2].*

Proof. Proposition 1.2 follows immediately from [10], Lemma D, (iv). □

2 Non-$\times\mu$-indivisible TKND-AVKF-fields

In the present section, we construct new examples of TKND-AVKF-fields that are not $\times\mu$-indivisible. First, we begin by proving the finiteness theorem of torsion points of abelian varieties [cf. Theorem B], which is a key ingredient of our construction:

Theorem 2.1. *Let p be a prime number; K a number field. Write L ($\subseteq \overline{\mathbb{Q}}$) for the field obtained by adjoining all roots of p to K [so L contains all roots of unity, and $K \subseteq L$ is a nonabelian metabelian Galois extension]. Let A be an abelian variety over L. Then, for each finite field extension $L \subseteq M$ ($\subseteq \overline{\mathbb{Q}}$), it holds that $A(M)_{\mathrm{tor}}$ is finite.*

Proof. First, by replacing K by a finite extension field of K, we may assume without loss of generality that

$$\zeta_{2p} \in K, \quad L = M,$$

and A descends to a semistable abelian variety A_0 over K [cf. [1], Exposé IX, Théorème 3.6]. Write

$$K' \stackrel{\text{def}}{=} \bigcup_{(m,p)=1} K(\mu_m(\overline{\mathbb{Q}})) \ (\subseteq L),$$

where m ranges over the positive integers coprime to p. Fix a prime of K that lies over p, and write

$$I_p \subseteq G_{K'} \subseteq G_K$$

for the inertia subgroup [determined up to conjugacy] associated to the prime. In light of the definition of L, by replacing K by a finite extension field of K again, we may assume without loss of generality that the natural composite

$$I_p \subseteq G_{K'} \twoheadrightarrow \mathrm{Gal}(L/K')$$

is *surjective.*

Next, we consider the mod l (respectively, l-adic) Galois representation associated to A. For each prime number $l \neq p$, let

$$W_l \subseteq A[l]^{G_L} \quad \text{(respectively, } W_l \subseteq (T_l A \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)^{G_L} \text{)}$$

be an irreducible $G_{K'}$-submodule, and write

$$\rho_l : G_{K'} \to GL(W_l)$$

for the mod l (respectively, l-adic) Galois representation that arises from the semistable abelian variety A_0 over K.

Next, we verify the following assertion:

> Claim 2.1.A: Let l be a prime number such that $l \neq p$. Then it holds that $W_l = W_l^{G_{K'}}$.

Indeed, since $W_l \subseteq A[l]^{G_L}$ (respectively, $W_l \subseteq (T_l A \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)^{G_L}$), it holds that ρ_l factors through the natural surjection $G_{K'} \twoheadrightarrow \mathrm{Gal}(L/K')$. Note that

- $\mathrm{Gal}(L/K')$ is an extension of pro-cyclic groups,
- A_0 is a semistable abelian variety over K,
- W_l is a finite dimensional Hausdorff topological vector space, and
- ρ_l is an irreducible $G_{K'}$-representation.

Then since the composite $I_p \subseteq G_{K'} \twoheadrightarrow \mathrm{Gal}(L/K')$ is surjective, it follows immediately from [1], Exposé IX, Proposition 3.5, together with Lemma 2.2 below, that $\rho_l(I_p) = \{1\}$. Thus, we conclude that $\rho_l(G_{K'}) = \{1\}$, hence that $W_l = W_l^{G_{K'}}$. This completes the proof of Claim 2.1.A.

Next, we consider the p-adic representation associated to A. Let

$$V_p \subseteq (T_p A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{G_L}$$

be a nonzero irreducible $G_{K'}$-submodule. Write

$$\rho_p : G_{K'} \to GL(V_p)$$

for the p-adic Galois representation that arises from the semistable abelian variety A_0 over K;

$$L_p \ (\subseteq \overline{\mathbb{Q}})$$

for the field obtained by adjoining all p-power roots of p to K'. On the other hand, since $\zeta_{2p} \in K$, it holds that ρ_p factors as the composite of the natural surjection

$$G_{K'} \twoheadrightarrow \mathrm{Gal}(L_p/K') \xrightarrow{\sim} \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p$$

— where "(1)" denotes the Tate twist — with a p-adic representation

$$\rho'_p : \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p \to GL(V_p).$$

Next, we verify the following assertion, which is a special case of Kubo-Taguchi's lemma [cf. [4], Lemma 2.2, (i)]:

> Claim 2.1.B: There exists an open subgroup $H \subseteq \mathbb{Z}_p(1)$ ($\subseteq \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p$) such that $V_p = V_p^H$.

Indeed, let $\sigma \in \mathbb{Z}_p(1)$ ($\subseteq \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p$) be an element. Then, for each $\tau \in \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p$, it holds that

$$\tau\sigma\tau^{-1} = \sigma^{\chi_p(\tau)},$$

where $\chi_p : (G_{K'} \twoheadrightarrow)\ \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p \twoheadrightarrow \mathbb{Z}_p \hookrightarrow \mathbb{Z}_p^\times$ denotes the p-adic cyclotomic character. Write $d \stackrel{\mathrm{def}}{=} \dim_{\mathbb{Q}_p} V_p$; $\{\lambda_1, \ldots, \lambda_d\}$ for the set of eigenvalues of $\rho'_p(\sigma)$. Let n be a positive integer such that $1 + p^n \in \mathrm{Im}(\chi_p)$. Then it follows immediately from the equality in the above display that

$$\{\lambda_1, \ldots, \lambda_d\} = \{\lambda_1^{1+p^n}, \ldots, \lambda_d^{1+p^n}\}.$$

Write

$$t \stackrel{\mathrm{def}}{=} \prod_{1 \le i \le d} (1 + p^n)^i - 1; \quad H \stackrel{\mathrm{def}}{=} t\mathbb{Z}_p(1).$$

Then it holds that $\lambda_i^t = 1$ for each positive integer i such that $1 \le i \le d$. In particular, since t is independent of the choice of σ, every element $\in \rho'_p(H)$ is *unipotent*. Note that

- $H \subseteq \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p$ is a pro-cyclic normal closed subgroup,
- V_p is a finite dimensional Hausdorff topological vector space, and
- ρ'_p is an irreducible $\mathbb{Z}_p(1) \rtimes \mathbb{Z}_p$-representation.

Thus, we conclude from Lemma 2.2 below that $V_p^H = V_p$. This completes the proof of Claim 2.1.B.

Finally, we verify that $A(L)_{\mathrm{tor}}$ is finite. It follows immediately from [the resp'd portion of] Claim 2.1.A and Claim 2.1.B, together with [3], Appendix, Theorem 3, that, for each prime number l, the subgroup of l-power torsion points of $A(L)$ is finite. On the other hand, it follows immediately from [the non-resp'd portion of] Claim 2.1.A, together with [3], Appendix, Theorem 2, that, for all but finitely many prime numbers l, the subgroup of l-torsion points of $A(L)$ is trivial. Thus, we conclude that $A(L)_{\mathrm{tor}}$ is finite. This completes the proof of Theorem 2.1. □

Lemma 2.2. *Let G be a profinite group; $H \subseteq G$ a pro-cyclic normal closed subgroup; V a finite dimensional irreducible Hausdorff topological G-vector space. Then, if the action of a topological generator of H on V is unipotent, then the action of H on V [obtained by restricting the action of G on V] is trivial.*

Proof. Let $\sigma \in H$ be a topological generator whose action on V is unipotent. Write

$$V^H \subseteq V, \quad V^\sigma \subseteq V$$

for the invariant subspaces associated to H, σ, respectively. Note that our assumptions that

- V is a Hausdorff topological vector space, and
- $\sigma \in H$ is a topological generator

imply that $V^H = V^\sigma$. Moreover, since the action of σ on the finite dimensional vector space V is unipotent, if $V \neq \{0\}$, then

$$V^H = V^\sigma \neq \{0\}.$$

On the other hand, observe that since $H \subseteq G$ is a normal closed subgroup, the action of G on V induces a natural action of G on the invariant subspace $V^H \subseteq V$. Thus, we conclude from our assumption that V is an irreducible topological G-vector space that $V^H = V$. This completes the proof of Lemma 2.2. □

Proposition 2.3. *Let p be a prime number; A a mixed characteristic Noetherian local domain of residue characteristic p; F an abelian extension field of the field of fractions K of A; $f \in F$ an element. Write $f^{\frac{1}{\infty}} \subseteq \overline{F}$ for the subset of all roots of f; E ($\subseteq \overline{F}$) for the field obtained by adjoining $f^{\frac{1}{\infty}}$ to F. Then every subfield of E is TKND.*

Proof. First, by replacing K, F, by extension fields of K, F, respectively, we may assume without loss of generality that

- $f \in K$,
- K is a mixed characteristic complete discrete valuation field whose residue field is an algebraically closed field of characteristic p, and
- $K^{\mathrm{tm}}(\mu(\overline{F})) \subseteq F$, where $K \subseteq K^{\mathrm{tm}}(\subseteq \overline{F})$ denotes the maximal tame extension [so, if $F \subsetneq E$, then the field extension $F \subseteq E$ is a $\mathbb{Z}_p$-extension].

Moreover, by replacing f by the multiple of the reciprocal of f with a suitable Teichmüller representative $\in K$, if necessary, we may assume without loss of generality that

$$f \in K \bigcap (\mathfrak{m}_F \bigcup 1 + \mathfrak{m}_F),$$

where $\mathfrak{m}_F$ denotes the maximal ideal of the ring of integers of the Henselian valuation field F.

Next, we verify the following assertion:

Claim 2.3.A: For each finite field extension $F \subseteq F^\dagger$, it holds that $F^{\times p^\infty} = (F^\dagger)^{\times p^\infty}$.

Indeed, Claim 2.3.A follows immediately from [5], Lemmas 2.5, 2.6, together with our assumptions on K.

Here, we consider the following commutative diagram

$$\begin{array}{ccccccc} & & & & F^\times & \longrightarrow & E^\times \\ & & & & \downarrow{\scriptstyle \kappa_F} & & \downarrow{\scriptstyle \kappa_E} \\ 0 & \longrightarrow & \mathrm{Hom}(\mathrm{Gal}(E/F), \mathbb{Z}_p) & \longrightarrow & \mathrm{Hom}(G_F, \mathbb{Z}_p) & \longrightarrow & \mathrm{Hom}(G_E, \mathbb{Z}_p), \end{array}$$

where the upper horizontal arrow denotes the natural injection; the vertical arrows κ_F and κ_E denote the Kummer maps; the lower horizontal sequence denotes the natural exact sequence. Note that $\mathrm{Ker}(\kappa_F) = F^{\times p^\infty}$, and $\mathrm{Ker}(\kappa_E) = E^{\times p^\infty}$. Write

$$P_f \subseteq E$$

for the subset consisting of the powers of elements $\in f^{\frac{1}{\infty}}$ $(\subseteq E)$.

Next, we verify the following assertion:

Claim 2.3.B: Suppose that $f \in 1 + \mathfrak{m}_F$ (respectively, $f \in \mathfrak{m}_F$). Then it holds that

$$E^{\times p^\infty} \subseteq F^{\times p^\infty} \cdot f^{\mathbb{Z}_p} \cdot P_f \quad \text{(respectively, } E^{\times p^\infty} \subseteq F^{\times p^\infty} \cdot P_f \text{).}$$

Indeed, if $F = E$, then we have nothing to prove. Thus, it suffices to consider the case where $F \subsetneq E$ [so $f \notin F^{\times p^\infty}$]. Let $g \in E^{\times p^\infty}$ be an element. In light of Claim 2.3.A, by replacing

K by a finite extension field of K, we may assume without loss of generality that $g \in F^{\times}$. Then since $f, g \in F \cap E^{\times p^{\infty}}$, it follows from the above commutative diagram that

$$\kappa_F(f),\ \kappa_F(g) \in \mathrm{Hom}(\mathrm{Gal}(E/F), \mathbb{Z}_p) \xrightarrow{\sim} \mathbb{Z}_p.$$

Note that since $f \notin F^{\times p^{\infty}}$, it holds that $\kappa_F(f) \neq 0$. Thus, we conclude that there exist $a \in \mathbb{Z}_p$ (respectively, $a \in \mathbb{Z}$) and $b \in p^{\mathbb{Z}_{\geq 0}}$ [where $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers] such that

$$\kappa_F(f^a) = \kappa_F(g^b).$$

This equality, together with our assumption that $\mu(\overline{F}) \subseteq F$, immediately implies that

$$g \in F^{\times p^{\infty}} \cdot f^{\mathbb{Z}_p} \cdot P_f \quad \text{(respectively, } g \in F^{\times p^{\infty}} \cdot P_f\ \text{)}.$$

This completes the proof of Claim 2.3.B.

Next, we verify the following assertion:

Claim 2.3.C: It holds that

$$\bigcup_{E \subseteq E^{\dagger}} (E^{\dagger})^{\times p^{\infty}} \subseteq E,$$

where $E \subseteq E^{\dagger}$ $(\subseteq \overline{F})$ ranges over the finite field extensions of E.

Indeed, Claim 2.3.C follows immediately from Claims 2.3.A, 2.3.B.

Next, we verify the following assertion:

Claim 2.3.D: $E \cap \overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}$ is an infinite field extension.

Indeed, observe that the image of the natural homomorphism $G_K \to G_{\mathbb{Q}}$ [determined up to composition with inner automorphisms] is isomorphic to the absolute Galois group G of a mixed characteristic Henselian discrete valuation field whose residue field is isomorphic to the algebraic closure of a finite field. Then since G is torsion-free [cf. Lemma 2.4, below], it holds that the image of the composite $G_E \subseteq G_K \to G_{\mathbb{Q}}$ is infinite. Thus, we conclude that $E \cap \overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}$ is an infinite field extension. This completes the proof of Claim 2.3.D.

Finally, let $M \subseteq E$ be a subfield. Then it follows immediately from Claim 2.3.C that

$$\bigcup_{M \subseteq M^{\dagger}} (M^{\dagger})^{\times p^{\infty}} \subseteq E,$$

where $M \subseteq M^{\dagger}$ $(\subseteq \overline{F})$ ranges over the finite field extensions of M. On the other hand, it follows immediately from Claim 2.3.D that $E \cap \overline{M} \subseteq \overline{M}$ is an infinite field extension. Thus, we conclude that M is TKND. This completes the proof of Proposition 2.3. □

Lemma 2.4. *Let K be a Henselian discrete valuation field with algebraically closed residues. Then it holds that G_K is torsion-free.*

Proof. Lemma 2.4 follows immediately from the fact that the cohomological dimension of G_K is equal to 1 [cf. [5], Lemma 3.1; [8], Chapter II, §3], hence, in particular, *finite.* $\square$

Next, we apply Theorem 2.1 and Proposition 2.3 to prove our main result:

Corollary 2.5. *In the notation of Theorem 2.1, it holds that L is not a $\times\mu$-indivisible field, and every subfield of L is a TKND-AVKF-field.*

Proof. First, observe that p is divisible in L. Then since $p \notin \mu(L)$, it holds that L is not $\times\mu$-indivisible. Next, observe that L coincides with the field obtained by adjoining all roots of p to the maximal cyclotomic extension of K. Then it follows immediately from Proposition 2.3 that every subfield of L is TKND. Finally, we conclude from Theorem 2.1, together with [7], Proposition 7, that L is AVKF, hence that every subfield of L is AVKF [cf. Remark 1.1.1]. This completes the proof of Corollary 2.5. $\square$

Remark 2.5.1. We maintain the notation of Corollary 2.5. Then it follows from Corollary 2.5 that every subfield of L satisfies the assumptions of various assertions in [2] [especially, [2], Theorems F, G].

Finally, we observe that there exists an example of a stably $\times\mu$-indivisible field that is not AVKF:

Proposition 2.6. *Write $K \stackrel{\text{def}}{=} \mathbb{Q}(\zeta_4)$. Then K^{ab} $(\subseteq \overline{\mathbb{Q}})$ is a stably $\times\mu$-indivisible field that is not AVKF.*

Proof. First, it follows immediately from Proposition 1.2 that K^{ab} is stably $\times\mu$-indivisible. Next, write E for the elliptic curve over K defined by the equation $y^2z = x^3 + xz^2$; $K \subseteq L$ $(\subseteq \overline{\mathbb{Q}})$ for the Galois extension obtained by adjoining the coordinates of all torsion points of E to K. Then it follows from the theory of complex multiplication that $L \subseteq K^{\text{ab}}$ [cf. [9], Theorem 2.3]. Thus, we conclude that all torsion points are divisible in $E(K^{\text{ab}})$, hence, in particular, that K^{ab} is not AVKF. This completes the proof of Proposition 2.6. $\square$

Remark 2.6.1. Thus, it follows from Corollary 2.5 and Proposition 2.6 that the notion of AVKF-field is neither stronger nor weaker than the notion of stably $\times\mu$-indivisible field. In particular, there exists no evident implication between [2], Corollary 6.5, (iii), and [10], Corollary E [cf. [2], Introduction].

Remark 2.6.2. On the other hand, observe that the field "L" that appears in Corollary 2.5 is a metabelian extension field of a number field, and the proof of Corollary 2.5 depends heavily on this property. Thus, one may pose the following question:

Question: Does there exist a subfield $L \subseteq \overline{\mathbb{Q}}$ such that

- L is a TKND-AVKF-field that is not $\times\mu$-indivisible;
- for any number field K, the field L may not be realized as a metabelian Galois extension field of K.

However, at the time of writing of the present paper, the author does not know whether the answer of this question is affirmative or not.

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