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Infinite ergodicity that preserves the Lebesgue measure

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ABSTRACT

In this study, we prove that a countably infinite number of one-parameterized one-dimensional dynamical systems preserve the Lebesgue measure and are ergodic for the measure. The systems we consider connect the parameter region in which dynamical systems are exact and the one in which almost all orbits diverge to infinity and correspond to the critical points of the parameter in which weak chaos tends to occur (the Lyapunov exponent converging to zero). These results are a generalization of the work by Adler and Weiss. Using numerical simulation, we show that the distributions of the normalized Lyapunov exponent for these systems obey the Mittag–Leffler distribution of order 1/2.

Time and space averages demonstrate an equality as a typical characteristic of ergodicity. However, the time average is not equivalent to the space average in infinite ergodic systems.[1] The Boole transformation is known as a one-dimensional map[2] that preserves the Lebesgue measure (infinite measure) and is ergodic. Here, the infinite measure means a measure that cannot be normalized as the standard probability measure. We call the invariant measure an *infinite ergodic measure* when the systems are ergodic with the infinite invariant measure. In this paper, we prove that a countably infinite number of one-parameterized one-dimensional maps that are generalized from the Boole transformation exactly preserve the Lebesgue measure (infinite measure) and are ergodic at certain parameters. Additionally, we show that in these maps, the normalized Lyapunov exponent obeys the Mittag–Leffler distribution of order 1/2 as well as the Boole transformation.

I. INTRODUCTION

Chaos theory has developed statistical physics through ergodic theory. In chaotic dynamics, future orbital states are difficult to predict from past information because the system is unstable or characterized by sensitivity to initial conditions. However, from its mixing property, a system can be characterized statistically using the invariant density function. Density function relates to microscopic dynamics, and their relation is important when macroscopic properties are derived from microscopic dynamics. Ergodicity plays a significant role in this derivation.

In the case of a dynamical system (X, T, μ) with a normalized ergodic invariant measure μ, which can be normalized to the unity, where X and T represent the phase space and a map, respectively, for an observable $f \in L^1(\mu)$, the time average $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i(x)$ converges with the phase average $\int_X f d\mu$ in almost all regions.[3] Here, $L^1(\mu)$ is a set of functions that are integrable in terms of the measure μ.

In systems with a normalized ergodic measure, their stability can be characterized using the Lyapunov exponent λ, which is defined as $\lambda \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log|T'(x_i)|$ when $\log|T'(x_i)| \in L^1(\mu)$ in a one-dimensional case. Normally, an orbit can be concluded as chaotic when the corresponding Lyapunov exponent is positive ($\lambda > 0$) and as stable when $\lambda < 0$.

The behavior of the Lyapunov exponent whose value is around zero characterizes the onset of chaos. In particular, for logistic map $x_{n+1} = a x_n(1 - x_n)$, at $a \simeq 3.57$, we can observe the universal

TABLE I. Statistical properties and invariant measures for α in the case of $K = 2N$ (or $2N + 1$).

α	$0(1/K^2) < \alpha < 1$	$\alpha = 1$	$1 < \alpha$
Statistical properties	Exact[7]	The present work	Almost all orbits diverge to infinity[7]
Invariant measures	Normalized ergodic measure[7]	The present work	
Lyapunov exponent	Positive[7]	Convergence to 0 as $\alpha \to 1$[7]	Positive[7]

critical phenomenon[4,5] at which the system becomes unstable from stable, called routes to chaos, and such kinds of critical phenomena have appeared in the fields of chaotic maps,[6,7] Hamiltonian dynamics,[8] intermittent systems,[9–12] differential equations,[13] coupled chaotic oscillators,[14] noise-induced systems,[15] certain experiments (Belousov–Zhabotinskii reaction, Rayleigh–Bénard convection, and Couette–Taylor flow),[16] and optomechanics.[17–19]

As maps that characterize the intermittent critical phenomenon, generalized Boole (GB) transformations were studied,[6] and we obtained the critical exponent of the Lyapunov exponent analytically. For GB transformations, at the onset of chaos, the Lyapunov exponent defined by the time average converges to zero as $\alpha \to 1$. The point $\alpha_c = 1$ is referred to as the critical point at which *Type* 1 intermittency (intermittency in which we have an eigenvalue of the Jacobian whose value is unity at the fixed point) occurs.[6]

The current authors proposed a countably infinite number of one-parameterized maps, which are called super-generalized Boole (SGB) transformations[7] $S_{K,\alpha}, K \in \mathbb{N}\backslash\{1\}, |\alpha| > 0$, and showed that the Lyapunov exponent converges to zero from a positive value as $\alpha \to 1$, and *Type* 1 intermittency occurs at $\alpha = 1$ for a countably infinite number of maps (SGB). That means at the critical point $\alpha = 1$, the onset of chaos appears. In addition, the statistical properties change drastically at $\alpha = 1$ as a boundary as shown in Table I. Thus, the property at $\alpha = 1$ is important from the viewpoints of the onset of chaos and ergodicity. However, the ergodic property at the critical point ($\alpha = 1$) is unsettled except for $K = 2$, which corresponds to the Boole transformation[2,20] $x_{n+1} = S_{2,1}(x_n) \stackrel{\text{def}}{=} x_n - 1/x_n$, where the dynamical system is proven to preserve the Lebesgue measure (infinite measure) and to be ergodic; the Boole transformation has the *infinite ergodic measure* (see Table II). Thus, it holds that $\int_{-\infty}^{\infty} f_1(x)\,dx = \int_{-\infty}^{\infty} f_1\left(x - \frac{1}{x}\right) dx$ for any L^1 function f_1 with respect to dx.

The Boole transformation $S_{2,1}$ is the critical map that connects the two different phases (the phase of $\alpha < 1$ and the one of $\alpha > 1$). With reference to the foundation of statistical mechanics, the Liouville measure on $\mathbb{R}^{2N}$ is vitally important and be regarded as the Lebesgue measure, which is invariant under the Hamiltonian dynamical system with N degrees of freedom.[21,22] Thus, it is of great interest to investigate the *ergodic* Lebesgue measure on $\mathbb{R}$, which is invariant under nonlinear transformations from a physical point of view.

For Boole transformation, the following interesting inconsistency can be observed because of the infinite ergodicity. For an observable $\log|S'_{2,1}|$, although the usual time average $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log|S'_{2,1}(x_i)|$ converges to zero,[1] the phase average is $\int_{-\infty}^{\infty} \log|S'_{2,1}(x)|\,dx = 2\pi$.[1] As such, although the system is considered ergodic, the time average does not coincide with the phase average.

In infinite ergodic systems, instead of the equality between the time average and the space average, distributional limit theorems[23–25] hold. For example, the Darling–Kac–Aaronson theorem states that if the observable f_2 is positive and $f_2 \in L^1(\nu)$, where ν is an infinite invariant measure, then the time average converges *in a distribution*.[23] These are examples of interesting phenomena that deviate from usual ergodic theory and standard statistical mechanics.

In an infinite ergodic measure system, following L^1 class observables converge to the Mittag–Leffler distribution, such as the Lempel–Ziv complexity,[26] the transformed observation function for the correlation function,[27] the normalized Lyapunov exponent,[1] and the normalized diffusion coefficient.[28] Moreover, non-L^1 class observables, such as the time average of position,[29] converge to a generalized arcsine distribution[24,25,30] or another distribution.[31]

Infinite densities that correspond to the infinite measure have been observed in physical systems in the context of the long time limit of the solution to the Fokker–Planck equation for Brownian motion,[32,33] semiclassical Monte Carlo simulations of cold atoms,[34] laser cooling,[35] and semi-Markov process.[36] Thus, infinite measure systems play an important role in not only mathematical but also physical systems.

To characterize the instability of systems with infinite measures, several quantities have been invented, such as Lyapunov pairs[1] and a generalized Lyapunov exponent.[37–39]

Regarding the above discussion, this study aims to clarify the ergodic properties of SGB transformations at the critical points for general K to connect continuously the parameter region in which the dynamical systems are mixing ($\alpha < 1$) and the one in which almost all orbits diverge to infinity ($\alpha > 1$). First, we extend the results

TABLE II. Statistical properties and invariant measures for α in the case of $K = 2$ in the previous studies.

α	$0 < \alpha < 1$	$\alpha = 1$	$1 < \alpha$
Statistical properties	Exact[7]	Ergodic[2]	Almost all orbits diverge to infinity[7]
Invariant measures	Normalized ergodic measure[7]	Infinite ergodic measure[2]	
Lyapunov exponent	Positive[7]	Zero[1]	Positive[7]

TABLE III. $S_{K,\pm 1}(x)$ for $K=2$, 3, 4, 5, and 6.

	$K=2$	3	4	5	6
$S_{K,\pm 1}(x)$	$\pm\left(x-\frac{1}{x}\right)$	$\pm 3\frac{x^3-3x}{3x^2-1}$	$\pm 4\frac{x^4-6x^2+1}{4x^3-4x}$	$\pm 5\frac{x^5-10x^3+5x}{5x^4-10x^2+1}$	$\pm 6\frac{x^6-15x^4+15x^2-1}{6x^5-20x^3+6x}$

of our previous study. Second, as the main result in the current study, we prove that the SGB transformations at $\alpha=\pm 1$ preserve the Lebesgue measure as an infinite measure and are ergodic. That is, we prove the infinite ergodicity for a countably infinite number of maps. Third, we clarify that a distributional limit theorem holds at these critical points by numerical experiments.

II. SUPERGENERALIZED BOOLE TRANSFORMATIONS

Before we present the main proof, we define the concept of SGB transformations,[7] introduce some definitions of ergodic properties, and explain the extension of previous results regarding the parameter range of *exactness*.

We define a function $F_K:\mathbb{R}\backslash B'\to\mathbb{R}\backslash B'$ such as

$$F_K(\cot\theta)\overset{\text{def}}{=}\cot(K\theta), \tag{1}$$

where $K\in\mathbb{N}\backslash\{1\}$ and B' represent a set of points $x\in\mathbb{R}$ such that for finite iteration $n\in\mathbb{Z}$, $F_K^n(x)$ reaches the singular point. F_K corresponds to the K-angle formula of the cot function. For example, $F_2(x)=\frac{1}{2}(x-\frac{1}{x})$ corresponds to the $\cot(2\theta)=\frac{1}{2}\left(\cot\theta-\frac{1}{\cot\theta}\right)$.[40]

Subsequently, SGB transformations $S_{K,\alpha}:\mathbb{R}\backslash B\to\mathbb{R}\backslash B$ are defined as follows:

$$x_{n+1}=S_{K,\alpha}(x_n)\overset{\text{def}}{=}\alpha KF_K(x_n), \tag{2}$$

where $|\alpha|>0$, $K\in\mathbb{N}\backslash\{1\}$, and B represent a set of points $x\in\mathbb{R}$ such that for finite iteration $n\in\mathbb{Z}$, $S_{K,\alpha}^n(x)$ reaches the singular point.

We define some ergodic properties and some concepts as follows.

Definition II.1 (measurable[41]). Let $(X,\mathscr{A},\mu)$ be a measure space. A transformation $S:X\to X$ is measurable if $S^{-1}(A)\in\mathscr{A}$ for all $A\in\mathscr{A}$, where S^{-1} denotes the inverse map of S.

Definition II.2 (nonsingular[41]). A measurable transformation $S:X\to X$ on a measure space $(X,\mathscr{A},\mu)$ is nonsingular if $\mu(S^{-1}(A))=0$ for all $A\in\mathscr{A}$ such that $\mu(A)=0$, where $\mu(A)$ denotes the measure of A.

Definition II.3 (measure preserving[41]). Let $(X,\mathscr{A},\mu)$ be a measure space and $S:X\to X$ a measurable transformation. Then, S is said to be measure preserving if

$$\mu(S^{-1}(A))=\mu(A)\quad\text{for all } A\in\mathscr{A}.$$

Definition II.4 (wandering set[23]). Let S be a nonsingular transformation of the measure space $(X,\mathscr{A},\mu)$. A set $W\subset X$ is called a wandering set if the sets $\{S^{-n}W\}_{n=0}^{\infty}$ are disjoint.

According to Aaronson,[23] define $\mathcal{W}=\mathcal{W}(S)$ as the collection of measurable wandering sets.

Definition II.5 (conservative[23]). Let $\mathcal{D}(S)=\bigcup(\mathcal{W}(S))$ be a measurable union of the collection of wandering sets for S. The nonsingular transformation S is called conservative if $\mu(X\backslash\mathcal{D}(S))=\mu(X)$.

Definition II.6 (ergodic[41]). Let $(X,\mathscr{A},\mu)$ be a measure space and let a nonsingular transformation $S:X\to X$ be given. Then S is called ergodic if every invariant set $A\in\mathscr{A}$ is such that either $\mu(A)=0$ or $\mu(X\backslash A)=0$.

Definition II.7 (mixing[41]). Let $(X,\mathscr{A},\mu)$ be a normalized measure space ($\mu(X)=1$) and $S:X\to X$ a measure preserving transformation. S is called mixing if

$$\lim_{n\to\infty}\mu(A\cap S^{-n}(B))=\mu(A)\mu(B)\quad\text{for all } A,B\in\mathscr{A}.$$

Definition II.8 (exact[41]). Let $(X,\mathscr{A},\mu)$ be a normalized measure space and $S:X\to X$ a measure preserving transformation such that $S(A)\in\mathscr{A}$ for each $A\in\mathscr{A}$. If

$$\lim_{n\to\infty}\mu(S^n(A))=1\text{ for every }A\in\mathscr{A},\mu(A)>0,$$

then S is called exact.

Among these ergodic properties, the following hierarchy holds: exact $\Rightarrow$ mixing $\Rightarrow$ ergodic.[41]

In a previous study, the authors proved[7] that SGB transformations preserve the Cauchy distribution corresponding to the normalized ergodic invariant measure (in this case, the ergodic invariant measure of the whole space is finite so that it can be normalized) and are *exact*[41] (stronger condition than mixing property) when parameters (K,α) are in Range A, as follows:

$$\begin{cases}0<\alpha<1\text{ in the case of }K=2N,N\in\mathbb{N},\\ \frac{1}{K^2}<\alpha<1\text{ in the case of }K=2N+1,\end{cases}$$

and that almost all orbits diverge to infinity for $\alpha>1$, as shown in Table I.

We mention briefly the extension of previous results.[7] That is, it is proven that SGB transformations preserve the normalized ergodic invariant measure and are exact when the parameters (K,α) are in Range B, defined as

$$\begin{cases}0<|\alpha|<1\text{ in the case of }K=2N,\\ \frac{1}{K^2}<|\alpha|<1\text{ in the case of }K=2N+1.\end{cases}$$

Moreover, orbits can diverge to infinity for $|\alpha|>1$. The details of the proof are given in Appendix A.

As such, SGB transformations preserve the normalized ergodic invariant measure when the parameters (K,α) are in Range B ($|\alpha|<1$), and the statistical properties of the systems change for $|\alpha|>1$. However, the ergodic property at the critical points $\alpha=\pm 1$ has been unsettled so far.

What happens at $\alpha=\pm 1$? Given the drastic change in statistical properties before and after the value of $\alpha=\pm 1$, the ergodic property of the *critical* SGB transformations at $\alpha=\pm 1$ is important. The Boole transformation, which corresponds to the case of $K=2,\alpha=1$, is known that it preserves the Lebesgue measure and is ergodic.[2] In Sec. III, we show that *all* the SGB transformations at

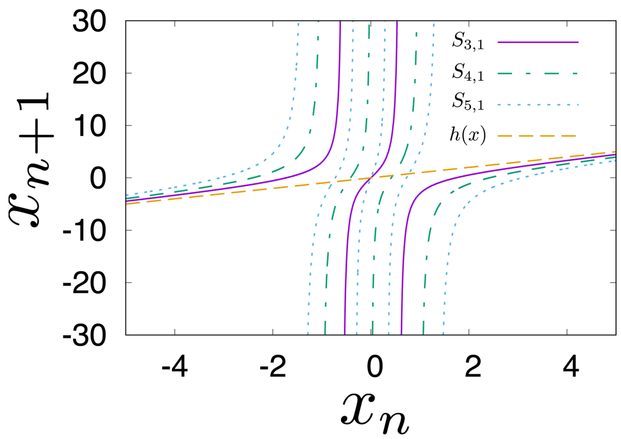

FIG. 1. Return maps of $S_{3,1}$, $S_{4,1}$, and $S_{5,1}$. The function $h(x) = x$ represents the set of fixed points.

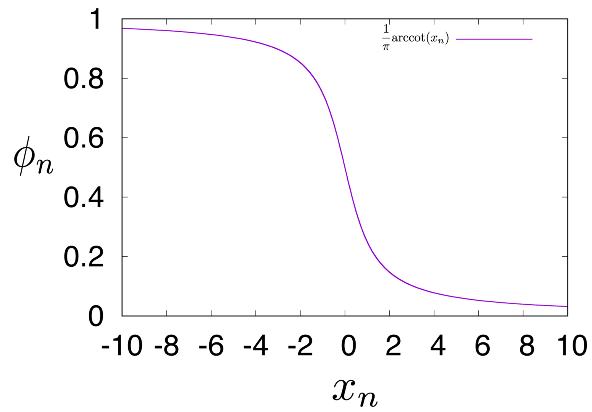

FIG. 2. Relation between x_n and ϕ_n.

$\alpha = \pm 1$ preserve the Lebesgue measure for any $K \in \mathbb{N}\backslash\{1\}$. Table III shows the explicit forms of $S_{K,\pm 1}$ for $K = 2, 3, 4, 5$, and 6. The forms of $S_{K,1}$ are shown in Fig. 1 for $K = 3, 4$, and 5.

III. INFINITE ERGODICITY FOR $\alpha = 1, -1$

In this section, we prove that SGB transformations preserve the Lebesgue measure and are ergodic at $\alpha = \pm 1$.

Theorem III.1. *The SGB transformations at $\alpha = \pm 1$ preserve the Lebesgue measure.*

Proof. The goal is to prove that

$$\left|S_{K,\pm 1}^{-1} I\right| = |I| \tag{3}$$

for any interval $I \subset \mathbb{R}\backslash B$, where $|\cdot|$ denotes the length of an interval (the Lebesgue measure of $\cdot$) and $S_{K,\pm 1}^{-1}$ denotes the inverse map of $S_{K,\pm 1}$. It is sufficient to verify this for intervals of $I = (0, \eta), \eta > 0$, and $I = (\eta, 0), \eta < 0$.[2]

(I) Case of $\alpha = 1$.

[In the following, we prove that Eq. (3) holds for $\eta > 0$; the proof for $\eta < 0$ is similar.] To simplify the proof, we introduce a variable θ defined as $\cot\theta \stackrel{\text{def}}{=} x$, where $\theta \in \mathrm{arccot}(\mathbb{R}\backslash B) \subset [0, \pi]$. We state that

$$S_{K,1}(x) = K\cot(K\theta). \tag{4}$$

For $S_{K,1}(y) = K\cot(K\theta_1) = 0$, $y = \cot\theta_1 = S_{K,1}^{-1}(0)$ satisfies the following relation:

$$\begin{aligned} K\theta_1 &= \frac{\pi}{2} \quad \mathrm{mod}\ \pi, \\ \therefore K\theta_1(m) &= \frac{\pi}{2} + m\pi, m \in \mathbb{Z}, \\ \theta_1(m) &= \frac{\pi}{2K} + \frac{m}{K}\pi. \end{aligned} \tag{5}$$

Given that $\theta_1(m) \in \mathrm{arccot}(\mathbb{R}\backslash B) \subset [0, \pi]$, $\theta_1(m)$ has to satisfy $0 \leq \theta_1(m) = \frac{\pi}{2K} + \frac{m}{K}\pi \leq \pi$. Thus, the range of possible values for m is $m = 0, 1, 2, \ldots, K-1$. For y, such that $S_{K,1}(y) = 0$, it follows that

$$y(m) = \cot\left(\frac{\pi}{2K} + \frac{m}{K}\pi\right), \quad m = 0, 1, 2, \ldots, K-1. \tag{6}$$

For $x = \cot\theta_2 = S_{K,1}^{-1}(\eta)$, it follows that

$$\begin{aligned} K\theta_2 &= \cot^{-1}\left(\frac{\eta}{K}\right) \quad \mathrm{mod}\ \pi, \\ \therefore K\theta_2(m) &= \cot^{-1}\left(\frac{\eta}{K}\right) + m\pi, \\ \theta_2(m) &= \frac{1}{K}\cot^{-1}\left(\frac{\eta}{K}\right) + \frac{m}{K}\pi. \end{aligned} \tag{7}$$

Here, given that

$$0 < \cot^{-1}\left(\frac{\eta}{K}\right) < \frac{\pi}{2}, \tag{8}$$

the variable $\theta_2(m)$ has to satisfy the following:

$0 \leq \theta_2(m) = \frac{1}{K}\cot^{-1}\left(\frac{\eta}{K}\right) + \frac{m}{K}\pi \leq \pi$, and the range of possible values for m is given by

$$-\frac{1}{2} < -\frac{1}{\pi}\cot^{-1}\left(\frac{\eta}{K}\right) \leq m \leq K - \frac{1}{\pi}\cot^{-1}\left(\frac{\eta}{K}\right) < K; \tag{9}$$

that is, $m = 0, 1, 2, \ldots, K-1$. Consequently, θ_2 and x are given by

$$\begin{aligned} \theta_2(m) &= \frac{1}{K}\cot^{-1}\left(\frac{\eta}{K}\right) + \frac{m}{K}\pi, \quad m = 0, 1, 2, \ldots, K-1, \\ x(m) &= \cot\left\{\frac{1}{K}\cot^{-1}\left(\frac{\eta}{K}\right) + \frac{m}{K}\pi\right\}, \end{aligned} \tag{10}$$

where $\eta = S_{k,1}(x(m)) = K\cot(K\theta_2(m))$. The $S_{K,1}$ increases monotonically and the cot function decreases monotonically for $\theta \in [0, \pi]$, and as such, the interval that is mapped from $(0, \eta)$ by $S_{K,1}^{-1}$ is

$$\begin{aligned} \bigcup_{m=0}^{K-1} (y(m), x(m)) = \bigcup_{m=0}^{K-1} \Bigg(&\cot\left(\frac{\pi}{2K} + \frac{m}{K}\pi\right), \\ &\cot\left\{\frac{1}{K}\cot^{-1}\left(\frac{\eta}{K}\right) + \frac{m}{K}\pi\right\}\Bigg). \end{aligned} \tag{11}$$

Then, the following is derived:

$$\left|S_{K,1}^{-1}(0,\eta)\right| = \sum_{m=0}^{K-1}\left[\cot\left\{\frac{1}{K}\cot^{-1}\left(\frac{\eta}{K}\right)+\frac{m}{K}\pi\right\} - \cot\left(\frac{\pi}{2K}+\frac{m}{K}\pi\right)\right]. \tag{12}$$

Now we consider $\sum_{m=0}^{K-1}\cot\left(\frac{\pi}{2K}+\frac{m}{K}\pi\right)$ as follows:

(i) Case of $K = 2N$.

For $\sum_{m=0}^{K-1}\cot\left(\frac{\pi}{2K}+\frac{m}{K}\pi\right)$, adding the terms corresponding to $m = l$ and $m = K-1-l$, $l = 0,\ldots,\frac{K}{2}-1$, we obtain

$$\begin{aligned}\cot\left(\frac{\pi}{2K}+\frac{l}{K}\pi\right) + \cot\left\{\frac{\pi}{2K}+\frac{(K-1)-l}{K}\pi\right\} \\ = \cot\left(\frac{(2l+1)\pi}{2K}\right)+\cot\left(\pi-\frac{(2l+1)\pi}{2K}\right) = 0.\end{aligned} \tag{13}$$

Thus, for $K = 2N$, the following relation holds:

$$\sum_{m=0}^{K-1}\cot\left(\frac{\pi}{2K}+\frac{m}{K}\pi\right) = 0. \tag{14}$$

(ii) Case of $K = 2N+1$.

We have

$$\begin{aligned}\sum_{m=0}^{K-1}\cot\left(\frac{\pi}{2K}+\frac{m}{K}\pi\right) &= \sum_{m=0}^{\frac{K-3}{2}}\cot\left(\frac{\pi}{2K}+\frac{m}{K}\pi\right)+\cot\left(\frac{K-1+1}{2K}\pi\right)+\sum_{m=\frac{K+1}{2}}^{K-1}\cot\left(\frac{\pi}{2K}+\frac{m}{K}\pi\right) \\ &= \sum_{m=0}^{\frac{K-3}{2}}\cot\left(\frac{\pi}{2K}+\frac{m}{K}\pi\right)+\sum_{m=\frac{K+1}{2}}^{K-1}\cot\left(\frac{\pi}{2K}+\frac{m}{K}\pi\right).\end{aligned} \tag{15}$$

Much as in (i), because the term corresponding to $m = l$ negates the term corresponding to $m = K-1-l$, $l = 0,\ldots,\frac{K-3}{2}$, it follows that

$$\sum_{m=0}^{K-1}\cot\left(\frac{\pi}{2K}+\frac{m}{K}\pi\right) = 0. \tag{16}$$

Thus, we have

$$\left|S_{K,1}^{-1}(0,\eta)\right| = \sum_{m=0}^{K-1}\cot\left\{\frac{1}{K}\cot^{-1}\left(\frac{\eta}{K}\right)+\frac{m}{K}\pi\right\}. \tag{17}$$

Now we calculate Eq. (17). $\{x(m)\}_{m=0}^{K-1}$ are the K roots of the equation $\eta = S_{K,1}(x)$. Given that the map $S_{K,1}(x)$ corresponds to the K-angle formula of the cot function, η is given by

$$\begin{aligned}\eta &= S_{K,1}(x(m)) \\ &= K\frac{x^K(m) + (K-2\text{ th and the smaller order terms})}{Kx^{K-1}(m)+(K-3\text{ th and the smaller order terms})}.\end{aligned} \tag{18}$$

Then, it follows that

$$x^K(m) - \eta x^{K-1}(m) + (K-2\text{th and the smaller order terms}) = 0. \tag{19}$$

By definition, $x(m)$ is a root of the above Kth-degree equation. According to the relation between the roots and coefficients of a Kth-degree equation, we can derive

$$\eta = \sum_{m=0}^{K-1}x(m) = \sum_{m=0}^{K-1}\cot\left\{\frac{1}{K}\cot^{-1}\left(\frac{\eta}{K}\right)+\frac{m}{K}\pi\right\}. \tag{20}$$

Therefore, given that

$$\left|S_{K,1}^{-1}(0,\eta)\right| = \eta, \tag{21}$$

Eq. (3) holds.

(II) Case of $\alpha = -1$.

Consider the case of $\eta > 0$ as in (I). In the case of $\alpha = -1$, we have that $x(m) = \cot\left\{\frac{1}{K}\cot^{-1}\left(\frac{-\eta}{K}\right)+\frac{m}{K}\pi\right\} = S_{K,-1}^{-1}(\eta)$. As the map $S_{K,-1}$ decreases monotonically,

$$\begin{aligned}\left|S_{K,-1}^{-1}(0,\eta)\right| &= \sum_{m=0}^{K-1}\left[\cot\left(\frac{\pi}{2K}+\frac{m}{K}\pi\right)\right. \\ &\quad \left. - \cot\left\{\frac{1}{K}\cot^{-1}\left(\frac{-\eta}{K}\right)+\frac{m}{K}\pi\right\}\right] \\ &= -\sum_{m=0}^{K-1}\cot\left\{\frac{1}{K}\cot^{-1}\left(\frac{-\eta}{K}\right)+\frac{m}{K}\pi\right\}.\end{aligned} \tag{22}$$

For the map $S_{K,-1}$, the following relation holds:

$$\eta = -K\frac{x^K(m)+(K-2\text{th and the smaller order terms})}{Kx^{K-1}(m)+(K-3\text{th and the smaller order terms})}$$

$$x^K(m)+\eta x^{K-1}(m)+(K-2\text{th and the smaller order terms}) = 0. \tag{23}$$

According to the relation between the roots and coefficients of a Kth-degree equation, we have the relation

$$-\eta = \sum_{m=0}^{K-1} x(m) = \sum_{m=0}^{K-1} \cot\left\{\frac{1}{K}\cot^{-1}\left(\frac{-\eta}{K}\right) + \frac{m}{K}\pi\right\},$$

$$\therefore -\sum_{m=0}^{K-1} \cot\left\{\frac{1}{K}\cot^{-1}\left(\frac{-\eta}{K}\right) + \frac{m}{K}\pi\right\} = \eta. \tag{24}$$

It follows that

$$\left|S_{K,-1}^{-1}(0,\eta)\right| = -\sum_{m=0}^{K-1} \cot\left\{\frac{1}{K}\cot^{-1}\left(\frac{-\eta}{K}\right) + \frac{m}{K}\pi\right\} = \eta, \tag{25}$$

and Eq. (3) holds. □

At $\alpha = \pm 1$, SGB transformations preserve the Lebesgue measure for *any* $K \geq 2$. Thus, for SGB transformations, the measure for the entire set cannot be normalized to unity (infinite measure). Consequently, we define the ergodicity for the system with the infinite measure as Definition II.6.

Theorem III.2. *SGB transformations at $\alpha = \pm 1$ are ergodic.*

Proof. For the map $S_{K,\pm 1}$, substituting $\cot(\pi\phi_n)$ with $x_n \in \mathbb{R}\backslash B$ gives the induced map $\bar{S}_{K,\pm 1} : X_1 \overset{\text{def}}{=} \frac{1}{\pi}\text{arccot}(\mathbb{R}\backslash B) \to X_1$ such that

$$\phi_{n+1} = \bar{S}_{K,\pm 1}(\phi_n) = \frac{1}{\pi}\cot^{-1}\left\{\pm K\cot(\pi K\phi_n)\right\}. \tag{26}$$

Figure 2 shows the relation between $\mathbb{R}\backslash B$ and X_1 in the range of $-10 < x_n < 10$.

We eliminate a countably infinite number of points whose measure is 0 from $(0, 1)$ to obtain the set X_1, deriving $X_1 \subset (0, 1)$. Consider the measure space $(X_1, \mathscr{B}, \mu_1)$, where $\mathscr{B}$ and μ_1 are the σ-algebra and the measure transformed from the Lebesgue measure by $x_n = \cot(\pi\phi_n)$, respectively. The map $\bar{S}_{K,\pm 1}$ has topological conjugacy with the map $S_{K,\pm 1}$ such that the ergodic properties of $\bar{S}_{K,\pm 1}$ are the same as those of $S_{K,\pm 1}$. In terms of the absolute value of the derivative of $\bar{S}_{K,\pm 1}$, the following holds:

$$\left|\bar{S}'_{K,\pm 1}(\phi)\right| = \frac{K^2\left\{1 + \cot^2(\pi K\phi)\right\}}{K^2\cot^2(\pi K\phi) + 1} > 1, {}^{\forall}\phi \in X_1. \tag{27}$$

Regarding the contraposition for Definition II.6, we show that

$$\text{for any set } A \in \mathscr{B} \text{ s.t. } \mu_1(A) \neq 0, \text{ and } \mu_1(A^c) \neq 0,$$
$$\Rightarrow A \text{ is an not invariant}, \tag{28}$$

where $\mu_1(A)$ and $\mu_1(A^c)$ denote the measure of A and A^c, respectively. Similar to the proof for the mixing property in generalized Boole transformations[6] and exactness in SGB transformations,[7] we define the open intervals $\{I_{j,n}\}$ for which the following relations hold:

$$\begin{aligned} &I_{j,n} \subset \left(\eta_{j,n}, \eta_{j+1,n}\right), \ \eta_{j,n} < \eta_{j+1,n}, \\ &n \in \mathbb{N}, \\ &0 \leq j \leq K^n - 1, \\ &\eta_{0,n} = 0 \text{ and } \eta_{K^n,n} = 1, \\ &\bar{S}^n_{K,\pm 1}(I_{j,n}) = X_1. \end{aligned} \tag{29}$$

Figure 3 illustrates the case of $\{I_{j,1}\}$ for $K = 3, 4$, and 5 at $\alpha = 1$.

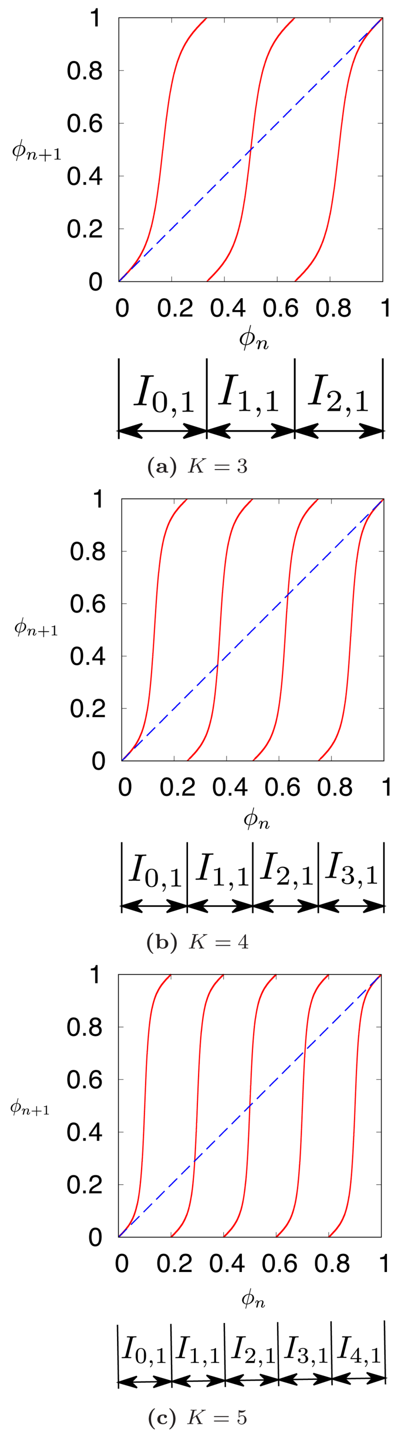

FIG. 3. The solid lines correspond to the transformation $\bar{S}_{K,1}$, which has exact topological conjugacy with the SGB transformation $S_{K,1}$, where $K = 3, 4$, and 5. The dashed line corresponds to the line $\phi_{n+1} = \phi_n$. (a), (b), and (c) correspond to the case of $K = 3, 4$, and 5, respectively.

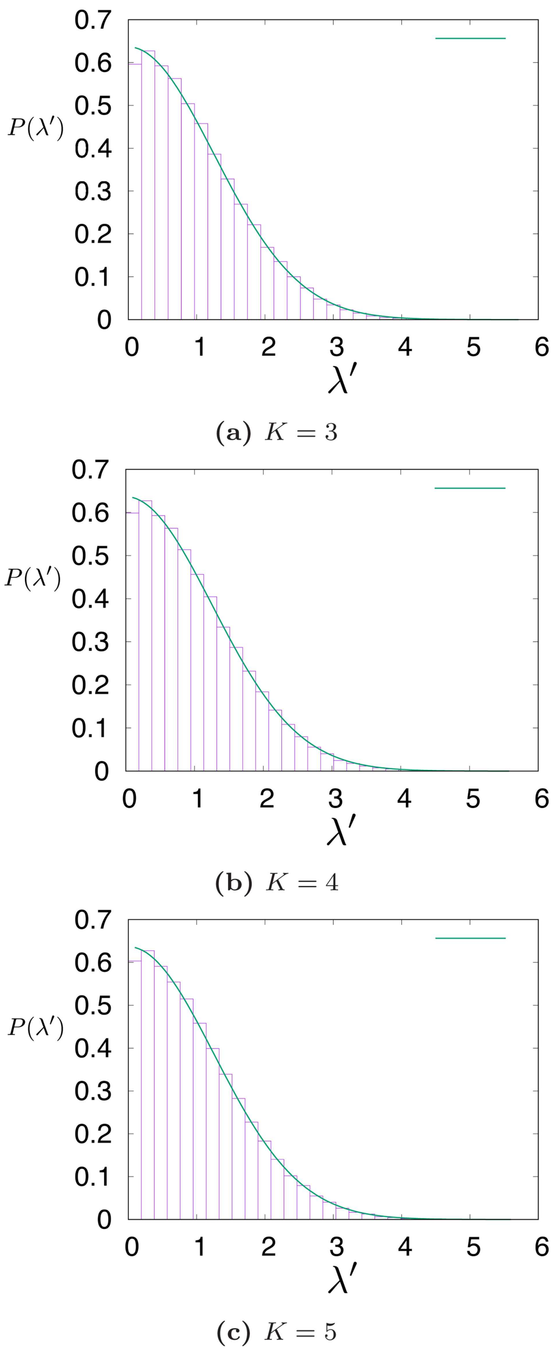

FIG. 4. Relation between density functions $P(\lambda')$ and the normalized Lyapunov exponents λ' in SGB transformations for $K = 3, 4,$ and 5 $(\alpha = 1)$. The number of initial points is 10^5, and the number of iterations is 10^5. The initial points are distributed to obey a normal distribution with mean and variance values of 0 and 1, respectively. The bar graph represents the numerical simulation of the normalized Lyapunov exponents, whereas the solid line represents the normalized Mittag–Leffler distributions of the order $\frac{1}{2}$. (a), (b), and (c) correspond to the case of $K = 3, 4,$ and 5, respectively.

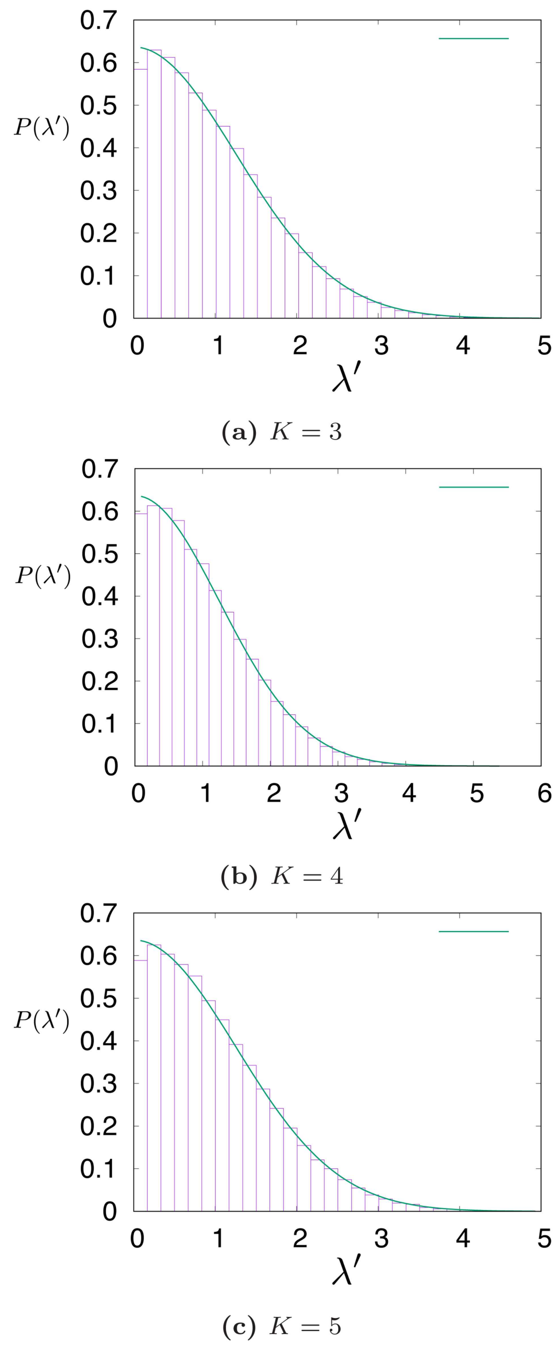

FIG. 5. Relation between density functions $P(\lambda')$ and the normalized Lyapunov exponents λ' in SGB transformations for $K = 3, 4,$ and 5 $(\alpha = -1)$. The number of initial points is 10^5, and the number of iterations is 10^5. The initial points are distributed to obey a normal distribution with mean and variance values of 0 and 1, respectively. The bar graph represents the numerical simulation of the normalized Lyapunov exponents, whereas the solid line represents the normalized Mittag–Leffler distributions of the order $\frac{1}{2}$. (a), (b), and (c) correspond to the case of $K = 3, 4,$ and 5, respectively.

Because the absolute value of the derivative $\bar{S}'_{K,\pm1}$ on any $I_{j,n}$ is larger than unity ($\because \{\phi | \cot^2(\pi K\phi) = \infty\} \notin X_1$), the length of the interval $I_{j,n}$ becomes infinitesimal, $n \to \infty$. Subsequently, given that the measure μ_1 is absolutely continuous to the Lebesgue measure, for any set A, such that $\mu_1(A) \neq 0$, it follows that

$$^{\exists}p, q \text{ s.t.} I_{p,q} \subset A. \tag{30}$$

From the definition of $I_{p,q}$, it follows that

$$\begin{aligned} \bar{S}^q_{K,\pm1} I_{p,q} &= X_1, \\ \therefore \bar{S}^q_{K,\pm1} A &= X_1. \end{aligned} \tag{31}$$

Next, for any set A, such that $\mu_1(A^c) \neq 0$, it follows that

$$A \neq X_1. \tag{32}$$

Then, for any set $A \subset X_1$ such that $\mu_1(A) \neq 0$ and $\mu_1(A^c) \neq 0$, it follows that

$$^{\exists}q \in \mathbb{N} \text{ s.t. } \bar{S}^q_{K,\pm1} A = X_1 \text{ and } A \neq X_1. \tag{33}$$

The implication is that the set A is *not* invariant. Therefore, Theorem III.2 holds. □

IV. NORMALIZED LYAPUNOV EXPONENT

The above discussion has clarified that SGB transformations preserve the *infinite ergodic measure* at $\alpha = \pm1$. In this section, we confirm the statistical properties of SGB transformations at $\alpha = \pm1$, which is a characteristic of infinite ergodic systems.

According to the Darling–Kac–Aaronson theorem,[23] the normalized time average of f_2 converges to the normalized *Mittag–Leffler distribution* (see the definition at Appendix B), as previously given[1,28,42] for the following: an infinite measure ν; a conservative, ergodic, measure preserving map T_2; a function f_2, such as $f_2 \in L^1(\nu), f_2 \geq 0, \int_{X_2} f_2 d\nu > 0$, where X_2 is a set on which the map T_2 is defined,

$$\frac{1}{a_n}\sum_{i=0}^{n-1} f_2 \circ {T_2}^i \to \left(\int_{X_2} f_2 d\nu\right) Y_\gamma, \tag{34}$$

where a_n is the return sequence (which is used to calculate the time average in order to derive the distributional limit theorems[23]) and Y_γ is a random variable that obeys the normalized Mittag–Leffler distribution of the order γ. In the case of the Boole transformation, by defining the return sequence $a_n \stackrel{\text{def}}{=} \frac{\sqrt{2n}}{\pi}$, the distribution of $\frac{1}{a_n}\sum_{i=0}^{n-1} \log |S'_{2,1}(x_i)|$, whose average is normalized to unity, converges to the normalized Mittag–Leffler distribution of order $1/2$.[23]

In the case of this SGB transformation at $\alpha = \pm1$, consider f_2 as $\log\left|\frac{dS_{K,\pm1}}{dx}\right|$. We clarify whether the normalized Lyapunov exponent converges to the normalized Mittag–Leffler distribution by numerical simulation.

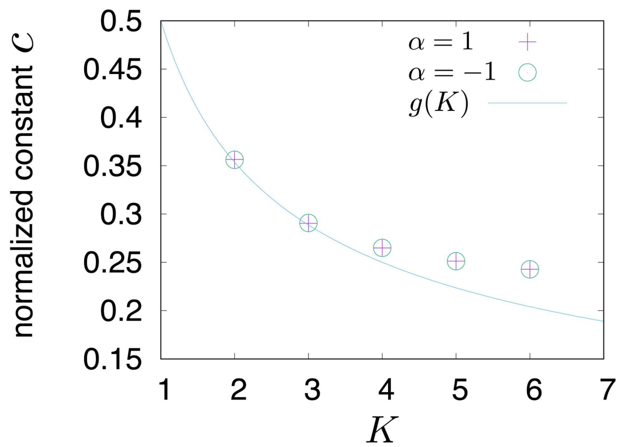

FIG. 6. Relation between normalization constant $c(K)$ and parameter K. The function $g(K)$ is rewritten as $g(K) = \frac{1}{2\sqrt{K}}$.

As previously established, $\log\left|\frac{dS_{K,\pm1}}{dx}\right| \geq 0$.[7] In the following, we assume such conditions as

$$\begin{aligned} &(1) \text{ transformations } S_{K,\pm1} \text{ are conservative,} \\ &(2)\ a_n \propto n^{\frac{1}{2}}, \text{ and} \\ &(3)\ \log\left|\frac{dS_{K,\pm1}}{dx}\right| \in L^1(\mu_2), \end{aligned} \tag{35}$$

as in the case of $(K, \alpha) = (2, 1)$,[23] where μ_2 is the Lebesgue measure.

We calculate the normalized Lyapunov exponents such as

$$\lambda' = \frac{c(K)}{\sqrt{n}}\sum_{i=0}^{n-1} \log\left|\frac{dS_{K,\pm1}}{dx}(x_i)\right|, \tag{36}$$

where $c(K)$ indicates the normalization constants to make the mean values of the Lyapunov exponent equal to unity. Figures 4(a)–4(c) and 5(a)–5(c) show the density function of the normalized Lyapunov exponents for $(K, \alpha) = (3, 1), (4, 1), (5, 1), (3, -1), (4, -1)$, and $(5, -1)$, respectively, which confirms that their normalized Lyapunov exponents are *distributed* according to the normalized Mittag–Leffler distribution of the order $1/2$.

Figure 6 shows the relation between normalization constants $c(K)$ and K at $\alpha = \pm1$. It shows that $c(K)$ tends to decrease as K increases. At $(K, \alpha) = (2, 1)$, $c(K) = \frac{1}{2\sqrt{2}} \simeq 0.354$, based on $a_n = \frac{\sqrt{2n}}{\pi}$.[1] Figure 6 is consistent with this result and from the fact that the points at $(K, \alpha) = (2, -1), (3, 1)$, and $(3, -1)$ are on $g(K) \stackrel{\text{def}}{=} \frac{1}{2\sqrt{K}}$, and that $\int \ln |S'_{2,-1}(x)|\, dx = \int \ln |S'_{3,\pm1}(x)|\, dx = 2\pi$, we conjecture that for $S_{2,-1}$, the return sequence a_n is given by $a_n = \frac{\sqrt{2n}}{\pi}$ and that for $S_{3,\pm1}$, $a_n = \frac{\sqrt{3n}}{\pi}$.

V. CONCLUSION

We showed the statistical ergodic properties of one-dimensional chaotic maps and the SGB transformations $S_{K,\alpha}$ at $\alpha = \pm1$ as shown

TABLE IV. Statistical properties and invariant measures for α in the case of $K = 2N$ (or $2N + 1$) proven in the current study.

$\lvert\alpha\rvert$	$0(1/K^2) < \lvert\alpha\rvert < 1$	$\lvert\alpha\rvert = 1$	$1 < \lvert\alpha\rvert$
Statistical properties	Exact	Ergodic	Almost all orbits diverge to infinity[7]
Invariant measures	Normalized ergodic measure	Infinite ergodic measure	

in Table IV. That is, we proved that for an infinite number of K, $S_{K,\pm 1}$ preserves the Lebesgue measure and that the dynamical systems are *ergodic* for $K \geq 2$. Given these results, we can obtain a class of countably infinite number of critical maps in the sense of the intermittency, which preserves the Lebesgue measure and is proven to be ergodic with respect to the Lebesgue measure.

Infinite measure systems play important roles in physical systems. Because of the existence of $S_{K,\pm 1}$, we can explain the natural change from the parameter region in which systems are *exact* (with a normalized ergodic measure) to the one in which orbits diverge to infinity, and at these points, we can observe the onset of chaos and intermittency. Thus, we can say that SGB transformations at $\alpha = \pm 1$ are *ideal* models for connecting infinite ergodic theory and physics.

In the case of $K = 2$ (the Boole transformation), Adler and Weiss proved its ergodicity in the unbounded region.[2] In our method, we proved the ergodicity by transforming the unbounded domain to the bounded domain using topological conjugacy. In a previous study,[7] we proved that SGB transformations are *exact* for $0 < \alpha < 1$ $(K = 2N)$ or $\frac{1}{K^2} < \alpha < 1$ $(K = 2N + 1)$, $N \in \mathbb{N}$, and change their statistical properties for $\alpha > 1$. The results of our study connect these two phases in the same way as previously reported for generalized Boole transformations.[6] We also demonstrated that the normalized Lyapunov exponents obey the Mittag–Leffler distribution of the order $\frac{1}{2}$ for $(K, \alpha) = (3, 1), (4, 1), (5, 1), (3, -1), (4, -1)$, and $(5, -1)$. In these numerical experiments, the form of the Mittag–Leffler distribution does not depend on the value of K, although there is a relation between $c(K)$ and K.

Various indicators have been proposed to characterize the instability when the corresponding Lyapunov exponent is zero, including the generalized Lyapunov exponent[37,38] and the Lyapunov pair.[1] These infinite critical SGB transformations can be expected to be used as representative indicator maps for detecting chaotic criticality because the ergodic properties are exactly obtained.

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APPENDIX A: PROOF OF EXACTNESS IN RANGE B

In this section, we extend the results of our previous study[7] related to exactness. In particular, we prove that SGB transformations preserve the ergodic invariant density function corresponding to the normalized ergodic invariant measure and are exact in Range B, which is newly defined as

$$\begin{cases} 0 < |\alpha| < 1 \text{ in the case of } K = 2N, \\ \frac{1}{K^2} < |\alpha| < 1 \text{ in the case of } K = 2N + 1. \end{cases}$$

We likewise prove that orbits diverge to infinity for $|\alpha| > 1$. To simplify the proof, we define Range A$'$ as

$$\begin{cases} -1 < |\alpha| < 0 \text{ in the case of } K = 2N, \\ -1 < |\alpha| < -\frac{1}{K^2} \text{ in the case of } K = 2N + 1. \end{cases}$$

We already know that SGB transformations preserve the Cauchy distribution and are exact when the parameters (K, α) are in Range A

$$\begin{cases} 0 < \alpha < 1 \text{ in the case of } K = 2N, \\ \frac{1}{K^2} < \alpha < 1 \text{ in the case of } K = 2N + 1. \end{cases}$$

We have also determined that orbits diverge to infinity for $\alpha > 1$.[7] Here, we hoped to prove similar tendencies in the case of Range A$'$ or $\alpha < -1$.

In the following, the extension from α to $|\alpha|$ can be proven in a similar manner as in our previous study.[7]

First, we show that SGB transformations preserve the Cauchy distribution when the parameters (K, α) are in Range B. If the density function at time n $(\rho_n(x))$ is denoted as

$$\rho_n(x) = \frac{1}{\pi} \frac{\gamma}{x^2 + \gamma^2},$$

then the density function at time $n + 1$, $\rho_{n+1}(x)$ is given by

$$\rho_{n+1}(x) = \frac{1}{\pi} \frac{|\alpha| K G_K(\gamma)}{x^2 + |\alpha|^2 K^2 G_K^2(\gamma)}$$

according to the Perron–Frobenius equation, where $G_K(\gamma)$ corresponds to the K-angle formula of the coth function defined as $G_K(\coth\theta) \stackrel{\text{def}}{=} \coth(K\theta)$.[7] The scale parameter γ is then changed in a single iteration as

$$\gamma \mapsto |\alpha| K G_K(\gamma).$$

In our previous study, the change in scale parameter γ is denoted as

$$\gamma \mapsto \alpha K G_K(\gamma), (K, \alpha) \in \text{Range A}.$$

By changing the parameter from α to $|\alpha|$, we can prove straightforwardly that SGB transformations $\{S_{K,\alpha}\}$ preserve the Cauchy distribution, and the scale parameter can be chosen uniquely when the parameters (K, α) are in Range B.

In terms of exactness, we apply a method similar to that in our previous study.[7] To prove the exactness, we consider the maps $\bar{S}_{K,\alpha}$, which are topologically conjugate with the maps $S_{K,\alpha}$, by changing the variables as $x_n = \cot(\pi\phi_n)$. In terms of $\bar{S}_{K,\alpha}$, it holds that

$$\bar{S}_{K,\alpha}(\phi) = \frac{\alpha K\{1 + \cot^2(\pi K\phi)\}}{\alpha^2 K^2 \cot^2(\pi K\phi) + 1} < 0$$

when the parameters (K, α) are in Range A$'$. In this way, $\bar{S}_{K,\alpha}(\phi)$ is also the monotonic function. Thus, we can prove that SGB transformations $\{S_{K,\alpha}\}$ are exact when the parameters (K, α) are in Range A$'$

considering the intervals $\{I_{j,n}\}$, defined as

$$\begin{aligned} I_{j,n} &\subset \left(\eta_{j,n}, \eta_{j+1,n}\right), \; \eta_{j,n} < \eta_{j+1,n}, \\ & n \in \mathbb{N}, \\ & 0 \le j \le K^n - 1, \\ \eta_{0,n} &= 0 \text{ and } \eta_{K^n,n} = 1, \\ \bar{S}^n_{K,\pm 1}(I_{j,n}) &= X_1. \end{aligned} \tag{A1}$$

In the case of $\alpha < -1$, changing the variable to $z_n = 1/x_n$ gives the map $z_{n+1} = \widetilde{S}_{K,\alpha}(z_n)$. It is clear that $\left|\frac{d\widetilde{S}_{K,\alpha}}{dz}(0)\right| = \frac{1}{|\alpha|} < 1$; here, orbits diverge to infinity.

From the above discussion, we know that SGB transformations preserve the Cauchy distribution corresponding to the normalized ergodic invariant measure and are *exact* when the parameters (K, α) are in Range B.

APPENDIX B: DEFINITION OF THE MITTAG–LEFFLER DISTRIBUTION

Definition B.1 Mittag–Leffler distribution[23]**.** Let $\alpha \in [0, 1]$. The random variable Y_α on $\mathbb{R}_+$ has the normalized Mittag–Leffler distribution of order α if

$$E\left(e^{zY_\alpha}\right) = \sum_{p=0}^{\infty} \frac{\Gamma(1+\alpha)^p z^p}{\Gamma(1+p\alpha)},$$

where $E(\cdot)$ denotes the expectation value of $\cdot$ and Γ denotes the Gamma function. The density function at $\alpha = 1/2$ is denoted as

$$f_{Y_{\frac{1}{2}}}(y) = \frac{2}{\pi} e^{-\frac{y^2}{\pi}}.$$

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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