Sums, products, ratios and intersections of two Cantor sets

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Abstract. We explain how questions on sums of two Cantor sets, products of two Cantor sets, ratios of two Cantor sets and intersections of two Cantor sets are related.

1. Introduction

Arithmetic sums and differences of two Cantor sets have been considered in many papers and in many different settings (e.g., [**1**], [**2**], [**3**], [**4**], [**5**], [**6**], [**7**], [**8**], [**9**], [**10**], [**15**], [**16**], [**17**], [**19**], [**20**], [**21**], [**22**], [**25**], [**26**], [**27**], [**29**]). It arises naturally in dynamical systems (e.g., [**24**]), in number theory (e.g., [**2**], [**8**]), and in spectral theory (e.g., [**3**], [**4**]). It also has a natural connection to the study of intersections of Cantor sets (e.g., [**11**], [**12**], [**13**], [**14**], [**18**], [**23**], [**28**]).

In [**30**], the author considered products of two Cantor sets and obtained the optimal estimates in terms of their thickness that guarantee that their product is an interval. This problem was motivated by the fact that the spectrum of the Labyrinth model, which is a two dimensional quasicrystal model, is given by a product of two Cantor sets. For the Labyrinth model, see [**31**].

In this article, we discuss how questions on sums of two Cantor sets, products of two Cantor sets, ratios of two Cantor sets and intersections of two Cantor sets are related.

2. Preliminaries

For any gap U of a Cantor set, we denote the right (resp. left) endpoint of U by U^R (resp. U^L). If gaps U_1, U_2 satisfy $U_1^R < U_2^L$, we denote $U_1 < U_2$.

Definition 2.1. Let K be a Cantor set. We define the *thickness* of K by

$$\inf_{U_1 < U_2} \max\left\{ \frac{U_2^L - U_1^R}{|U_1|}, \frac{U_2^L - U_1^R}{|U_2|} \right\},$$

where the infimum is taken for all pairs of gaps of K, with at least one of them being a finite gap. We denote this value by $\tau(K)$.

3. Intersections of two Cantor sets

We say that Cantor sets K and L are *interleaved* if neither K nor L lies in a complementary domain of the other. In [**23**], Newhouse proved the following so-called Gap Lemma:

LEMMA 3.1 (Gap Lemma). *Let $K, L \subset \mathbb{R}$ be interleaved Cantor sets with $\tau(K) \cdot \tau(L) \geqslant 1$. Then $K \cap L$ contains at least one element.*

It is natural to ask the cardinality of $K \cap L$. In [**11**] and [**13**], the following was shown (independently):

THEOREM 3.1. *Let $K, L \subset \mathbb{R}$ be interleaved Cantor sets. Assume that $\tau(K) \geqslant \tau(L)$. Then, if*

$$\tau(K) > \frac{\tau(L)^2 + 3\tau(L) + 1}{\tau(L)^2} \quad \text{and} \quad \tau(L) > \frac{(2\tau(K) + 1)^2}{\tau(K)^3},$$

$K \cap L$ contains a Cantor set. Furthermore, for any $M, N > 0$ with $M \geqslant N$, if

$$M < \frac{N^2 + 3N + 1}{N^2} \quad \text{or} \quad N < \frac{(2M + 1)^2}{M^3},$$

there exist interleaved Cantor sets $K, L \subset \mathbb{R}$ such that $\tau(K) = M, \tau(L) = N$ and $K \cap L$ consists of a point.

4. Sums of two Cantor sets

The following is a direct consequence of Gap Lemma (for the proof, see e.g. [**30**]):

THEOREM 4.1. *Let K and L be Cantor sets with $\tau(K) \cdot \tau(L) \geqslant 1$. Then, $K + L$ is a disjoint union of finitely many closed intervals.*

THEOREM 4.2. *Suppose that K and L are Cantor sets with $\tau(K) \cdot \tau(L) \geqslant 1$. Assume that the size of the largest gap of K (resp. L) is not greater than the diameter of L (resp. K). Then $K + L$ is a closed interval.*

5. Products of two Cantor sets

It is easy to see that if $\tau(K) \cdot \tau(L) > 1$ then $K \cdot L$ is a union of countably many or finitely many closed intervals (for the proof see [**30**]). In [**30**], the author considered products of two Cantor sets and obtained the optimal estimate in terms of thickness that $K \cdot L$ is an interval.

DEFINITION 5.1. Let $K \subset \mathbb{R}$ be a Cantor set. We call K a

(1) 0-*Cantor set* if $K_+, K_- \neq \phi$, $\inf K_+ = 0$ and $\inf K_- = 0$;
(2) 0^+-*Cantor set* if $\min K = 0$.

THEOREM 5.1 (Theorem 1.2 in [**30**]). *Let K, L be 0^+-Cantor sets. Then, $K \cdot L$ is an interval if*

$$\tau(L) \geqslant \frac{2\tau(K) + 1}{\tau(K)^2} \quad \text{or} \quad \tau(K) \geqslant \frac{2\tau(L) + 1}{\tau(L)^2}.$$

In particular, if

$$\tau(K) = \tau(L) \geqslant \frac{1+\sqrt{5}}{2},$$

then $K \cdot L$ is an interval. Furthermore, for any $M, N > 0$ with

$$N < \frac{2M+1}{M^2} \quad \textit{and} \quad M < \frac{2N+1}{N^2},$$

there exist 0^+-Cantor sets K, L such that $\tau(K) = M$, $\tau(L) = N$ and $K \cdot L$ is a disjoint union of $\{0\}$ and countably many closed intervals.

Theorem 5.2 (Theorem 1.4 in [**30**]). *Let K, L be 0-Cantor sets. Then, if*

$$2\left(\tau(K)+1\right)\left(\tau(L)+1\right) \leqslant \left(\tau(K)\tau(L)-1\right)^2,$$

$K \cdot L$ is an interval. In particular, if

$$\tau(K) = \tau(L) \geqslant 1+\sqrt{2},$$

then $K \cdot L$ is an interval. Furthermore, for any $M, N > 0$ with

$$2(M+1)(N+1) > (MN-1)^2,$$

there exist 0-Cantor sets K, L such that $\tau(K) = M$, $\tau(L) = N$, and $K \cdot L$ is a disjoint union of two intervals.

To ensure that $K \cdot L$ may contain countably many disjoint closed intervals, we have the following estimate:

Theorem 5.3 (Theorem 1.5 in [**30**]). *Let $M, N > 0$ be real numbers with $M \geqslant N$. Then, if*

$$M < \frac{N^2+3N+1}{N^2} \quad \textit{or} \quad N < \frac{(2M+1)^2}{M^3}, \tag{5.1}$$

there exist 0-Cantor sets K, L such that $\tau(K) = M$, $\tau(L) = N$ and $K \cdot L$ is a disjoint union of $\{0\}$ and countably many closed intervals.

Compare with Theorem 3.1. The author could not prove that the above estimate is optimal. Namely, the following is open:

Conjecture 5.1. Let K, L be 0-Cantor sets. Assume that $\tau(K) \geqslant \tau(L)$. If

$$\tau(K) > \frac{\tau(L)^2+3\tau(L)+1}{\tau(L)^2} \quad \text{and} \quad \tau(L) > \frac{(2\tau(K)+1)^2}{\tau(K)^3},$$

then $K \cdot L$ is a disjoint union of finitely many closed intervals.

6. Ratios of two Cantor sets

It is natural to consider ratios of two Cantor sets. For any Cantor sets $K, L \subset \mathbb{R}$, we denote $K/(L \setminus \{0\})$ simply by K/L. The following is an immediate consequence of Theorem 3.1.

Theorem 6.1. *Let $K, L \subset \mathbb{R}$ be 0-Cantor sets. Assume that $\tau(K) \geqslant \tau(L)$. If*

$$\tau(K) > \frac{\tau(L)^2+3\tau(L)+1}{\tau(L)^2} \quad \textit{and} \quad \tau(L) > \frac{(2\tau(K)+1)^2}{\tau(K)^3},$$

then $K/L = \mathbb{R}$*. Furthermore, for any* $M, N > 0$ *with* $M \geqslant N$*, if*

$$M < \frac{N^2+3N+1}{N^2} \ \ or \ \ N < \frac{(2M+1)^2}{M^3},$$

there exist 0*-Cantor sets* $K, L \subset \mathbb{R}$ *such that* $\tau(K) = M, \tau(L) = N$ *and* K/L *is a union of* $\{0\}$ *and countably many closed intervals.*

Proof. Note that, for $t \in \mathbb{R} \setminus \{0\}$,

$$t \in K/L \iff K \cap tL \neq \{0\}.$$

We have $0 \in K \cap tL$. It is easy to see that K and tL are interleaved. Therefore, by Theorem 3.1 we have $K \cap tL \neq \{0\}$. As for the latter half, the example given in the proof of Theorem 1.5 in [**30**] works. □

In the case that K, L are both 0^+-Cantor sets, we have the following (the proof is analogous):

Theorem 6.2. *Let* K, L *be* 0^+*-Cantor sets. If*

$$\tau(L) \geqslant \frac{2\tau(K)+1}{\tau(K)^2} \ \ or \ \ \tau(K) \geqslant \frac{2\tau(L)+1}{\tau(L)^2},$$

then $K/L = [0, \infty)$*. Furthermore, for any* $M, N > 0$ *with* $M \geqslant N$*, if*

$$N < \frac{2M+1}{M^2} \ \ and \ \ M < \frac{2N+1}{N^2},$$

there exist 0^+*-Cantor sets* $K, L \subset \mathbb{R}$ *such that* $\tau(K) = M, \tau(L) = N$ *and* K/L *is a union of* $\{0\}$ *and countably many closed intervals.*

7. Open problem

In [**2**], Astels considered sums of three or more Cantor sets and obtained the optimal estimate that the sumset is an interval. It would be very interesting if one can obtain the optimal estimate that $f(K_1, K_2, \cdots, K_n)$ is an interval, for a wider class of functions $f : \mathbb{R}^n \to \mathbb{R}$.

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