Non-autonomous iterated function systems and fractals

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Abstract

We consider non-autonomous iterated function systems (NAIFSs). NAIFSs are generalization of the iterated function systems and are an important method to construct generalized fractals (limit sets). However, many researchers only deal with NAIFSs defined on bounded sets (in some sense) or compact sets. The aim of this paper is to present a method to construct the limit sets (and the limit measures) generated by NAIFSs (with weights) defined on complete (separable) metric spaces, which has not been discussed before. If an NAIFS (with weights) satisfies a "good" condition, we can construct the limit sets (and the limit measures) generated by the NAIFS (with weights). We also discuss some basic properties. Besides, this paper presents some examples of NAIFSs.

1 Introduction

Benoit Mandelbrot recognized importance to study fractals around 1970 and many researchers have studied fractals in many directions. Mathematically speaking, there are some methods to construct and study fractals. One of the most important methods to construct fractals is the application of iterated function systems (for short, IFSs). Especially, the fractals (limit sets) generated by IFSs with finitely many mappings are studied since around 1970 ([8], [2], [15], [6], [4], [9], [10]). However, some researchers have studied not only the limit sets generated by IFSs but also the limit sets generated by non-autonomous iterated function systems (for short, NAIFSs). There are some results on the limit sets generated by NAIFSs ([3], [7], [14], [1], [12], [13], [5]). This indicates that we can analyze not only the limit sets of (autonomous) IFSs but also ones of NAIFSs. Note that the authors of those papers only deal with NAIFSs defined on bounded sets (in some sense) or compact sets.

The author found that we can generalize Hutchinson's idea to the setting of NAIFSs (with weights) on complete (separable) metric spaces which are possibly unbounded. The aim of this paper is to present a method to construct the limit sets (and the limit measures) generated by NAIFSs (with weights) on complete (separable) metric spaces based on Hutchinson's idea.

We first consider an NAIFS defined on a complete metric space and show that we can define the limit sets as the limit of a sequence associate with the NAIFS under a simple condition. Since the sequence of limit sets satisfies the property which is similar to the self-similarity, we can regard it as generalized fractals in some sense. Note that if we assume a stronger condition, we also show the exponential convergence to the limit sets. In addition, under the same condition, we construct the projection mapping. it is important to analyze geometrical properties of the limit sets. Moreover, to construct a generalization of self-similar measures, we next consider an NAIFSs with (positive) weights defined on a complete separable metric space which is possibly unbounded. By the above ideas, we also define the limit measures as the limit of a sequence associate with the NAIFS with weights under the same condition. Note that if we assume a stronger condition, we also show the exponential convergence to the limit measures. In addition, we also show that the support of the limit measure is compact and is equal to the limit set under the same condition.

2 Preliminaries and main results

In this section, we present the setting and the main theorems in this paper .

2.1 Definition and main results on NAIFSs

Let I be a set and (X, ρ) be a complete metric space.

Definition 2.1. We say that $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS) if

(i) $\{J_n\}_{n\in\mathbb{N}}$ is a sequence in $\{J \subset I \mid J \text{ is finite}\}$, and

(ii) $\{f_i \colon X \to X\}_{i\in I}$ is a family of contractive mappings on X with the uniform contraction constant $c \in (0,1)$, that is, there exists $c \in (0,1)$ such that for all $i \in I$ and $x, y \in X$,
$$\rho(f_i(x), f_i(y)) \leq c\, \rho(x, y).$$

Note that there exists $z_i \in X$ such that z_i is the unique fixed point of f_i for each $i \in I$ since X is complete and f_i is a contractive on X for each $i \in I$. Let d_H be the Hausdorff distance on $\mathcal{K}(X)$ which is defined by

$$d_H(A, B) := \inf\{\epsilon > 0 \mid A \subset B_\epsilon, B \subset A_\epsilon\} \quad (A, B \in \mathcal{K}(X)),$$

where for each $\epsilon > 0$ and $A \subset X$, we set $A_\epsilon := \{y \in X \mid \exists a \in A, \rho(y, a) \leq \epsilon\}$. For each $n \in \mathbb{N}$, $F_n \colon \mathcal{K}(X) \to \mathcal{K}(X)$ is defined by

$$F_n(A) := \bigcup_{i\in J_n} f_i(A),$$

which is well-defined since each f_i is continuous on X and each J_n is finite. Each mapping F_n is often called the Barnsley operator ([11]).

Let (X, ρ) a metric space and $s \in (0,1)$. We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x with an exponential convergence rate s if there exists $C > 0$ such that for each $n \in \mathbb{N}$, we have $\rho(x_n, x) \leq C s^n$.

We now present the two main results in this paper.

Theorem 2.2 (Main Theorem 1). Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS). Suppose that there exists $x_0 \in X$ such that

$$\sum_{n\in\mathbb{N}} \left\{\max_{i\in J_n} \rho(x_0, z_i)\right\} c^n < \infty. \tag{2.1}$$

Then, there exists the unique sequence of compact subsets $\{K_m\}_{m\in\mathbb{N}}$ in $\mathcal{K}(X)$ such that for each $m \in \mathbb{N}$ and $A \in \mathcal{K}(X)$, we have

$$\lim_{n\to\infty} (F_m \circ F_{m+1} \circ \cdots \circ F_{m+n-1})(A) = K_m \quad \text{in } \mathcal{K}(X).$$

In addition, for each $m \in \mathbb{N}$, we have

$$K_m = F_m(K_{m+1}).$$

Moreover, suppose that there exists $x_0 \in X$,

$$a := \limsup_{n \longrightarrow \infty} \sqrt[n]{\max_{i \in J_n} \rho(x_0, z_i)} < \frac{1}{c}. \tag{2.2}$$

Then, for all $m \in \mathbb{N}$, $r \in \{r > 0 \mid c \leq r < 1, ac < r\}$ and $A \in \mathcal{K}(X)$, $(F_m \circ F_{m+1} \circ \cdots \circ F_{m+n-1})(A)$ converges to K_m as n tends to infinity in $\mathcal{K}(X)$ with an exponential convergence rate r.

Theorem 2.3 (Main Theorem 2). Let $(\{f_i\}_{i \in I}, \{J_n\}_{n \in \mathbb{N}})$ satisfy the setting (NAIFS). Suppose that there exists $x_0 \in X$ such that x_0 satisfies the inequality (2.1). Then, $\mathrm{diam}(f_{w|_n}(K_{m+n}))$ converges to zero as n tends to the infinity. In addition, for each $m \in \mathbb{N}$, there exists a surjective mapping $\pi_m \colon \Pi_{j=m}^{\infty} J_j \to K_m$ such that

$$\{\pi_m(w)\} = \bigcap_{n \in \mathbb{N}} f_{w|_n}(K_{m+n}),$$

where $w = w_m w_{m+1} \cdots \in \prod_{j=m}^{\infty} J_j$ and $f_{w|_n} = f_{w_m} \circ f_{w_{m+1}} \circ \cdots \circ f_{w_{m+n-1}}$. Besides, π_m is uniformly continuous on $\Pi_{j=m}^{\infty} J_j$.

Moreover, suppose that there exists $x_0 \in X$ such that x_0 satisfies the inequality (2.2). Then, for each $r \in \{r > 0 \mid c \leq r < 1, ac < r\}$, $\mathrm{diam}(f_{w|_n}(K_{m+n}))$ converges to zero as n tends to the infinity with an exponential convergence rate r.

2.2 Definition and main results on NAIFSs with weights

Let I be a set and (X, ρ) be a complete separable metric space.

Definition 2.4. We say that $(\{f_i\}_{i \in I}, \{J_n\}_{n \in \mathbb{N}}, \{p_n\}_{n \in \mathbb{N}})$ satisfy the setting (wNAIFS) if

(i) $\{J_n\}_{n \in \mathbb{N}}$ is a sequence in $\{J \subset I \mid J \text{ is finite}\}$,

(ii) $\{f_i \colon X \to X\}_{i \in I}$ is a family of contractive mappings on X with the uniform contraction constant $c \in (0,1)$, that is, there exists $c \in (0,1)$ such that for all $i \in I$ and $x, y \in X$,

$$\rho(f_i(x), f_i(y)) \leq c\, \rho(x, y), \text{ and}$$

(iii) for each $n \in \mathbb{N}$, p_n is $[0, \infty)$-valued functions on I with $p_n(i) > 0$ if and only if $i \in J_n$, and p_n satisfies

$$\sum_{i \in J_n} p_n(i) = 1.$$

Let $\mathcal{P}_1(X)$ be the set of Borel probability measures defined on the complete separable metric space (X, ρ) for which there exists $a \in X$ such that the function $x \mapsto \rho(a, x)$ is integrable. Note that for each $b \in X$ and $P \in \mathcal{P}_1(X)$, we have $\int_X \rho(b, x) P(\mathrm{d}x) < \infty$.

Let $\mathrm{Lip}_1(X)$ be the set of $\mathbb{R}$-valued functions f on X such that $\rho(f(x), f(y)) \leq \rho(x, y)$ for all $x, y \in X$. Let d_{MK} be the Monge-Kantrovich distance on $\mathcal{P}_1(X)$ which is defined by

$$d_{MK}(\mu, \nu) := \sup \left\{ \int_X f \mathrm{d}\mu - \int_X f \mathrm{d}\nu \mid f \in \mathrm{Lip}_1(X) \right\} \quad (\mu, \nu \in \mathcal{P}_1(X)).$$

$M_n\colon \mathcal{P}_1(X) \to \mathcal{P}_1(X)$ $(n \in \mathbb{N})$ is defined by

$$M_n(\mu)(B) := \sum_{i \in J_n} p_n(i)\ \mu(f_i^{-1}(B)) \quad (B \in \mathcal{B}(X)),$$

where $\mathcal{B}(X)$ is the set of all Borel sets in X. Note that for each $n \in \mathbb{N}$, M_n is well-defined since

$$\begin{aligned}
&\int_X \rho(x,a)\ \mathrm{d}M_n(\mu) = \sum_{i \in J_n} p_n(i) \int_X \rho(x,a)\ \mathrm{d}(\mu \circ f_i^{-1}) \\
&= \sum_{i \in J_n} p_n(i) \int_X \rho(f_i(x),a)\ \mathrm{d}\mu \le \sum_{i \in J_n} p_n(i) \int_X \rho(f_i(x), f_i(z_i)) + \rho(f_i(z_i),a)\ \mathrm{d}\mu \\
&= \sum_{i \in J_n} p_n(i) \left\{ \int_X c\ \rho(z_i,x)\ \mathrm{d}\mu + \rho(z_i,a) \right\} < \infty.
\end{aligned}$$

Each mapping M_n is often called the Foias operator ([11]).

We now present the following two more main results in this paper.

Theorem 2.5 (**Main Theorem 3**). Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}}, \{p_n\}_{n\in\mathbb{N}})$ satisfy the setting (wNAIFS). Suppose that there exists $x_0 \in X$ such that x_0 satisfies the inequality (2.1). Then, there exists the unique sequence of probability measures $\{\nu_m\}_{m\in\mathbb{N}}$ in $\mathcal{P}_1(X)$ such that for each $m \in \mathbb{N}$ and $\mu \in \mathcal{P}_1(X)$,

$$\lim_{n\to\infty} (M_m \circ M_{m+1} \circ \cdots \circ M_{m+n-1})(\mu) = \nu_m \quad \text{in } \mathcal{P}_1(X).$$

In addition, for each $m \in \mathbb{N}$, we have

$$\nu_m = M_m(\nu_{m+1}).$$

Moreover, suppose that there exists $x_0 \in X$ such that x_0 satisfies the inequality (2.2). Then, for each $m \in \mathbb{N}$, $r \in \{r > 0 \mid c \le r < 1, ac < r\}$ and $\mu \in \mathcal{P}_1(X)$, $(M_m \circ M_{m+1} \circ \cdots \circ M_{m+n-1})(\mu)$ converges to ν_m as n tends to infinity in $\mathcal{P}_1(X)$ with an exponential convergence rate r.

Theorem 2.6 (**Main Theorem 4**). Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}}, \{p_n\}_{n\in\mathbb{N}})$ satisfy the setting (wNAIFS). If $\mu \in \mathcal{P}_1(X)$ has a compact support, then $\mathrm{supp}(M_n(\mu)) = F_n(\mathrm{supp}(\mu))$ for each $n \in \mathbb{N}$. In addition, if there exists $x_0 \in X$ such that x_0 satisfies the inequality (2.1). then for each $m \in \mathbb{N}$, we have $\mathrm{supp}(\nu_m) = K_m$ and if $\mu \in \mathcal{P}_1(X)$ has a compact support, then

$$\lim_{n\to\infty} \mathrm{supp}(M_m \circ M_{m+1} \circ \cdots \circ M_{m+n-1}(\mu)) = \mathrm{supp}(\nu_m) \quad \text{in } \mathcal{K}(X).$$

Moreover, suppose that there exists $x_0 \in X$ such that x_0 satisfies the inequality (2.2). If $\mu \in \mathcal{P}_1(X)$ has a compact support, then for each $r \in \{r > 0 \mid c \le r < 1, ac < r\}$, $\mathrm{supp}(M_m \circ M_{m+1} \circ \cdots \circ M_{m+n-1}(\mu))$ converges to $\mathrm{supp}(\nu_m)$ in $\mathcal{K}(X)$ as n tends to infinity with an exponential convergence rate r.

3 Some comments

In this section, we give some comments on the main results.

3.1 Bounded case

In Theorem 2.2, we assume that the set $\{z_i \in X \mid i \in I\}$ is bounded instead of the assumption (2.1). By slightly modifying the proof of Theorem 2.2, we have the following result.

Theorem 3.1. Let $(\{f_i\}_{i\in I}, \{J_n\}_{n\in\mathbb{N}})$ satisfy the setting (NAIFS). Suppose that $Z := \{z_i \in X \mid i \in I\}$ is bounded. Then, $\{K_m \in \mathcal{K}(X) \mid m \in \mathbb{N}\}$ is a bounded set in $(\mathcal{K}(X), d_H)$ and there exists the unique sequence of compact subsets $\{K_m\}_{m\in\mathbb{N}}$ in $\mathcal{K}(X)$ such that for each $m \in \mathbb{N}$ and $A \in \mathcal{K}(X)$, we have

$$\lim_{n\to\infty}(F_m \circ F_{m+1} \circ \cdots \circ F_{m+n-1})(A) = K_m \quad \text{in } \mathcal{K}(X)$$

with an exponential convergence rate c. In addition, for each $m \in \mathbb{N}$, we have

$$K_m = F_m(K_{m+1}).$$

By Theorem 3.1, we also construct the sequence of the limit sets as the limit of the sequence in $\mathcal{K}(X)$ generated by the Barnsley operators of an NAIFS defined on a bounded set. In fact, if the set of the unique fixed point of the contractive mappings is bounded, then we reduce to the case of an NAIFS defined on a bounded set. Besides, by Theorem 3.1, we deduce that for all $m \in \mathbb{N}$ and $A \in \mathcal{K}(X)$, $(F_m \circ F_{m+1} \circ \cdots \circ F_{m+n-1})(A)$ converges to K_m as n tends to infinity in $\mathcal{K}(X)$ with an exponential convergence rate c. This is the same case of [14].

3.2 Examples of the setting (NAIFS)

The following counterexample shows that if we do not assume the condition (2.1), the conclusion in Theorem 2.2 does not hold in general.

Example 3.2. Let $f_j \colon \mathbb{R} \to \mathbb{R}$ $(j \in \mathbb{N})$ is defined by

$$f_j(x) := c(x - a_j) + a_j = cx + (1 - c)a_j \ (x \in \mathbb{R}),$$

where $c \in (0, 1)$ and $a_j \geq 0$. Let $I := \mathbb{N}$ and $J_n := \{n\}$. Note that for each $j \in \mathbb{N}$, a_j is the unique fixed point of f_j (i.e. $z_j = a_j$) and $F_j(\{x\}) = \{f_j(x)\}$ for each $x \in \mathbb{R}$.

Then, for all $m, k \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$f_m \circ \cdots \circ f_{m+k-1}(x) = c^k x + (1 - c)\sum_{l=0}^{k-1} c^l a_{m+l}. \tag{3.1}$$

Let $c = 1/2$, $a_i = 2^i$ and $x \in \mathbb{R}$. Note that $\sum_{i\in\mathbb{N}} |z_i|\ c^i = \infty$. Then, for each $k \in \mathbb{N}$, the equation (3.1) is the following.

$$f_m \circ \cdots \circ f_{m+k-1}(x) = \frac{1}{2^k}x + \frac{1}{2}\sum_{l=0}^{k-1}\frac{2^{m+l+1}}{2^l} = \frac{1}{2^k}x + k2^m,$$

which deduce that for each $m \in \mathbb{N}$ and $x \in \mathbb{R}$, $F_m \circ \cdots \circ F_{m+k-1}(\{x\})$ does not converge as k tends to infinity.

In addition, the following example shows that there exists the setting (NAIFS) with the condition (2.1) and (2.2) but $\{z_i \mid i \in \mathbb{N}\}$ and $\{K_m \in \mathcal{K}(X) \mid m \in \mathbb{N}\}$ are not bounded.

Example 3.3. In the previous Example 3.2, we set $c = 1/2$, $a_i = i$ and $x = 0$. Then, we have

$$\sum_{i\in\mathbb{N}} |z_i|\ c^i = \sum_{i\in\mathbb{N}} \frac{i}{2^i} < \infty \quad \text{and} \quad \limsup_{n\to\infty} \sqrt[n]{z_n} = \limsup_{n\to\infty} \sqrt[n]{n} = 1 < c^{-1}.$$

We deduce that this example satisfies the condition (2.1) and (2.2). Note that $\{z_i \mid i \in \mathbb{N}\} = \mathbb{N}$ is not bounded. In addition, by the equation (3.1), for all $m, k \in \mathbb{N}$, we have

$$f_m \circ \cdots \circ f_{m+k-1}(x) = \sum_{l=0}^{k-1} \frac{m+l}{2^l}.$$

Therefore, we have $K_m = \{2m+2\}$ for all $m \in \mathbb{N}$ since

$$\sum_{l=0}^{k} \frac{m+l}{2^l} = \sum_{l=0}^{k} \frac{m}{2^l} + \sum_{l=0}^{k} \frac{l}{2^l} \longrightarrow 2m+2$$

as k tends to infinity. Thus, we deduce that $\{K_m \in \mathcal{K}(X) \mid m \in \mathbb{N}\}$ is also unbounded.

Moreover, the following example shows that there exists the setting (NAIFS) which satisfies the condition (2.1) but do not satisfies the condition (2.2).

Example 3.4. In the previous Example 3.2, we set $c = 1/2$ and $a_i = 2^i/(i+1)^2$. Then, we have

$$\sum_{i\in\mathbb{N}} c^i |z_i| = \sum_{i\in\mathbb{N}} \frac{1}{(i+1)^2} < \infty,$$

which deduce that this example satisfies the condition (2.1). However, we have

$$\limsup_{n\to\infty} \sqrt[n]{z_n} = \limsup_{n\to\infty} \sqrt[n]{2^n/(n+1)^2} = \limsup_{n\to\infty} 2/\sqrt[n]{(n+1)^2} = 2 = c^{-1}.$$

We deduce that this example does not satisfy the condition (2.2).

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