Examples of third noise problems for action evolutions with infinite past

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1 Introduction

An *action evolution* is a pair (X, N) of processes $X = (X_k)_{k\in\mathbb{Z}}$ and $N = (N_k)_{k\in\mathbb{Z}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the stochastic recursive equation

$$X_k = N_k X_{k-1} \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z}, \tag{1.1}$$

where the *observation* $X = (X_k)_{k\in\mathbb{Z}}$ takes values in a measurable space V which evolves at each time being acted by the *driving process* $N = (N_k)_{k\in\mathbb{Z}}$ which is an iid random mappings of V into itself. Here $N_k X_{k-1}$ means the evaluation of a random mapping N_k at X_{k-1}; we always write fv simply for the evaluation $f(v)$. We may call X a *two-sided random orbit* of the random dynamical system generated by the iid random mappings N.

Let us give a precise definition. Let Σ be a measurable space consisting of mappings from V to itself and let $\mathcal{P}(\Sigma)$ denote the set of probability measures on Σ. For $\mu \in \mathcal{P}(\Sigma)$, we call (X, N) a *μ-evolution* if it satisfies (1.1) and N_k at each time k has common law μ and is independent of the past $\mathcal{F}^{X,N}_{k-1}$ defined as

$$\mathcal{F}^{X,N}_{k-1} := \sigma(X_j, N_j : j \le k-1). \tag{1.2}$$

It is obvious that (X, N) is a μ-evolution if and only if the Markov property

$$\mathbb{P}\Big((X_k, N_k) \in \cdot \mid \mathcal{F}^{X,N}_{k-1}\Big) = Q\Big((X_{k-1}, N_{k-1}); \cdot\Big) \quad \text{a.s. for } k \in \mathbb{Z} \tag{1.3}$$

holds with the joint transition probability being given as

$$Q\Big((x, g); \cdot\Big) = \mu\Big\{f \in \Sigma : (fx, f) \in \cdot\Big\}. \tag{1.4}$$

Consider a μ-evolution (X, N). Since for $j < k$ we know that $\sigma(N_k, N_{k-1}, \ldots, N_{j+1})$ is independent of $\mathcal{F}^X_j := \sigma(X_j, X_{j-1}, \ldots)$, the *driving noise* $\mathcal{F}^N_k := \sigma(N_k, N_{k-1}, \ldots)$ is independent of the *remote past noise* $\mathcal{F}^X_{-\infty} := \bigcap_j \mathcal{F}^X_j$. We sometimes encounter a *third noise*, which we define as a sequence of random variables $(U_k)_{k\in\mathbb{Z}}$ such that

$$\mathcal{F}^X_k \subset \mathcal{F}^N_k \vee \mathcal{F}^X_{-\infty} \vee \sigma(U_k) \quad \text{a.s. and} \quad \sigma(U_k) \subset \mathcal{F}^{X,N}_k \quad \text{a.s. for } k \in \mathbb{Z} \tag{1.5}$$

holds with the three σ-fields $\mathcal{F}^N_k$, $\mathcal{F}^X_{-\infty}$ and $\sigma(U_k)$ being independent. Here, for σ-fields $\mathcal{F}_1, \mathcal{F}_2, \ldots$ we always write $\mathcal{F}_1 \vee \mathcal{F}_2 \vee \cdots$ for $\sigma(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \cdots)$. In addition, we can sometimes find a *reduced driving noise*, which we define as a sequence of σ-fields $(\mathcal{G}^N_k)_{k\in\mathbb{Z}}$ such that

$$\mathcal{F}^X_k = \mathcal{G}^N_k \vee \mathcal{F}^X_{-\infty} \vee \sigma(U_k) \quad \text{a.s. and} \quad \mathcal{G}^N_k \subset \mathcal{F}^N_k \quad \text{a.s. for } k \in \mathbb{Z} \tag{1.6}$$

holds. The former identity of (1.6) will be called the *resolution of the observation*.

Iterating the equation (1.1), we have $X_k = N_{k,j} X_j$ with

$$N_{k,j} := N_k N_{k-1} \cdots N_{j+1} \tag{1.7}$$

for $j < k$, and thus we may expect in general that

$$\mathcal{F}_k^X \subset \bigcap_{j<k} \left(\mathcal{F}_k^N \vee \mathcal{F}_j^X \right) \overset{?}{\subset} \mathcal{F}_k^N \vee \left(\bigcap_{j<k} \mathcal{F}_j^X \right) = \mathcal{F}_k^N \vee \mathcal{F}_{-\infty}^X. \tag{1.8}$$

This inclusion $\overset{?}{\subset}$, however, is false in general, or in other words, there may exist a non-trivial third noise; see [6, (1) of Remark 1.4] for the famous errors by Kolmogorov and Wiener. See also [2, Section 2.5] for related discussions.

In many results, the third noise is always a random variable with a uniform law on a certain set which is not given a priori. We discuss several examples with proofs, for better understanding the third noise problems.

2 Random translations on a torus

Let us consider the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0,1)$, which is a compact commutative group with respect to addition in $\mathbb{T}$, or addition in $\mathbb{R}$ mod 1. For a probability meausure μ on $\mathbb{T}$, we consider a μ-evolution

$$X_k = N_k + X_{k-1} \quad \text{a.s. for } k \in \mathbb{Z}, \tag{2.1}$$

where X and N take values in $\mathbb{T}$ and we understand $N_k + X_{k-1}$ as addition in $\mathbb{T}$. Note that $\mathcal{F}_k^N \subset \mathcal{F}_k^X$ for all $k \in \mathbb{Z}$, since $N_k = X_k - X_{k-1}$ in $\mathbb{T}$ (or in $\mathbb{R}$ mod 1).

The resolution problem itself originates from Yor [9] who did a thorough study about this action evolution on a torus with inhomogeneous noise. His results were generalized to action evolutions on general compact groups by Akahori–Uenishi–Yano [1] and Hirayama–Yano [3]; see also [8] for a survey of this topic.

Let us write $\mu\nu \in \mathcal{P}(\mathbb{T})$ for the convolution of μ and $\nu \in \mathcal{P}(\mathbb{T})$:

$$(\mu\nu)(A) = \iint_{\mathbb{T}\times\mathbb{T}} 1_A(x+y)\mu(\mathrm{d}x)\nu(\mathrm{d}y). \tag{2.2}$$

We equip $\mathcal{P}(\mathbb{T})$ with the topology of weak convergence; $\mu_n \to \mu$ if and only if $\int \varphi \mathrm{d}\mu_n \to \int \varphi \mathrm{d}\mu$ for all continuous function φ on $\mathbb{T}$. We write ω_G for the normalized Haar measure on G if G is a compact group. We write ω_V for the uniform law on V if V is a finite set. There is no confusion for finite groups.

Let us present some examples of Yor [9].

Example 2.1 (deterministic translation). Consider a μ-evolution (X, N) with $\mu = \delta_a$ for $a \in \mathbb{T}$:

$$X_k = a + X_{k-1} \quad \text{a.s. for } k \in \mathbb{Z}. \tag{2.3}$$

Then the driving noise is trivial, i.e., $\mathcal{F}_k^N = \{\emptyset, \Omega\}$ a.s. for $k \in \mathbb{Z}$, and we have

$$\mathcal{F}_k^X = \mathcal{F}_{-\infty}^X \quad \text{a.s. for } k \in \mathbb{Z}, \tag{2.4}$$

which gives the resolution of the observation and shows no third noise. In fact, since $X_k - ka = X_{k-1} - (k-1)a$ a.s., we have $X_k = X_0 + ka$ a.s. for $k \in \mathbb{Z}$, which shows that $\mathcal{F}_k^X = \mathcal{F}_{-\infty}^X = \sigma(X_0)$ a.s.

Example 2.2 (lazy irrational translation). Consider a μ-evolution (X, N) with $\mu = \omega_{\{0,a\}}$. We call it a *lazy translation*, because it sometimes stays ($X_k = X_{k-1}$) or translates ($X_k = a + X_{k-1}$) according as $N_k = 0$ or a. Suppose $a \notin \mathbb{Q}$. It then holds that the remote past noise is trivial, i.e., $\mathcal{F}_{-\infty}^X = \{\emptyset, \Omega\}$ a.s. for $k \in \mathbb{Z}$, and that

$$\mathcal{F}_k^X = \mathcal{F}_k^N \vee \sigma(X_k) \quad \text{a.s. for } k \in \mathbb{Z}, \tag{2.5}$$

where, for each $k \in \mathbb{Z}$, the X_k is independent of $\mathcal{F}_k^N$ and has uniform law on $\mathbb{T}$; this gives the resolution of the observation and shows that $(X_k)_{k \in \mathbb{Z}}$ is a third noise.

Let us prove this claim. Set $N_{k,j} = N_k + N_{k-1} + \cdots + N_{j+1}$ for $j < k$. Since $X_j = X_k - N_{k,j}$ and $N_j = X_j - X_{j-1}$ for $j < k$, the identity (2.5) is obvious. Let us prove the independence. We utilize the characters: $\chi_m(z) = \mathrm{e}^{2\pi i m z}$ for $z \in \mathbb{T}$ and $m \in \mathbb{Z}$. By the irrationality of a, we have $\left|\frac{1+\chi_m(a)}{2}\right| < 1$ for all $m \neq 0$, and hence we have

$$\int_{\mathbb{T}} \chi_m(z) \mu^n(\mathrm{d}z) = \left(\int_{\mathbb{T}} \chi_m(z) \mu(\mathrm{d}z) \right)^n = \left(\frac{1 + \chi_m(a)}{2} \right)^n \underset{n \to \infty}{\longrightarrow} 0 \quad \text{for any } m \neq 0. \tag{2.6}$$

This shows $\mu^n \to \omega_{\mathbb{T}}$ by the theory of Fourier series. For $B \in \mathcal{F}_{-\infty}^X$, for any continuous function φ on $\mathbb{T}$, for $k > j > l$ and for $A \in \sigma(N_k, N_{k-1}, \ldots, N_{j+1})$, we have

$$\begin{aligned} \mathbb{E}[1_A 1_B \varphi(X_k)] =& \mathbb{E}[1_A 1_B \varphi(N_{k,j} + N_{j,l} + X_l)] && (2.7) \\ =& \mathbb{E}\left[1_A 1_B \int_{\mathbb{T}} \varphi(N_{k,j} + z + X_l) \mu^{j-l}(\mathrm{d}z) \right] && (2.8) \\ \underset{l \to -\infty}{\longrightarrow}& \mathbb{E}\left[1_A 1_B \int_{\mathbb{T}} \varphi(z) \omega_{\mathbb{T}}(\mathrm{d}z) \right] && (2.9) \\ =& \mathbb{P}(A)\mathbb{P}(B) \int_{\mathbb{T}} \varphi(z) \omega_{\mathbb{T}}(\mathrm{d}z), && (2.10) \end{aligned}$$

since for any sequences $\{x_n\}, \{x_n'\} \subset \mathbb{T}$ we have $\delta_{x_n} \mu^{j+n} \delta_{x_n'} \to \omega_{\mathbb{T}}$ as $n \to \infty$. This shows that, for each $k \in \mathbb{Z}$, the X_k has uniform law on $\mathbb{T}$ and that the three σ-fields $\mathcal{F}_k^N$, $\mathcal{F}_{-\infty}^X$ and $\sigma(X_k)$ are independent. By the identity (2.5), we have $\mathcal{F}_{-\infty}^X \subset \mathcal{F}_k^N \vee \sigma(X_k)$ a.s. Since the remote past noise $\mathcal{F}_{-\infty}^X$ is independent of $\mathcal{F}_k^N \vee \sigma(X_k)$, we see that it is independent of itself, which shows its triviality. The proof is now complete.

Example 2.3 (lazy rational translation). Consider a μ-evolution (X, N) with $\mu = \omega_{\{0,a\}}$ for $a = 1/3 \in \mathbb{T}$. Set $H = \{0, a, 2a\}$ and $C = [0, a)$. Note that H is a subgroup of $\mathbb{T}$ and C is isomorphic to the quotient set $\mathbb{T}/H$. Define a mapping $\mathbb{T} \ni z \mapsto (z^H, z^C) \in H \times C$ so that $z = z^H + z^C$, or in other words, if $z = (n + t)a$ for $n = 0, 1, 2$ and $t \in [0, 1)$, then

$z^H = na$ and $z^C = ta$. Let (X, N) be a μ-evolution. It then holds that there exists a C-valued random variable Z_C such that $X_k^C = Z_C$ a.s. for $k \in \mathbb{Z}$. Moreover, we have the decomposition

$$X_j = -N_{k,j} + X_k^H + Z_C \quad \text{a.s. for } j < k \tag{2.11}$$

and the resolution of the observation

$$\mathcal{F}_k^X = \mathcal{F}_k^N \vee \mathcal{F}_{-\infty}^X \vee \sigma(X_k^H) \quad \text{a.s. for } k \in \mathbb{Z}, \tag{2.12}$$

where $\mathcal{F}_{-\infty}^X = \sigma(Z_C)$ a.s. and, for each $k \in \mathbb{Z}$, the X_k^H is independent of $\mathcal{F}_k^N \vee \mathcal{F}_{-\infty}^X$ and has uniform law on H; as a consequence $(X_k^H)_{k\in\mathbb{Z}}$ is a third noise.

Let us prove this claim. Since $X_k = N_k + X_{k-1}$ and $N_k \in \{0, a\} \subset H$, we have

$$X_k^H = N_k + X_{k-1}^H, \quad X_k^C = X_{k-1}^C \quad \text{a.s.}, \tag{2.13}$$

which shows existence of $Z_C(= X_0^C)$ such that $X_k^C = Z_C$ a.s. for $k \in \mathbb{Z}$. Identity (2.11) is now obvious. Since

$$\int_{\mathbb{T}} \chi_m(z)\mu^n(\mathrm{d}z) = \left(\frac{1 + \chi_m(a)}{2}\right)^n \underset{n\to\infty}{\longrightarrow} \begin{cases} 1 & (m \in 3\mathbb{Z}) \\ 0 & (m \notin 3\mathbb{Z}), \end{cases} \tag{2.14}$$

we obtain $\mu^n \to \omega_H$ by the theory of Fourier series. For $B \in \mathcal{F}_{-\infty}^X$, for any continuous function φ on H, for $l < j < k$ and for $A \in \sigma(N_k, N_{k-1}, \ldots, N_{j+1})$, we have

$$\mathbb{E}[1_A 1_B \varphi(X_k^H)] = \mathbb{E}[1_A 1_B \varphi(N_{k,j} + N_{j,l} + X_l^H)] \underset{l\to-\infty}{\longrightarrow} \mathbb{P}(A)\mathbb{P}(B) \int_H \varphi(z)\omega_H(\mathrm{d}z), \tag{2.15}$$

in the same way as the previous example. This shows that, for each $k \in \mathbb{Z}$, the X_k^H has uniform law on H and that the three σ-fields $\mathcal{F}_k^N$, $\mathcal{F}_{-\infty}^X$ and $\sigma(X_k^H)$ are independent. By (2.11), we obtain

$$\mathcal{F}_k^X = \mathcal{F}_k^N \vee \sigma(Z_C) \vee \sigma(X_k^H) \quad \text{a.s. for } k \in \mathbb{Z}. \tag{2.16}$$

From this identity we obtain $\mathcal{F}_{-\infty}^X \subset \sigma(Z_C) \vee (\mathcal{F}_k^N \vee \sigma(X_k^H))$ a.s. By the independence of $\mathcal{F}_{-\infty}^X$ and $\mathcal{F}_k^N \vee \sigma(X_k^H)$, we can deduce that $\mathcal{F}_{-\infty}^X \subset \sigma(Z_C)$ a.s. (see [2, Section 2.2]). We now obtain $\mathcal{F}_{-\infty}^X = \sigma(Z_C)$ a.s. and thus the proof is complete.

Example 2.4 (lazy rational-irrational translation on a two-dimensional torus). Let us consider the two-dimensional torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$, which is also a compact commutative group under componentwise addition. Consider a μ-evolution with $\mu = \omega_{\{(0,0),(a,b)\}}$ for $a = 1/3$ and $b \notin \mathbb{Q}$. Set $H = \{(0, x), (a, x), (2a, x) : x \in \mathbb{T}\}$ and $C = \{(ta, x) : t \in [0, 1), x \in \mathbb{T}\} = [0, 1/3) \times \mathbb{T}$. Note that H is a subgroup of $\mathbb{T}$ and C is isomorphic to the quotient set $\mathbb{T}^2/H$. We can then deduce the same resolution of the observation by the same argument as the previous example.

3 Finite-state action evolutions

For a finite set V and for the set Σ of mappings of V into itself, we may consider the action evolution

$$X_k = N_k X_{k-1} \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z}, \tag{3.1}$$

where X takes values in V and N does in Σ. As we have some difficulty in obtaining resolution of the observation, we would like to consider a *multiparticle evolution*. For $m \in \mathbb{N}$, we understand that any mapping $f : V \to V$ operates $\boldsymbol{x} = (x^1, \dots, x^m) \in V^m$ componentwise, i.e., $f\boldsymbol{x} = (fx^1, \dots, fx^m)$. We call $(\mathbb{X}, N)$ an *m-particle μ-evolution* if it satisfies

$$\mathbb{X}_k = N_k \mathbb{X}_{k-1} \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z}, \tag{3.2}$$

or in other words

$$X_k^i = N_k X_{k-1}^i \quad \mathbb{P}\text{-a.s. for } k \in \mathbb{Z} \text{ and } i = 1, \dots, m, \tag{3.3}$$

where $\mathbb{X} = (\mathbb{X}_k)_{k\in\mathbb{Z}}$ with $\mathbb{X}_k = (X_k^1, \dots, X_k^m)$ takes values in V^m and $N = (N_k)_{k\in\mathbb{Z}}$ takes values in Σ with N_k at each time k having common law μ and being independent of $\mathcal{F}_{k-1}^{\mathbb{X},N}$.

Write $S(\mu) = \{f : \mu\{f\} > 0\}$ for the support of μ and write $\langle S(\mu)\rangle = \bigcup_{n=1}^{\infty} S(\mu)^n$ for the semigroup generated by $S(\mu)$. For monoparticle action evolutions, Yano [7] has proved that there is no third noise if and only if $S(\mu)$ is *sync*, i.e., there exists $g \in \langle S(\mu)\rangle$ such that $\#g(V) = 1$. Recently Ito–Sera–Yano [4] has obtained the resolution of the observation for m_μ-particle action evolution where $m_\mu = \min\{\#g(V) : g \in \langle S(\mu)\rangle\}$.

For $\mu, \nu \in \mathcal{P}(\Sigma)$ and for $\lambda \in \mathcal{P}(V)$, we define

$$(\mu\nu)(A) = \int_\Sigma \int_\Sigma 1_A(fg)\mu(\mathrm{d}f)\nu(\mathrm{d}g), \quad A \subset \Sigma, \tag{3.4}$$

$$(\mu\lambda)(B) = \int_\Sigma \int_V 1_B(fx)\mu(\mathrm{d}f)\lambda(\mathrm{d}x), \quad B \subset V. \tag{3.5}$$

We sometimes write μf simply for $\mu\delta_f$, etc. Let us present an example of the main theorem of Ito–Sera–Yano [4].

Example 3.1. Let $V = \{1, 2, 3, 4\}$. We write $[y^1, y^2, y^3, y^4]$ for the mapping f of V into itself such that $fi = y^i$ for $i = 1, 2, 3, 4$. Consider the three mappings:

$$f = [2, 2, 4, 4], \quad g = [3, 3, 1, 1], \quad h = [1, 3, 3, 1]. \tag{3.6}$$

Consider $\mu = \omega_{\{\mathrm{id}, f, g, h\}}$, where id denotes the identity mapping of V. It is obvious that $S(\mu) = \{\mathrm{id}, f, g, h\}$ and that $m_\mu = 2$.

(i) The unique minimal two-sided ideal K of $\langle S(\mu)\rangle$, which is called the *kernel* of $\langle S(\mu)\rangle$, admits the *Rees decomposition* given as

$$K = \langle\{f, g, h\}\rangle = LGR, \quad L = \{e, f\}, \quad G = \{e, g\}, \quad R = \{e, h\}, \tag{3.7}$$

where $e := g^2 = [1,1,3,3]$ and the product mapping $L \times G \times R \ni (a,b,c) \mapsto abc \in K$ is bijective. We denote its inverse by $K \ni k \mapsto (k^L, k^G, k^R) \in L \times G \times R$.

(ii) The convolution product of μ satisfies

$$\mu^n \to \eta^L \omega_G \eta^R, \quad \eta^L = \frac{2}{3}\delta_e + \frac{1}{3}\delta_f, \quad \eta^R = \frac{2}{3}\delta_e + \frac{1}{3}\delta_h. \tag{3.8}$$

(iii) It is easy to see that the measure

$$\Lambda = \frac{1}{3}\delta_{(1,3)} + \frac{1}{3}\delta_{(3,1)} + \frac{1}{6}\delta_{(2,4)} + \frac{1}{6}\delta_{(4,2)} \tag{3.9}$$

is a unique μ-invariant probability (i.e., $\mu\Lambda = \Lambda$) on $V_\times^2 = \{(x^1,x^2) \in V^2 : x^1 \neq x^2\}$. Set

$$W_\mu = S(\Lambda) = \{(1,3),(3,1),(2,4),(4,2)\} \ (\subset V_\times^2), \tag{3.10}$$

then we easily have the representations

$$W_\mu = LG(1,3) \quad \text{and} \quad \Lambda = \eta^L \omega_G (1,3). \tag{3.11}$$

We now obtain that the product mapping $L \times G \ni (a,b) \mapsto ab(1,3) \in W_\mu$ is bijective. We denote its inverse by $W_\mu \ni \boldsymbol{x} \mapsto (\boldsymbol{x}^L, \boldsymbol{x}^G) \in L \times G$.

(iv) Let us consider a stationary biparticle μ-evolution $(\mathbb{X}, N)$ such that $\mathbb{X}$ has a common law Λ. Then we have the factorization

$$\mathbb{X}_j = \mathbb{X}_j^L (M_{k,j}^G)^{-1} U_k^G (1,3) \quad \text{a.s. for } j < k \tag{3.12}$$

with $U_k^G = \mathbb{X}_k^G \overset{\mathrm{d}}{=} \omega_G$, $M_j^G = \mathbb{X}_j^G (\mathbb{X}_{j-1}^G)^{-1}$ and $M_{k,j}^G = M_k^G M_{k-1}^G \cdots M_{j+1}^G$. Consequently, we obtain the resolution of the observation

$$\mathcal{F}_k^{\mathbb{X}} = \mathcal{G}_k^N \vee \sigma(U_k^G) \quad \text{a.s. with } \mathcal{G}_k^N = \sigma(\mathbb{X}_j^L, M_j^G : j \le k), \tag{3.13}$$

where $\mathcal{F}_{-\infty}^X$ is trivial and the two σ-fields $\mathcal{F}_k^N (\supset \mathcal{G}_k^N)$ and $\sigma(U_k^G)$ are independent.

Let us prove these claims.

(i) Let us write $S = \langle S(\mu) \rangle$. Since $g^2 = e^2 = e$ and $ge = eg = g$, the set $G = \{e,g\}$ is a group with e the unit element. Noting that

$$ef = he = e, \quad f^2 = fe = f, \quad gf = hg = hf = g, \quad h^2 = eh = h, \tag{3.14}$$

we have so that LGR is a two-sided ideal of S. For any $k \in LGR$, we have $e \in SkS$ so that LGR is a minimal ideal. Uniqueness of a minimal ideal is a known fact (see, e.g., [5, Theorem 2.12]). The injectivity of the product mapping $L \times G \times R \ni (a,b,c) \mapsto abc \in K$ is obvious because $ef = he = e$ and G is a group.

(ii) From a known fact (see, e.g., [5, Theorem 2.2]), the set $\mathcal{K}$ of subsequential limits of $\{\mu^n\}$ is given either as [$\mathcal{K} = \{\eta, \mu\eta\}$ with $\eta = \eta^L \eta^R$ and $\mu\eta = \eta^L g \eta^R$] or as [$\mathcal{K} = \{\eta\}$ with

$\eta = \eta^L \omega_G \eta^R]$, where $\eta^L = \eta\{k^L : k \in K\}$ and $\eta^R = \eta\{k^R : k \in K\}$. If $\mathcal{K} = \{\eta, \mu\eta\}$ were the case, then it would follow that $f = ff \in S(\mu)S(\eta) = S(\mu\eta) = LgR$, which would contradict the fact that $f^G = e$. Hence we see that $\mathcal{K} = \{\eta\}$, which shows $\mu^n \to \eta$. Let $\eta^L = p\delta_e + q\delta_f$ with $p = 1 - q \in [0, 1]$. Since $\eta^R e = \delta_e$, we have $\mu\eta^L\omega_G = \eta^L\omega_G$. We have

$$\mu\eta^L\omega_G = \frac{1}{4}\{2p\delta_e + (1+q)\delta_f + (1+q)\delta_g\}\,\omega_G = \frac{1}{4}\{(2+p)\delta_e + (1+q)\delta_f\}\,\omega_G, \tag{3.15}$$

which shows that $\eta^L = \frac{2}{3}\delta_e + \frac{1}{3}\delta_f$. By the same way we obtain $\eta^R = \frac{2}{3}\delta_e + \frac{1}{3}\delta_h$.

(iii) Since $e(1,3) = (1,3)$ and $g(1,3) = (3,1)$, the group G acts on the two-point set $\{1,3\}$ as permutations. Noting that $f(1,3) = (2,4)$ and $f(3,1) = (4,2)$, we have obtained the injectivity of the product mapping $L \times G \ni (a,b) \mapsto ab(1,3) \in W_\mu$.

(iv) Let $k \in \mathbb{Z}$ be fixed. Since $\mathbb{X}_k \in S(\Lambda) = W_\mu$, we can decompose it as $\mathbb{X}_k = \mathbb{X}_k^L \mathbb{X}_k^G (1,3)$. Since $\mathbb{X}_k \overset{\mathrm{d}}{=} \Lambda = \eta^L\omega_G(1,3)$, we see that $\mathbb{X}_k^L$ and $\mathbb{X}_k^G$ are independent and $\mathbb{X}_k^G \overset{\mathrm{d}}{=} \omega_G$.

For $j < k$, since $M_{k,j}^G = \mathbb{X}_k^G(\mathbb{X}_j^G)^{-1}$, we have

$$\mathbb{X}_j = \mathbb{X}_j^L\mathbb{X}_j^G(1,3) = \mathbb{X}_j^L(M_{k,j}^G)^{-1}\mathbb{X}_k^G(1,3) \quad \text{a.s.}, \tag{3.16}$$

which shows (3.12) and (3.13). We here omit the proof of the fact $\mathcal{G}_k^N \subset \mathcal{F}_k^N$; see [4] for the details.

Let us prove the independence of the two σ-fields $\mathcal{F}_k^N$ and $\sigma(\mathbb{X}_k^G)$. For $j < k$, let $A \in \sigma(N_k, N_{k-1}, \dots, N_{j+1})$ and let φ be a function on G. We now have

$$\mathbb{E}\big[1_A\varphi(\mathbb{X}_k^G)\big] = \mathbb{E}\big[1_A\varphi(M_{k,j}^G\mathbb{X}_j^G)\big] = \mathbb{P}(A)\int_G \varphi(a)\omega_G(\mathrm{d}a), \tag{3.17}$$

since $\mathbb{X}_j^G$ is independent of $\sigma(N_k, N_{k-1}, \dots, N_{j+1})$ and has uniform distribution on G. This shows the desired independence.

References

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