Asymptotic solution of Bowen equation for perturbed potentials defined on shift spaces

Haruyoshi Tanaka

Department of Mathematics and Statistics, Wakayama Medical University

Abstract

We study the asymptotic solution of the equation of the pressure function $s \mapsto P(s\phi_\epsilon)$ for a perturbed potential ϕ_ϵ defined on a shift space with countable state. We show that if the perturbed potential ϕ_ϵ has an asymptotic expansion for a small parameter ϵ and some conditions are satisfied, then the solution $s = s(\epsilon)$ of $P(s\phi_\epsilon) = 0$ has also an asymptotic behaviour with same order. In addition, we also give the case where the order of the expansion of the solution $s = s(\epsilon)$ is less than the order of the expansion of the perturbed potential ϕ_ϵ. Our results can be applied to problems concerning asymptotic behaviors of Hausdorff dimensions obtained from Bowen formula.

1 Preliminaries

In this section we will recall the notion of thermodynamic formalism and some facts of Ruelle transfer operators which were manly introduced by Sarig [4, 5, 6].

Let $G = (V, E, i(\cdot), t(\cdot))$ be a directed multigraph endowed with countable vertex set V, countable edge set E, and two maps $i(\cdot)$ and $t(\cdot)$ from E to V. For each $e \in E$, $i(e)$ is called the initial vertex of e and $t(e)$ called the terminal vertex of e. Denoted by E^∞ the one-sided shift space $\{\omega = \omega_0\omega_1 \cdots \in \prod_{k=0}^{\infty} E : t(\omega_k) = i(\omega_{k+1}) \text{ for any } k \geq 0\}$. The shift transformation $\sigma : E^\infty \to E^\infty$ is defined by $(\sigma\omega)_k = \omega_{k+1}$ for any $k \geq 0$. For $\theta \in (0, 1)$, a metric d_θ on E^∞ is given by $d_\theta(\omega, \upsilon) = \theta^{\inf\{k \geq 0 : \omega_k \neq \upsilon_k\}}$. The metric space (E^∞, d_θ) is compact if E is finite. On the other hand, this metric space is complete and separable and however may be not compact when E is infinite.

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $C(E^\infty, \mathbb{K})$ be the set of all $\mathbb{K}$-valued continuous functions on E^∞, and $F_\theta(E^\infty, \mathbb{K})$ the set of all $\mathbb{K}$-valued d_θ-Lipschitz continuous functions on E^∞. We define $C_b(E^\infty, \mathbb{K})$ as the set of all functions $f \in C(E^\infty, \mathbb{K})$ with $\|f\|_\infty < \infty$ and $F_{\theta,b}(E^\infty, \mathbb{K})$ as the set of all functions $f \in F_\theta(E^\infty, \mathbb{K})$ with $\|f\|_\theta < \infty$, where $\|f\|_\theta = \|f\|_\infty + [f]_\theta$, $\|f\|_\infty = \sup_{\omega \in E^\infty} |f(\omega)|$ and $[f]_\theta = \sup\{|f(\omega) - f(\upsilon)|/d_\theta(\omega, \upsilon) : \omega, \upsilon \in E^\infty,\ \omega \neq \upsilon,\ \omega_0 = \upsilon_0\}$. It is known that if E is finite then the equalities $C(E^\infty, \mathbb{K}) = C_b(E^\infty, \mathbb{K})$ and $F_\theta(E^\infty, \mathbb{K}) = F_{\theta,b}(E^\infty, \mathbb{K})$ hold. For simplicity, the notation $\mathbb{K}$ is omitted from these definitions when $\mathbb{K} = \mathbb{C}$.

The incidence matrix $A = (A(ee'))$ of E^∞ is an $E \times E$ zero-one matrix defined by $A(ee') = 1$ if $t(e) = i(e')$ and $A(ee') = 0$ if $t(e) \neq i(e')$. The matrix A is said to be *finitely irreducible* if there exists a finite subset F of $\bigcup_{k=0}^\infty E^k$ such that for any $e, e' \in E$, ewe' is a path on the graph G for some $w \in F$. This matrix A is called *finitely primitive* if there exist an integer $n \geq 1$ and a finite subset F of E^k for some $k \geq 0$ such that for any $e, e' \in E$, ewe' is a path on the graph G for some $w \in F$. Note that A is finitely irreducible if and only if the dynamics (E^∞, σ) is topologically transitive and A has the *big images and pre-images property* [5], i.e. there exists a finite set F of E such that for any $e \in E$, $A(e'e)A(ee'') = 1$ for some $e', e'' \in F$. Similarity, A is finitely primitive if and only if (E^∞, σ) is topologically mixing and A has the big images and pre-images property.

Assume that the incidence matrix of E^∞ is finitely irreducible and $\psi : E^\infty \to \mathbb{R}$ is in $F_\theta(E^\infty, \mathbb{R})$. We recall the *topological pressure* $P(\psi)$ of ψ defined by

$$P(\psi) = \lim_{k \to \infty} \frac{1}{k} \log \sum_{w \in E^k : [w] \neq \emptyset} \exp(\sup_{\omega \in [w]} \sum_{j=0}^{k-1} \psi(\sigma^j \omega)),$$

where $[w] = \{\omega \in E^\infty : \omega_0 \cdots \omega_{k-1} = w\}$ denotes the cylinder of a word $w \in E^k$. This limit exists in $(-\infty, +\infty]$ (see [2]). A σ-invariant Borel probability measure μ on E^∞ is said to be a *Gibbs measure* of a function $\psi : E^\infty \to \mathbb{R}$ if there exist constants $c \geq 1$ and $P \in \mathbb{R}$ such that for any $\omega \in E^\infty$ and $k \geq 1$

$$c^{-1} \leq \frac{\mu([\omega_0 \omega_1 \cdots \omega_{k-1}])}{\exp(-kP + \sum_{j=0}^{k-1} \psi(\sigma^j \omega))} \leq c.$$

For the existence of this measure, see Theorem 1.1 below (see also [5]).

For a real-valued function ψ defined on E^∞, the Ruelle operator $\mathcal{L}_\psi$ associated to ψ is defined by

$$\mathcal{L}_\psi f(\omega) = \sum_{e \in E : t(e) = i(\omega_0)} e^{\psi(e \cdot \omega)} f(e \cdot \omega)$$

if this series converges in $\mathbb{C}$ for a complex-valued function f on E^∞ and for $\omega \in E^\infty$. Here $e \cdot \omega$ is the concatenation of e and ω, i.e. $e \cdot \omega = e\omega_0\omega_1 \cdots \in E^\infty$. It is known that if E^∞ is finitely irreducible and ψ is in $F_\theta(E^\infty, \mathbb{R})$ with finite pressure, then $\mathcal{L}_\psi$ is a bounded linear operator both on $F_{\theta,b}(E^\infty)$ and on $C_b(E^\infty)$.

The following is a version of Ruelle-Perron-Frobenius Theorem for $\mathcal{L}_\psi$:

Theorem 1.1 ([1, 4]) *Let $G = (V, E, i(\cdot), t(\cdot))$ be a directed multigraph such that A is finitely irreducible. Assume that $\psi \in F_\theta(E^\infty, \mathbb{R})$ with $P(\psi) < \infty$. Then there exists a unique triplet $(\lambda, h, \nu) \in \mathbb{R} \times F_{\theta,b}(E^\infty) \times C_b(E^\infty)^*$ such that the following are satisfied:*

(1) *The number* λ *is positive and a simple maximal eigenvalue of the operator* $\mathcal{L}_\psi : F_{\theta,b}(E^\infty) \to F_{\theta,b}(E^\infty)$.

(2) *The operator* $\mathcal{L}_\psi : F_{\theta,b}(E^\infty) \to F_{\theta,b}(E^\infty)$ *has the decomposition*

$$\mathcal{L}_\psi = \lambda\mathcal{P} + \mathcal{R}$$

with $\mathcal{PR} = \mathcal{RP} = O$. *Here the operator* $\mathcal{P}$ *is a projection onto the one-dimensional eigenspace of the eigenvalue* λ. *Moreover, this has the form* $\mathcal{P}f = \int_{E^\infty} f h\, d\nu$ *for* $f \in C_b(E^\infty)$, *where* $h \in F_{\theta,b}(E^\infty, \mathbb{R})$ *is the corresponding eigenfunction of* λ *and* ν *is the corresponding eigenvector of* λ *of the dual* $\mathcal{L}^*_\psi$ *with* $\nu(h) = 1$. *In particular,* h *satisfies* $0 < \inf_\omega h(\omega) \le \sup_\omega h(\omega) < \infty$ *and* ν *is a Borel probability measure on* E^∞.

(3) *The spectrum of* $\mathcal{R} : F_{\theta,b}(E^\infty) \to F_{\theta,b}(E^\infty)$ *is contained in* $\{z \in \mathbb{C} : |z - \lambda| \ge \rho\}$ *for some small* $\rho > 0$.

Note that the eigenvalue λ is equal to $\exp(P(\psi))$ and $h\nu$ becomes the Gibbs measure of the potential ψ. For simplicity, we sometimes call h the Perron eigenfunction of $\mathcal{L}_\varphi$ and ν the Perron eigenvalue of $\mathcal{L}^*_\varphi$.

2 Known result : the case when E is finite

We give an asymptotic solution of $P(s\varphi(\epsilon,\cdot)) = 0$ for $s \in \mathbb{R}$ under the finite state space given in [7]. Recall that if $\sharp E < +\infty$, $\varphi \in F_\theta(E^\infty, \mathbb{R})$ and $\varphi < 0$ are satisfied, then the equation $P(s\varphi) = 0$ has a unique solution $s \ge 0$. Then we have the following.

Theorem 2.1 ([7, Theorem 2.6]) *Assume that* E *is finite and the incidence matrix* A *of* E^∞ *is (finitely) irreducible. Assume also that functions* $g(\epsilon,\cdot) : E^\infty \to \mathbb{R}$ *has the form* $g(\epsilon,\cdot) = g + g_1\epsilon + \cdots + g_n\epsilon^n + \tilde{g}_n(\epsilon,\cdot)\epsilon^n$ *for some Hölder continuous functions* $g = g_0, g_1, g_2, \ldots, g_n, \tilde{g}_n(\epsilon,\cdot)$ *from the metric space* (E^∞, d_θ) *to* $\mathbb{R}$ *with* $0 < \|g\|_\infty < 1$ *and* $\lim_{\epsilon \to 0} \|\tilde{g}_n(\epsilon,\cdot)\|_\infty = 0$. *Take a unique solution* $s = s_0 \ge 0$ *of the equation* $P(s \log|g|) = 0$. *Then for any small* $\epsilon > 0$, *the solution* $s = s(\epsilon)$ *of the equation* $P(s\log|g(\epsilon,\cdot)|) = 0$ *exists uniquely. Moreover, there exist numbers* $s_1, s_2, \ldots, s_n \in \mathbb{R}$ *such that* $s(\epsilon)$ *has an* n*-order asymptotic expansion*

$$s(\epsilon) = s_0 + s_1\epsilon + \cdots + s_n\epsilon^n + o(\epsilon^n).$$

as $\epsilon \to 0$.

Corollary 2.2 *Under the same conditions of Theorem 2.1, we also assume that the remainder* $\tilde{g}_n(\epsilon,\cdot)$ *of* $g(\epsilon,\cdot)$ *is equal to zero. Then the solution* $s(\epsilon)$ *has an* m*-order asymptotic expansion for any* $m \ge 1$.

Remark 2.3 Each coefficient s_k in Theorem 2.1 is precisely decided (see the proof of Theorem 2.6 in [7]). Indeed, this number is given for $k = 1, 2, \dots, n$ inductively by

$$s_k = \frac{-1}{\mu(\log|g|)} \left(\sum_{i=1}^{k-1} \nu_{k-i}(\mathcal{L}_{s_0 \log|g|}((\log|g|)h))s_i + \sum_{j=1}^{k} \nu_{k-j}(\mathcal{M}_j h) \right),$$

where μ is the Gibbs measure of $s_0 \log|g|$, $\mathcal{M}_1, \dots, \mathcal{M}_n$ are coefficients of the expansion $\mathcal{L}_{s(\epsilon)\log|g(\epsilon,\cdot)|} = \mathcal{L}_{s_0 \log|g|} + \sum_{j=1}^{n} \mathcal{M}_j \epsilon^j + (s(\epsilon) - s_0)\mathcal{L}_{s_0 \log|g|}((\log|g|) \cdot\ ) + o(\epsilon^n)$ of the Ruelle operator $\mathcal{L}_{s(\epsilon)\log|g(\epsilon,\cdot)|}$, $\nu_0, \cdots, \nu_n$ are coefficients of the expansion $\nu(\epsilon, f) = \nu_0(f) + \sum_{i=1}^{n-1} \nu_i(f)\epsilon^i + o(\epsilon^{n-1})$ for $f \in F_{\theta,b}(E^\infty)$ of the Perron eigenvector $\nu(\epsilon, \cdot)$ of the dual of $\mathcal{L}_{s(\epsilon)\log|g(\epsilon,\cdot)|}$, and h is the Perron eigenfunction of $\mathcal{L}_{s_0 \log|g|}$.

3 Main results : the case when E is infinite

First we will state one of our main results. Assume that there exist functions $g, g_1, \dots, g_n \in F_{\theta,b}(E^\infty, \mathbb{R})$, $g(\epsilon, \cdot), \tilde{g}_n(\epsilon, \cdot) \in F_{\theta(\epsilon)}(E^\infty, \mathbb{R})$ and numbers $c_1, c_2, c_3 > 0$, $q \in (0, 1]$ and $c_4(\epsilon) > 0$ with $\lim_{\epsilon \to 0} c_4(\epsilon) = 0$ such that

$$g(\epsilon, \cdot) = g + g_1\epsilon + \cdots + g_n\epsilon^n + \tilde{g}_n(\epsilon, \cdot)\epsilon^n \tag{1}$$

$$\|g\|_\infty < 1 \tag{2}$$

$$\text{either } \inf_{\omega \in [e]} g(\omega) > 0 \text{ or } \sup_{\omega \in [e]} g(\omega) < 0 \quad \text{for each } e \in E \tag{3}$$

$$||g(\omega)| - |g(\upsilon)|| \le c_1 |g(\omega)| d_\theta(\omega, \upsilon) \quad \text{for } \omega, \upsilon \in E^\infty \text{ with } \omega_0 = \upsilon_0 \tag{4}$$

and the inequalities

$$|g_k(\omega)| \le c_2 |g(\omega)|^q \quad \text{for } k = 1, 2, \dots, n \text{ and } \omega \in E^\infty \tag{5}$$

$$|g_k(\omega) - g_k(\upsilon)| \le c_3 |g(\omega)|^q d_\theta(\omega, \upsilon) \quad \text{for } \omega, \upsilon \in E^\infty \text{ with } \omega_0 = \upsilon_0 \tag{6}$$

$$|\tilde{g}_n(\epsilon, \omega)| \le c_4(\epsilon)|g(\omega)|^q \quad \text{for } \omega \in E^\infty \tag{7}$$

hold for any $\omega \in E^\infty$, $k = 1, 2, \dots, n$ and any small $\epsilon > 0$. Let

$$\underline{s} := \inf\{s \ge 0 \ :\ P(s \log|g|) < +\infty\}$$

and put $S(n) := \max(\underline{s} + (1 - q)n, \underline{s}/q)$. We also assume that

$$\text{there exists } s_0 > S(n) \text{ such that } P(s_0 \log|g|) = 0. \tag{8}$$

Then we have the following:

Theorem 3.1 ([8]) *Fix a nonnegative integer n. Let $G = (V, E, i(\cdot), t(\cdot))$ be a graph multigraph satisfying that V, E are countable and the incidence matrix of E^∞ is finitely* *irreducible. Assume that the conditions (1)-(8) are satisfied. Then exists a unique solution* $s = s(\epsilon)$ *of the equation* $P(s \log |g(\epsilon, \cdot)|) = 0$ *for any small* $\epsilon > 0$*, and there exist numbers* $s_1, \ldots, s_n \in \mathbb{R}$ *such that* $s(\epsilon)$ *has an* n*-order asymptotic expansion*

$$s(\epsilon) = s_0 + s_1\epsilon + \cdots + s_n\epsilon^n + \tilde{s}_n(\epsilon)\epsilon^n$$

with $\tilde{s}_n(\epsilon) \to 0$ *as* $\epsilon \to 0$.

Note that each coefficient s_k is decided as well as in Remark 2.3.

Next we give a sufficient condition for a case that $s(\epsilon)$ does not have an $(n+1)$-order asymptotic expansion under the conditions the conditions (1)-(8). We introduce the following conditions:

$$E \text{ is infinitely countable} \tag{9}$$

$$\tilde{g}_n(\epsilon, \cdot) \equiv 0 \tag{10}$$

$$q < 1 \tag{11}$$

$$|g(\omega)| \leq |g(\epsilon, \omega)| \text{ for any } \omega \in E^\infty \text{ and for any small } \epsilon > 0 \tag{12}$$

$$\text{there exist } q_0 \in [q, \frac{nq+1}{n+1}) \text{ and } c_5 > 0 \text{ such that } g_1(\omega) \geq c_5 \frac{|g(\omega)|^{q_0}}{\text{sign}(g(\omega))} \tag{13}$$

$$s_0 < \underline{q} + (1 - q_0)(n+1). \tag{14}$$

Proposition 3.2 ([8]) *Under the same conditions of Theorem 3.1, we also assume the conditions (9)-(14) are satisfied. Then the remainder of the expansion* $s(\epsilon) = s_0 + s_1\epsilon + \cdots + s_n\epsilon^n + \tilde{s}_n(\epsilon)\epsilon^n$ *satisfies* $\lim_{\epsilon \to \infty} |\tilde{s}_n(\epsilon)|/\epsilon = +\infty$*. (Compare with Corollary 2.2).*

4 Exmaples

4.1 Nussbaum-Priyadarshi-Lunel's infinite iterative function systems

We refer to [3]. We assume the following (i)-(ix):

(i) $G = (\{v\}, E = \{1, 2, \cdots\})$ is an infinitely directed graph with singleton vertex.

(ii) (J, d) is a compact metric space. Moreover, J is *perfect set*, namely for any $x \in J$ there exists a sequence $x_k \in J$ with $x_k \neq x$ $(k \geq 1)$ such that $\lim_{k \to \infty} d(x_k, x) = 0$.

(iii) For any $e \in E$, $T_e : J \to J$ is a Lipschitz map satisfying $\sup_{e \in E} \text{Lip}(T_e) =: r < 1$.

(iv) Each T_e is an *infinitesimal similitude* on J, i.e. for each $x \in J$, for any sequences (x_k) and (y_k) on J with $x_k \neq y_k$ for each $k \geq 1$ and $x_k \to x$ and $y_k \to x$, the limit

$$\lim_{k \to \infty} \frac{d(T_e(x_k), T_e(y_k))}{d(x_k, y_k)} =: DT_e(x)$$

exists in $\mathbb{R}$ and is independent of the particular sequences (x_k) and (y_k).

(v) $DT_e(x) > 0$ For any $e \in E$ and $x \in J$.

(vi) There exist constants $c > 0$ and $\beta > 0$ such that for any $x, y \in J$, $|DT_e(x) - DT_e(y)| \leq c|DT_e(x)|d(x,y)^\beta$.

(vii) There exist $t > 0$ and $x \in J$ such that $\sum_{e \in E}(DT_e(x))^t < +\infty$.

(viii) For any $\eta > 0$ there exists $c(\eta) \geq 1$ with $\lim_{\eta \to +0} c(\eta) = 1$ such that for each $e \in E$ and $x, y \in J$ with $0 < d(x,y) < \eta$,

$$c(\eta)^{-1} DT_e(x) \leq \frac{d(T_e(x), T_e(y))}{d(x,y)} \leq c(\eta) DT_e(x).$$

(ix) For each $k \geq 1$, the limit set K_k of the finite iterated function system $(T_1, T_2, \ldots, T_k)$ satisfies that the restricted map $T_e(K_k) \cap T_{e'}(K_k) = \emptyset$ for each $1 \leq e < e' \leq k$ with $e \neq e'$, and $T_e|_{K_k} : K_k \to J$ is one to one for $1 \leq e \leq k$.

Such a system is firstly introduced by Nussbaum, Priyadarshi and Lunel in [3]. For convenience, we call such a system $(J, (T_e))$ an *NPL system*.

The coding map $\pi : E^\infty \to J$ is defined as $\{\pi\omega\} = \{\bigcap_{k=0}^\infty T_{\omega_0} \circ \cdots \circ T_{\omega_k}(J)\}$. Let K be the limit set of the system $(G, J, (O_e))$ which is given by $K = \pi(E^\infty)$. Put $\varphi(\omega) = \log DT_{\omega_0}(\pi\sigma\omega)$. Then a version of Bowen's formula is described as follows:

Theorem 4.1 *Assume that $(J, (T_e))$ is an NPL system and K is its limit set. Then we have $\dim_H K = \inf\{s > 0 : P(s\varphi) < 0\}$. Moreover, if $(J, (T_e))$ is strongly regular, i.e. $0 < P(s\varphi) < +\infty$ for some $s > 0$, then $\dim_H K = s$ if and only if $P(s\varphi) = 0$.*

Fix $n \geq 0$. To formulate asymptotic perturbation of NPL systems, we consider the following conditions:

(I) A pair $(J, (T_e))$ is a strongly regular NPL system satisfying that J is a compact subset of a Banach space $(X, \|\cdot\|)$. Moreover, there exists an open connected subset O of X containing J such that T_e is extended to a map of $C^n(O, X)$ and DT_e is extended to a map of $C^{n+\beta}(O, \mathbb{R})$.

(II) A pair $(J, (T_e(\epsilon, \cdot)))$ is an NPL system with a small parameter $\epsilon > 0$ satisfying that there exists a number $s \in (\underline{s}/\dim_H K, 1]$ if $n = 0$ or $s \in (1 - (\dim_H K - \underline{s})/n, 1]$ if $n \geq 1$ such that the following conditions (a)-(d) are satisfied:

(a) For each $e \in E$, $T_e(\epsilon, \cdot)$ has the n-asymptotic expansion:

$$T_e(\epsilon, \cdot) = T_e + \sum_{k=1}^n T_{e,k}\epsilon^k + \tilde{T}_{e,n}(\epsilon, \cdot)\epsilon^n \quad \text{on } J$$

for some mappings $T_{e,k} \in C^{n-k+1}(O, X)$ $(k = 1, 2, \ldots, n)$ and $\tilde{T}_{e,n}(\epsilon, \cdot) \in C^1(O, X)$ with $\sup_{e \in E} \sup_{x \in J} \|\tilde{T}_{e,n}(\epsilon, x)\| \to 0$.

(b) For each $e \in E$, $DT_e(\epsilon, \cdot)$ has an n-order asymptotic expansion:

$$DT_e(\epsilon, \cdot) = DT_e + \sum_{k=1}^{n} S_{e,k}\epsilon^k + \tilde{S}_{e,n}(\epsilon, \cdot)\epsilon^n \quad \text{on } J$$

for some mappings $S_{e,k} \in C^{n-k+\beta}(O, \mathbb{R})$ $(k = 1, 2, \ldots, n)$ and $\tilde{S}_{e,n}(\epsilon, \cdot) \in C^{\beta(\epsilon)}(O, \mathbb{R})$ with $\sup_{e\in E} \sup_{x\in J} |\tilde{S}_{e,n}(\epsilon, x)| \to 0$ with $\beta(\epsilon) > 0$ for $\epsilon > 0$.

(c) There exist $q \in (\max(1 - (\dim_H K - \underline{s})/n, \underline{s}/ \dim_H K), 1]$ and a constant $c > 0$ such that for any $e \in E$, $l = 0, 1, \ldots, n$, $x \in J$, $y \in O$ and $k = 1, 2, \ldots, n - l + 1$,

$$\|T_{e,l}^{(n-l+1)}(x) - T_{e,l}^{(n-l+1)}(y)\| \le c(DT_e(x))^q \|x - y\|^\beta,$$
$$\|T_{e,l}^{(k)}(x)\| \le c(DT_e(x))^q.$$

(d) There exists a map $c(\epsilon) > 0$ with $\lim_{\epsilon \to 0} c(\epsilon) = 0$ such that for any $e \in E$, $x \in J$ and $\epsilon > 0$

$$\|\tilde{S}_{e,n}(\epsilon, x)\| \le c(\epsilon)(DT_e(x))^q.$$

By applying Theorem 3.1 to the function $g(\epsilon, \omega) := \log DT_{\omega_0}(\epsilon, \pi(\epsilon, \sigma\omega))$, we obtain the following, where $\pi(\epsilon, \cdot)$ denotes the coding map of $(J, (T_e(\epsilon, \cdot)))$.

Theorem 4.2 ([8]) *Under the above conditions (I) and (II) for NPL systems* $(J, (T_e(\epsilon, \cdot)))$ *with small parameter* $\epsilon > 0$*, the Hausdorff dimension of the limit set* $K(\epsilon)$ *of* $(J, (T_e(\epsilon, \cdot)))$ *has the form* $\dim_H K(\epsilon) = \dim_H K + s_1\epsilon + \cdots + s_n\epsilon^n + o(\epsilon^n)$ *for some numbers* $s_1, \ldots, s_n \in \mathbb{R}$.

4.2 Linear countable IFS

Let $a > 1$. Let E be the set of all positive integers. We take an infinite graph $G = (\{v\}, E)$, $J_v = [0, 1]$ and $O_v = (-\eta, 1 + \eta)$ for a small $\eta > 0$. For $e \in E$ and $\epsilon \ge 0$, we define a function $T_e(\epsilon, \cdot) : O_v \to O_v$ by

$$T_e(\epsilon, x) = \Big(\frac{1}{5^e} + \frac{1}{a^e}\epsilon\Big)x + b(e),$$

where we choose $b(e) = 1 - 1/2^{e-1}$. Put $g(\epsilon, \omega) = (1/5^{\omega_0}) + 1/a^{\omega_0}\epsilon$. Denoted by $K(\epsilon)$ the limit set of the IFS $(T_e(\epsilon, \cdot))_{e\in E}$ which is defined as $K(\epsilon) = \bigcup_{\omega\in E^\infty} \bigcap_{n=0}^{\infty} T_{\omega_0}(\epsilon, \cdot) \circ \cdots \circ T_{\omega_n}(\epsilon, \cdot)(J_v)$. It is not hard to check that the function $g(\epsilon, \cdot)$ satisfies all conditions (1)-(8) by putting $g(\omega) = 1/5^{\omega_0}$, $g_1(\omega) = 1/a^{\omega_0}$, $g_2 = \cdots = g_n = 0$, and $\tilde{g}_n(\epsilon, \cdot) = 0$. Moreover the topological pressure of $s \log |g|$ has the equation $P(s \log |g|) = \log \sum_{e\in E} (1/5^e)^s$ for $s > 0$. Therefore, a Bowen formula [2] implies that $P(s(0) \log |g|) = 0$ if and only if $\sum_{e\in E}(1/5^e)^{s(0)} = 1$ if and only if $\dim_H K(0) = s(0) = \log 2/ \log 5$. Moreover, $\underline{s} = \inf\{s : P(s \log |g|) < +\infty\}$ is equal to 0.

Theorem 4.3 ([8]) *Assume the above conditions for $T_e(\epsilon,\cdot)$. Then we have the following:*

(1) If $a \geq 5$ then the Hausdorff dimension $s(\epsilon) = \dim_H K(\epsilon)$ has an n-order asymptotic expansion $s(\epsilon) = s(0) + s_1\epsilon + \cdots + s_n\epsilon^n + o(\epsilon^n)$ for any $n \geq 0$. Each coefficient s_k $(k = 1, 2, \ldots, n)$ is decided as

$$s_k = \sum_{\substack{0\le v\le u, 0\le q\le u-v:\\ (v,q)\neq(0,1)}} \sum_{j=0}^{\min(v,q)} s_{q,u-v}\frac{a_{v,j,s(0)}}{(q-j)!}(-\log 5)^{q-j}\sum_{e=1}^{\infty} e^{q-j}\left(\frac{5^v}{2a^v}\right)^e, \tag{15}$$

where $s_{q,u-v}$ and $a_{v,j,s(0)}$ are defined by

$$(s(\epsilon)-s(0))^k = \begin{cases} s_{1,0} + s_{1,1}\epsilon + \cdots + s_{1,n-1}\epsilon^{n-1} + o(\epsilon^{n-1}) & (k=1) \\ s_{k,0} + s_{k,1}\epsilon + \cdots + s_{k,n-1}\epsilon^{n-1} + s_{k,n}\epsilon^n + o(\epsilon^n) & (k\geq 2) \end{cases} \quad \textit{with}$$

$$s_{k,i} = \begin{cases} 1 & (k=i=0) \\ \displaystyle\sum_{\substack{j_1,\cdots,j_{i-1}\ge 0:\\ j_1+\cdots+j_{i-1}=k\\ j_1+2j_2+\cdots+(i-1)j_{i-1}=i}} \frac{s_1^{j_1}\cdots s_{i-1}^{j_{i-1}}}{j_1!\cdots j_{i-1}!} & (k\geq 1 \textit{ and } k\leq i\leq n) \\ 0 & (\textit{otherwise}), \end{cases}$$

and

$$\binom{t}{v} = \sum_{j=0}^{v} a_{v,j,s(0)}(t-s(0))^j \qquad \textit{with}$$

$$a_{v,j,s(0)} = \begin{cases} \dbinom{s(0)}{v} & (j=0) \\ \displaystyle\sum_{\substack{0\le i_1,\cdots,i_{v-j}\le v-1\\ i_1<\cdots<i_{v-j}}} \frac{1}{v!}\prod_{p=1}^{v-j}(s(0)-i_p) & (l\geq 1 \textit{ and } 0\leq j<v) \\ 1/v! & (v\geq 1 \textit{ and } j=v) \\ 0 & (v<j), \end{cases}$$

where $\binom{t}{v}$ is the binomial coefficient. In particular,

$$s_1 = \frac{\log 2}{(\log 5)^2}\frac{5}{4a-10}$$

$$s_2 = \frac{25\log 2}{(\log 5)^3}\left(\frac{1}{2(2a-5)^2} - \frac{a\log 2}{(2a-5)(4a^2-5)^2} + \frac{\log(2/5)}{8a^2-100}\right).$$

(2) If $1 < a < 5$ then take the largest integer $k \geq 0$ satisfying $a \leq 5/2^{1/(k+1)}$. In this case,

$s(\epsilon)$ has the form

$$s(\epsilon) = \begin{cases} s(0) + s_1\epsilon + \cdots + s_k\epsilon^k + \hat{s}(\epsilon)\epsilon^{k+1}\log\epsilon & (a = 5/2^{1/(k+1)} \text{ for some } k \geq 0) \\ s(0) + s_1\epsilon + \cdots + s_k\epsilon^k + \hat{s}(\epsilon)\epsilon^{\frac{\log 2}{\log(5/a)}} & (\text{otherwise}) \end{cases}$$

with $|\hat{s}(\epsilon)| \asymp 1$ as $\epsilon \to 0$, i.e. $c^{-1} \leq |\hat{s}(\epsilon)| \leq c$ for any $\epsilon > 0$ for some $c \geq 1$, where each s_i is defined by (15).

References

[1] J. Buzzi and O. Sarig, Uniqueness of equilibrium measures for countable Markov shifts and multidimensional piecewise expanding maps. Erg. Th. Dynam. Sys., **23**, 1383–1400 (2003).

[2] R. D. Mauldin and M. Urbański, Graph Directed Markov Systems : Geometry and dynamics of limit sets, Cambridge (2003).

[3] R. D. Nussbaum, A. Priyadarshi, and S. V. Lunel, Positive operators and Hausdorff dimension of invariant sets. Trans. Amer. Math. Soc. **364**, no. 2, 1029-1066 (2012).

[4] O. M. Sarig, Thermodynamic formalism for countable Markov shifts, Erg. Th. Dynam. Sys. **19**, no. 6, 1565–1593 (1999).

[5] O. M. Sarig, Existence of Gibbs measures for countable Markov shifts, Proc. Amer. Math. Soc. **131**, no. 6, 1751-1758 (2003).

[6] O. M. Sarig, Lecture Notes on Thermodynamic Formalism for Topological Markov Shifts, Penn State (2009).

[7] H. Tanaka, An asymptotic analysis in thermodynamic formalism, Monatsh. Math. **164**, 467–486 (2011).

[8] H. Tanaka, Asymptotic solution of Bowen equation for perturbed potentials defined on shift spaces, preprint (2019).

Department of Mathematics and Statistics
Wakayama Medical University
580, Mikazura, Wakayama-city, Wakayama, 641-0011, Japan
htanaka@wakayama-med.ac.jp

和歌山県立医科大学　医学部　教養・医学教育大講座　田中　晴喜