THE HAUSDORFF DIMENSION OF THE REGION OF MULTIPLICITY ONE OF OVERLAPPING ITERATED FUNCTION SYSTEMS ON THE INTERVAL

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1. Introduction

Let us consider iterated function systems on the unit interval $I = [0,1]$ generated by two contractive similarity transformations

$$f_0(x) = ax, \qquad f_1(x) = ax + (1-a) \tag{1}$$

with similarity ratio $0 < a < 1$. If a is grater than or equal to $1/2$, the limit set of the iterated function system $S(a) = \{f_0, f_1\}$ is the interval itself and we say that such an iterated function system is overlapping. We consider overlapping iterated function systems, and study the subset of points of the limit set having unique addresses which we denote by $J_1(S(a))$. Fig.1 shows $J_1(S(a))$ for values of a between $1/2$ and the golden ratio $g = (\sqrt{5}-1)/2$.

We explicitly determine the Hausdorff dimension of $J_1(S(a))$ for values of a described below. For $k = 1, 2, \ldots$, let b_k denote the unique value of $1/2 < a < 1$ satisfying

$$f_0 f_1^k f_0(1) = f_1(0). \tag{2}$$

Likewise, let c_k denote the unique value of $1/2 < a < 1$ satisfying

$$f_0 f_1^{k+1}(0) = f_1(0). \tag{3}$$

The main theorem of this paper is the following.

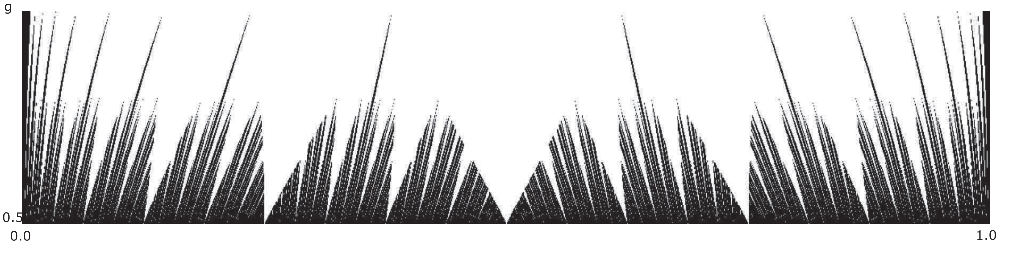

Figure 1. $J_1(S(a))$ for a between $1/2$ and the golden ratio g

k	b_k	c_k	λ_k	$\log \lambda_k$	$\log(2^{k+2}-6)/(k+3)$
1	0.5698402822	0.6180339754	1.0	0.0	
2	0.5356873572	0.5436890423	1.6180339887	0.4812118251	0.4605170186
3	0.5172810853	0.5187900364	1.8392867552	0.6093778634	0.5430160897
4	0.5083449185	0.5086604059	1.9275619755	0.6562559792	0.5800632872
5	0.5040674508	0.5041382611	1.9659482366	0.6759746921	0.6005026306
6	0.5020004213	0.5020170510	1.9835828434	0.6849047264	0.6134956575
7	0.5009901822	0.5009941757	1.9919641966	0.6891211854	0.6226536669
8	0.5004921257	0.5004931390	1.9960311797	0.6911607989	0.6295995634
9	0.5002452433	0.5002454817	1.9980294703	0.6921614300	0.6351404166
10	0.500122398	0.5001224577	1.9990186327	0.6926563765	0.6397154038

TABLE 1. b_k,c_k,λ_k,$\log \lambda_k$ and $\log(2^{k+2}-6)/(k+3)$

Theorem 1.1. *For any a with $b_k \le a \le c_k$ $(k \ge 2)$, the Hausdorff dimension of $J_1(S(a))$ is given by*

$$\dim_H J_1(S(a)) = -\frac{\log \lambda_k}{\log a},$$

where λ_k is the largest eigenvalue of the matrix A_k given in Section 3.

Theorem 1.2. *For any a with $b_k \le a \le c_k$ $(k \ge 2)$, the Hausdorff dimension of $J_1(S(a))$ satisfies*

$$\dim_H J_1(S(a)) \ge -\frac{\log(2^{k+2}-6)}{(k+3)\log a}.$$

Table 1 shows the values of b_k, c_k, λ_k, $\log \lambda_k$ and $\log(2^{k+2}-6)/(k+3)$ for k up to 10. To prove the theorem, we define a graph directed Markov system. The matrix A_k is its incidence matrix.

2. PRELIMINARY

2.1. **Multiplicity function.** Let Σ be a finite set of symbols. We denote by Σ^n the set of codes of length n of symbols in Σ. The set of all finite codes is denoted by $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$. The length of $\omega \in \Sigma^*$ is denoted by $|\omega|$. Given an infinite code

$$\omega = \omega_1\omega_2 \cdots \in \Sigma^\infty,$$

we denote the finite code consisting of the first n symbols of ω by

$$\omega|_n = \omega_1\omega_2 \cdots \omega_n.$$

We deal with iterated function systems (IFS). Let X be a non-empty compact subset of the Euclidean space $\mathbf{R}^d$. A similarity iterated function system is a family of contracting similarity transformations

$$f_i : X \to X \quad (i \in \Sigma).$$

Let $S = \{f_i : I \to I \mid i \in \Sigma\}$ be a similarity iterated function system of the unit interval. Given a code $\omega = \omega_1\omega_2 \cdots \omega_n \in \Sigma^n$, we define $f_\omega : I \to I$ by

$$f_\omega = f_{\omega_1} \circ f_{\omega_2} \circ \cdots \circ f_{\omega_n}.$$

The code map

$$\pi : \Sigma^\infty \to I$$

is defined by

$$\pi(\omega) = \bigcap_{n=1}^{\infty} f_{\omega|_n}(I) \quad (\omega \in \Sigma^\infty).$$

Its image $\pi(\Sigma^\infty)$ is called the limit set of the iterated function system, which we denote by $J(S)$.

If an iterated function system S satisfies

$$f_i(J(S)) \cap f_j(J(S)) = \emptyset$$

for any i, j with $i \neq j$, we say that S is totally disconnected. If not, we say that S is overlapping. If S is totally disconnected, the code map π is one-to-one and every point $x \in J(S)$ has a unique address $\pi^{-1}(x)$. But in case of overlapping iterated function system, π is not one-to-one and some limit points $x \in J(S)$ have more than one address. The multiplicity function

$$m : I \to \mathbf{N} \cup \{\infty\}$$

is given by

$$m(x) = \sharp\{\omega \in \Sigma^\infty \mid \pi(\omega) = x\} \quad (x \in I).$$

For $k = 0, 1, \dots,$ we define $J_k(S)$ by

$$J_k(S) = \{x \in I \mid m(x) = k\}.$$

Then the limit set decomposes into a disjoint union as

$$J(S) = J_1(S) \cup J_2(S) \cup \cdots \cup J_\infty(S).$$

For totally disconnected iterated function systems, we have $J_1(S) = J(S)$. Here we are interested in $J_1(S)$ for overlapping iterated function systems.

Now let us consider the iterated function system given by (1). If $a < 1/2$, the system is totally disconnected. The limit set $J(S(a)) = J_1(S(a))$ is the Cantor set, and its Hausdorff dimension is given by the Hutchinson's theorem ([3]). For $a = 1/2$, $J(S) = I = J_1(S(1/2)) \cup J_2(S(1/2))$ where $J_2(S(1/2))$ is countable. The Hausdorff dimension of $J_1(S(1/2))$ is therefore 1. When $a > 1/2$, the Hausdorff dimension of $J_1(S(a))$ is generally difficult to determine. But in the cases described in Theorem 1.1 we can determine the Hausdorff dimension.

Assume that $a > 1/2$. We define

$$F = f_0(I) \cap f_1(I) = [1 - a, a],$$

and

$$F^* = \bigcup_{\mu \in \{0,1\}^*} f_\mu(F).$$

Proposition 2.1. *Consider an iterated function system $S(a) = \{f_0, f_1\}$ given by (1). If $a > 1/2$, then we have*

$$\bigcup_{m \geq 2} J_m(S(a)) = F^*.$$

Proposition 2.2. *If a is greater than or equal to the golden ratio g, then $J_1(S(a)) = \{0, 1\}$.*

2.2. **Graph directed Markov systems.** In the proof of the main theorem we use the concept of graph directed Markov system. A graph directed Markov system is based on a directed multigraph and an associated incidence matrix, (V, E, A, i, t). The set of vertices V and the set of directed edges E are assumed to be finite. The function $A : E \times E \to \{0, 1\}$ is called an incidence matrix. It determines which edges may follow a given edge. For each edge e, $i(e)$ is the initial vertex of e and $t(e)$ is the terminal vertex of e. So, it holds that $A_{uv} = 1$ if and only if $t(u) = i(v)$. We will consider finite and infinite code spaces of edges consistent with the incidence matrix. We define the infinite code space by

$$E_A^\infty = \{\eta \in E^\infty \mid A_{\eta_i \eta_{i+1}} = 1 \text{ for all } i \geq 1\}.$$

We also define

$$E_A^n = \{\eta \in E^n \mid A_{\eta_i \eta_{i+1}} = 1 \text{ for all } i \geq 1\}.$$

The space of codes of finite length is denoted by $E_A^* = \bigcup_{n=1}^\infty E_A^n$.

We say that A is irreducible if for all $a, b \in E$, there exists $\eta \in E_A^*$ such that $a\eta b \in E_A^*$.

A graph directed Markov system (GDMS) consists of the following

- a directed multigraph, (V, E, i, t),
- an incidence matrix A,
- a set of nonempty compact spaces $\{X_v \subset \mathbf{R}^d \mid v \in V\}$,
- for every $e \in E$, a similarity transformation $f_e : X_{i(e)} \to X_{t(e)}$ with a Lipschitz constant K $(0 < K < 1)$.

Briefly the set

$$S = \{f_e : X_{t(e)} \to X_{i(e)} \mid e \in E\}$$

is called a GDMS. When the vertex set V is a singleton, S is an iterated function system. We can generalize the code map to GDMS.

Definition 2.3. *The code map* $\pi : E_A^\infty \to \bigcup_{v \in V} X_v$ *is defined by*

$$\pi(\eta) = \bigcap_{l=1}^\infty f_{\eta|_l}(X_{t(\eta_l)}) \quad (\eta \in E_A^\infty).$$

The limit set of the graph directed Markov system S *is defined to be the image of the code map,*

$$J(S) = \bigcup_{\eta \in E_A^\infty} \pi(\eta).$$

With respect to the product topology, the code space E_A^∞ is compact and the code map π is continuous. Hence, the limit set $J(S)$ is compact. Since $f_{\eta|_l}(X_{t(\eta_l)})$ shrinks to a point uniformly as $l \to \infty$, we can also express the limit set as

$$J(S) = \bigcap_{l=1}^\infty \bigcup_{\eta \in E_A^l} f_\eta(X_{t(\eta_l)}).$$

2.3. **The Hausdorff dimension.** We need a couple of conditions to evaluate the Hausdorff dimension of the limit set. The first one is the open set condition.

Definition 2.4. *We say that a GDMS* $S = \{f_e : X_{t(e)} \to X_{i(e)} \mid e \in E\}$ *satisfies the open set condition if there exists a nonempty open set* $U \subset \bigcup_{v \in V} X_v$ *such that for all* $e, e' \in E$ $(e \neq e')$,

$$f_e(U \cap X_{t(e)}) \cap f_{e'}(U \cap X_{t(e')}) = \emptyset \qquad \textit{and} \qquad \bigcup_{e \in E} f_e(U \cap X_{t(e)}) \subset U.$$

The second condition we need is the bounded distortion property. We denote the derivative of f at x by f'_x, and define $|f'(x)| = \max\{|f'_x(y)|\}$, where the maximum is taken over all unit vectors y in the tangent space.

Definition 2.5. *A GDMS* $S = \{f_e : X_{t(e)} \to X_{i(e)} \mid e \in E\}$ *satisfies the bounded distortion property if there exists* $K \geq 1$ *such that* $\frac{|f'_\eta(x)|}{|f'_\eta(y)|} \leq K$ *for all* $\eta \in E_A^n$ $(n = 1, 2, \dots)$ *and* $x, y \in X_{t(\eta_n)}$.

If a GDMS satisfies these conditions and the incidence matrix is irreducible, we can evaluate the Hausdorff dimension of the limit set as follows ([2]).

Theorem 2.6 (Mauldin and Urbański)**.** *Suppose that a GDMS* $S = \{f_e : X_{t(e)} \to X_{i(e)} \mid e \in E\}$ *satisfies the open set condition and the bounded distortion property, and that the incidence matrix* A *is irreducible. Define* $P(t)$ *by*

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\eta \in E_A^n} ||f'_\eta||^t.$$

Then the Hausdorff dimension of the limit set is given by

$$\dim_H J(S) = \sup\{t > 0 \mid P(t) > 0\} = \inf\{t > 0 \mid P(t) < 0\}.$$

When all the transformations are similarity transformations, we note that it is also possible to evaluate the Hausdorff dimension by a method similar to the proof of Hutchinson's theorem ([3]).

3. The structure of the GDMS

First, we say the following.

Lemma 3.1. *Let* b_k *denote the unique value of* $1/2 < a < 1$ *satisfying* (2)*. Likewise, let* c_k *denote the unique value of* $1/2 < a < 1$ *satisfying* (3)*. Then,*

$$\frac{1}{2} < \cdots < b_k < c_k < \cdots < b_2 < c_2 < b_1 < c_1.$$

Furthermore, the sequences b_k *and* c_k *converge to* $1/2$ *as* k *increases.*

Now we consider the iterated function system $S(a) = \{f_0, f_1\}$ defined by (1). We denote its limit set by $J(S(a))$.

If $a = 1/2$, for any k, we have

$$J(S(a)) = I = \bigcup_{\omega \in \{0,1\}^k} f_\omega(I), \tag{4}$$

where the intervals $f_\omega(I)$ line up from 0 to 1 according to the order of binary integers $\omega_1 \omega_2 \cdots \omega_k$ (2). (See Fig. 2(top).)

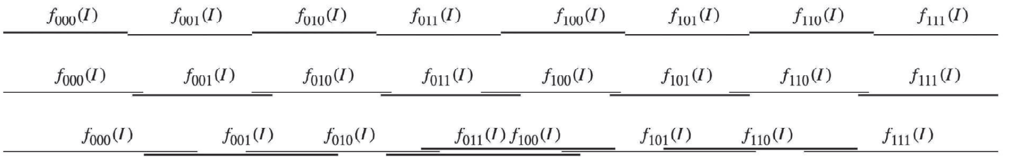

FIGURE 2. The intervals $f_\omega(I)$ for $\omega \in \{0,1\}^3$ when $a = 0.5$ (top), $a = 0.52$ (middle), and $a = 0.58$ (bottom).

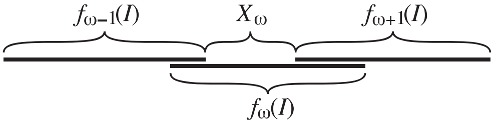

FIGURE 3. X_ω

If a is slightly larger than $1/2$, (4) still holds with slightly overlapping intervals as shown in Fig. 2(middle). In the following, for any intervals I and I', we define $I < I'$ if and only if $x < y$ for any $x \in I$ and $y \in I'$.

Lemma 3.2. *Suppose that $1/2 < a < b_{k-1}$. Then, we have $f_{\omega-1}(I) < f_{\omega+1}(I)$ for all $\omega \in \{0,1\}^{k+1} \setminus \{0\cdots 0, 1\cdots 1\}$.*

We define X_ω to be the interval between $f_{\omega-1}(I)$ and $f_{\omega+1}(I)$. (See Fig. 3.)

Definition 3.3. *Suppose that $1/2 < a < b_{k-1}$. For any $\omega \in \{0,1\}^{k+1}$, we define the interval X_ω by*

$$\begin{aligned} X_\omega &= f_\omega(I) - \mathrm{int}(f_{\omega-1}(I)) - \mathrm{int}(f_{\omega+1}(I)) \quad (\omega \neq 0\cdots 0, 1\cdots 1), \\ X_{0\cdots 0} &= f_{0\cdots 0}(I) - \mathrm{int}(f_{0\cdots 01}(I)), \\ X_{1\cdots 1} &= f_{1\cdots 1}(I) - \mathrm{int}(f_{1\cdots 10}(I)). \end{aligned}$$

Now we define the GDMS $S_k(a)$ for $b_k \le a \le c_k$ ($k = 1, 2, \dots$). The multigraph with associated incidence matrix (V_k, E_k, A_k, i, t) is defined as follows. The vertex set is

$$V_k = \{0,1\}^{k+1}.$$

Elements of V_k are codes of length $k+1$; we also regard them as $(k+1)$-digit binary numbers. The edge set E_k is then defined by

$$E_k = \{(\omega, \phi_0(\omega)) \in V_k \times V_k \mid \omega \neq 1\cdots 1\} \cup \{(\omega, \phi_1(\omega)) \in V_k \times V_k \mid \omega \neq 0\cdots 0\},$$

where the maps $\phi_0, \phi_1 : V_k \to V_k$ are defined by

$$\phi_0(\omega) = \left\lfloor \frac{\omega}{2} \right\rfloor, \qquad \phi_1(\omega) = \left\lfloor \frac{\omega}{2} \right\rfloor + 2^k.$$

Here, $\left\lfloor \frac{\omega}{2} \right\rfloor$ is the maximum integer not greater than $\omega/2$. The incidence matrix

$$A_k : E_k \times E_k \to \{0,1\}$$

is given by

$$A_k((\omega,\phi_l(\omega)),(\omega',\phi_m(\omega'))) = \begin{cases} 1 & (\phi_l(\omega)=\omega') \\ 0 & (\text{otherwise}). \end{cases}$$

We define $S_k(a)$ by

$$S_k(a) = \{f_e : X_{t(e)} \to X_{i(e)} \mid e \in E_k\},$$

where

$$f_e = \begin{cases} f_0|_{X_{t(e)}} & \text{if } e = (\omega, \phi_0(\omega)) \\ f_1|_{X_{t(e)}} & \text{if } e = (\omega, \phi_1(\omega)). \end{cases}$$

That the image of f_e is contained in $X_{i(e)}$ is seen from the following lemma.

Lemma 3.4. *The GDMS $S_k(a)$ satisfies the open set condition. In fact, the open set*

$$U = \bigcup_{\omega \in V_k} \operatorname{int}(X_\omega),$$

satisfies

$$f_e(U \cap X_{t(e)}) \cap f_{e'}(U \cap X_{t(e')}) = \emptyset \tag{5}$$

for $e, e' \in E$ $(e \neq e')$, and

$$\bigcup_{e \in E_k} f_e(U \cap X_{t(e)}) \subset U. \tag{6}$$

Lemma 3.5. *Suppose that $b_k \leq a < c_k$. Then, for $l = 0, 1, \dots,$ we have*

$$\bigcup_{\eta \in E^l_{A_k}} f_\eta(X_{t(\eta_l)}) \cup \bigcup_{|\mu| \leq l+k} f_\mu(F) = I. \tag{7}$$

Moreover, $f_\eta(X_{t(\eta_l)})$ for $\eta \in E^l_{A_k}$ and $f_\mu(F)$ for $|\mu| \leq l+k$ can meet only at a boundary point.

Lemma 3.6. *Suppose that $b_k \leq a < c_k$. Then we have $J(S_k(a)) = J_1(S(a))$.*

4. Proof of the theorems

Suppose that $b_k \leq a \leq c_k$. By Lemma 3.6, if $b_k \leq a < c_k$, then we have $J_1(S(a)) = J(S_k(a))$. If $a = c_k$, $J_1(S(a)) \subset J(S_k(a))$ and their difference $J(S_k(a)) \setminus J_1(S(a))$ consists of boundary points of $f_\nu(F)$ $(\nu \in \{0,1\}^*)$, and is countable. In either case, we have

$$\dim_H J_1(S(a)) = \dim_H J(S_k(a)).$$

Recall that we define the multigraph with the associated incidence matrix (V_k, E_k, A_k, i, t) in Section 3. First, we can show that

$$\dim_H J(S_k(a)) \leq -\frac{\log \lambda_k}{\log a}. \tag{8}$$

The GDMS satisfies the open set condition by Lemma 3.4. It also satisfies the bounded distortion property since all of the transformations are similarity transformations. Theorem 2.6 asserts that the equality would hold in (8) if A_k were irreducible.

Since the incidence matrix A_k is not irreducible, we modify the GDMS. The multigraph with the associated incidence matrix, (V'_k, E'_k, A'_k, i, t), is defined as follows. The vertex set is given by $V'_k = V_k \setminus \{0\cdots0, 1\cdots1\}$. The edges of E'_k are those of E_k not involving the vertices $0\cdots0, 1\cdots1$. The incidence matrix A'_k is the restriction of A_k to $E'_k \times E'_k$.

Given $\omega \in \{0,1\}^{k+1}$, the map ϕ_0 (resp. ϕ_1) shifts the digits of ω to the right and append 0 (resp. 1) to the left:

$$\phi_0(\omega_1 \ldots \omega_k \omega_{k+1}) = 0\omega_1 \ldots \omega_k$$
$$\phi_1(\omega_1 \ldots \omega_k \omega_{k+1}) = 1\omega_1 \ldots \omega_k$$

To see that the modified incidence matrix A'_k is irreducible, we show for any $p, q \in V'_k$, there exists a path from p to q within $E^*_{A'_k}$. Define $r_0, r_1 \in \{0,1\}$ by

$$r_0 \neq p_1, \qquad r_1 \neq q_{k+1}.$$

Then we have

$$\phi_{q_1} \cdots \phi_{q_{k+1}} \phi_{r_1} \phi_{r_0}(p) = q,$$

and for all $i = 1, \ldots, k$, we have

$$\phi_{q_i} \cdots \phi_{q_{k+1}} \phi_{r_1} \phi_{r_0}(p) \in V'_k.$$

This shows that A'_k is irreducible.

We have

$$\dim_H J(S'_k(a)) \leq \dim_H J(S_k(a)), \tag{9}$$

and

$$\dim_H J(S'_k(a)) = -\frac{\log \lambda'_k}{\log a}, \tag{10}$$

where λ'_k is the largest eigenvalue of A'_k.

The eigenvalues of A_k are $0, 1$, and the eigenvalues of A'_k. This can be seen by

$$\begin{aligned}
&\det(A_k - sE) \\
&= \det \left(\begin{array}{cc|ccccc|cc}
1-s & 1 & 0 & & \cdots & 0 & 0 & 0 & 0 \\
0 & -s & 1 & & 1 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & & & & & & 0 & 0 \\
\vdots & \vdots & & & A'_k - sE' & & & 0 & 0 \\
\vdots & \vdots & & & & & & \vdots & \vdots \\
0 & 0 & & & & & & 0 & 0 \\
\hline
0 & 0 & \cdots & & 0 & 1 & 1 & -s & 0 \\
0 & 0 & 0 & & \cdots & 0 & 0 & 1 & 1-s
\end{array}\right) \\
&= s^2(1-s)^2 \det(A'_k - sE').
\end{aligned}$$

Combining (8), (9), and (10), we obtain

$$-\frac{\log \lambda_k}{\log a} = -\frac{\log \lambda'_k}{\log a} = \dim_H J(S'_k(a)) \leq \dim_H J(S_k(a)) \leq -\frac{\log \lambda_k}{\log a}.$$

This completes the proof of Theorem 1.1.

The proof of Theorem 1.2 is as follows. Since all elements of $(A'_k)^{(k+3)}$ are greater than or equal to 1, for every integer m, we have

$$\operatorname{tr}(A'_k)^{(k+3)m} \geq (2^{k+2} - 6)^m.$$

Therefore, by the Perron-Frobenius theorem, we obtain

$$\begin{aligned}(k+3)\log\lambda'_k &= \lim_{m\to\infty}\frac{1}{m}\log\operatorname{tr}(A'_k)^{(k+3)m}\\ &\geq \log(2^{k+2}-6).\end{aligned}$$

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