Classification of generic random holomorphic dynamical systems associated with analytic families of rational maps

Hiroki Sumi
Graduate School of
Human and Environmental Studies
Kyoto University
Japan
E-mail: sumi@math.h.kyoto-u.ac.jp
http://www.math.h.kyoto-u.ac.jp/~sumi/index.html

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Definition 1.

(1) Let $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cong S^2$ be the Riemann sphere endowed with the spherical distance d.

(2) Let $\text{Rat} := \{f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid f$ is non-constant and holomorphic$\}$ endowed with the distance η, where $\eta(f,g) = \sup_{z \in \hat{\mathbb{C}}} d(f(z), g(z))$. Note that (Rat, η) is a complete separable metric space.

(3) For a metric space Y, we denote by $\mathfrak{M}_1(Y)$ the space of all Borel probability measures on Y.

(4) For a subset Y of Rat, we set

$$\mathfrak{M}_{1,c}(Y) := \{\tau \in \mathfrak{M}_1(Y) \mid \operatorname{supp} \tau \text{ is a compact subset of } Y\}.$$

(5) For a $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$, we set

$$G_\tau := \{\gamma_n \circ \cdots \circ \gamma_1 \mid n \in \mathbb{N}, \gamma_j \in \operatorname{supp} \tau (\forall j)\}.$$

Note that this is a semigroup whose product is the composition of maps.

(6) We say that an element $\tau \in \mathfrak{M}_{1,c}(\mathrm{Rat})$ is weakly mean stable if there exist an $n \in \mathbb{N}$, an $m \in \mathbb{N}$, non-empty open subsets $U_1, \ldots, U_m$ of $\hat{\mathbb{C}}$, a non-empty compact subset K of $\hat{\mathbb{C}}$ with $K \subset \cup_{j=1}^m U_j$, and a constant c with $0 < c < 1$ such that the following (a) (b) (c) hold.

(a) For each $(\gamma_1, \ldots, \gamma_n) \in (\operatorname{supp} \tau)^n$, we have

$$\gamma_n \circ \cdots \circ \gamma_1 (\cup_{j=1}^m U_j) \subset K.$$

Moreover, for each $j = 1, \ldots, m$, for all $x, y \in U_j$ and for each $(\gamma_1, \ldots, \gamma_n) \in (\operatorname{supp} \tau)^n$, we have

$$d(\gamma_n \circ \cdots \circ \gamma_1(x), \gamma_n \circ \cdots \circ \gamma_1(y)) \leq c d(x, y).$$

(b) Let $D_\tau := \bigcap_{h \in G_\tau} h^{-1}(\hat{\mathbb{C}} \setminus \cup_{j=1}^m U_j)$. Then $\sharp D_\tau < \infty$.

(c) For each minimal set L of τ with $L \subset D_\tau$, there exist a $z \in L$ and an $\alpha \in G_\tau$ such that $\alpha(z) = z$ and $|\alpha'(z)| > 1$(if $z = \infty$ then we consider the condition $|(\varphi \circ \alpha \circ \varphi^{-1})'(0)| > 1$ instead of the condition $|\alpha'(z)| > 1$, where $\varphi(z) = 1/z$).
Here, a non-empty compact subset L of $\hat{\mathbb{C}}$ is said to be a minimal set of τ if for each $z \in L$, $\overline{\cup_{h \in G_\tau} \{h(z)\}} = L$.

(7) For each $Y \subset \mathrm{Rat}$, we endow $\mathfrak{M}_{1,c}(Y)$ with the topology such that a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ in $\mathfrak{M}_{1,c}(Y)$ tends to an element $\tau \in \mathfrak{M}_{1,c}(Y)$ if and only if

($*$) for each bounded continuous function $\varphi : Y \to \mathbb{R}$, we have

$$\int_Y \varphi \, d\tau_n \to \int_Y \varphi \, d\tau \text{ as } n \to \infty,$$

and

(∗∗) $\operatorname{supp}\tau_n \to \operatorname{supp}\tau$ as $n \to \infty$ with respect to the Hausdorff metric in the space of all non-empty compact subsets of Y.

Theorem 2 ([4]). *Let Y be one of the following (1)–(4).*

(1) $\{f \in \mathrm{Rat} \mid f$ *is a polynomial with* $\deg(f) \geq 2\}$.

(2) $\{$"$z \mapsto \lambda z(1-z)$"$\in \mathrm{Rat} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$.

(3) $\{$"$z \mapsto z - \lambda \frac{f(z)}{f'(z)}$"$\in \mathrm{Rat} \mid \lambda \in \mathbb{C}, |\lambda - 1| < 1\}$
where f is a polynomial with $\deg(f) \geq 2$.

Remark: *This family is related to "random relaxed Newton's methods for f" in which we can find roots of any polynomial f more easily than deterministic Newton's method* ([4]).

(4) $\{$"$z \mapsto z + \lambda f(z)$"$\in \mathrm{Rat} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$
where f is a polynomial with $\deg(f) \geq 2$
such that for each $z_0 \in \mathbb{C}$ with $f(z_0) = 0$, we have $f'(z_0) \neq 0$.

Then, there exists an open and dense subset A of $\mathfrak{M}_{1,c}(Y)$ such that for each $\tau \in A$, we have the following (I)(II)(III).

(I) *τ is weakly mean stable.*

(II) *There exists $c_\tau < 0$ s.t. for all but countably many $z \in \hat{\mathbb{C}}$, for $(\otimes_{n=1}^{\infty}\tau)$-a.e. $(\gamma_1, \gamma_2, \ldots) \in Y^{\mathbb{N}}$, we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|D(\gamma_n \circ \cdots \circ \gamma_1)_z\| \leq c_\tau < 0.$$

(III) *For all but countably many $z \in \hat{\mathbb{C}}$, for $(\otimes_{n=1}^{\infty}\tau)$-a.e. $\gamma = (\gamma_1, \gamma_2, \ldots) \in Y^{\mathbb{N}}$, there exists a minimal set $L = L(z, \gamma)$ of τ which is either*

- (a) *"attracting for τ", or*
- (b) *included in D_τ with $\chi(\tau, L) < 0$, where $\chi(\tau, L)$ denotes the Lyapunov exponent of (τ, L),*

such that

$$d(\gamma_n \circ \cdots \circ \gamma_1(z), L) \to 0 \text{ as } n \to \infty.$$

Theorem 3. *Let* f *be a polynomial with* $\deg(f) \geq 2$ *and let* $Q := \{x \in \mathbb{C} \mid f(x) = 0\}$. *Suppose that* $f'(x) \neq 0$ $(\forall x \in Q)$. *Suppose also that for each* $a, b \in Q$ *with* $a \neq b$, *we have* $f'(a) \neq f'(b)$. *Let* $Y = \{$ "$z \mapsto z + \lambda f(z)$" $\in \mathrm{Rat} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$.
For each $\tau \in \mathfrak{M}_{1,c}(Y)$ *and* $x \in Q$, *let* $\chi(\tau, x) := \int_Y \log |h'(x)| \, d\tau(h)$.
Let A *be the set of elements* $\tau \in \mathfrak{M}_{1,c}(Y)$ *satisfying that*

- τ *is weakly mean stable with* $D_\tau \subset Q$,
- *for each* $x \in Q$, *we have* $\chi(\tau, x) \neq 0$, *and*
- *if* $x \in Q$ *and* $\chi(\tau, x) > 0$, *then for each* $h \in \operatorname{supp} \tau$, *we have* $h'(x) \neq 0$.

Then, *we have the following* (i)(ii).

(i) A *is open and dense in* $\mathfrak{M}_{1,c}(Y)$ *and statements* (II) *and* (III) *in Theorem 2 hold for each* $\tau \in A$.

(ii) *For any two subsets* Q_1, Q_2 *of* Q, *let* $A_{Q_1,Q_2} = \{\tau \in A \mid \chi(\tau, x) > 0 (\forall x \in Q_1)$ *and* $\chi(\tau, x) < 0 (\forall x \in Q_2)\}$. *Then* A_{Q_1,Q_2} *is a non-empty open subset of* A. *Moreover, we have* $A = \coprod_{(Q_1,Q_2)} A_{Q_1,Q_2}$ *(disjoint union).*

Theorem 4. *Let* $a, x_1, x_2 \in \mathbb{C}$ *with* $x_1 \neq x_2$.
Let $f(z) = a(z - x_1)(z - x_2)$.
Let $Y = \{$ "$z \mapsto z + \lambda f(z)$" $\in \mathrm{Rat} \mid \lambda \in \mathbb{C} \setminus \{0\}\}$.
Let A *be the set of elements* $\tau \in \mathfrak{M}_{1,c}(Y)$ *satisfying that*

- τ *is weakly mean stable with* $D_\tau \subset \{x_1, x_2\}$,
- *for each* $i = 1, 2$, *we have* $\chi(\tau, x_i) \neq 0$, *and*
- *if* $i \in \{1, 2\}$ *and* $\chi(\tau, x_i) > 0$, *then for each* $h \in \operatorname{supp} \tau$, *we have* $h'(x_i) \neq 0$.

Then, *we have the following* (i)–(iii).

(i) A *is open and dense in* $\mathfrak{M}_{1,c}(Y)$ *and statements* (II) *and* (III) *in Theorem 2 hold for each* $\tau \in A$.

(ii) *For any* $\tau \in A$, *we have* $D_\tau \neq \emptyset$.

(iii) *For each* $\gamma = (\gamma_1, \gamma_2, \ldots,) \in Y^{\mathbb{N}}$, *let* F_γ *be the set of pointz* $z \in \hat{\mathbb{C}}$ *satisfying that there exists a neighborhood* U *of* z *in* $\hat{\mathbb{C}}$ *such that* $\{\gamma_n \circ \cdots \circ \gamma_1\}_{n=1}^{\infty}$ *is equicontinuous on* U. Then *for* $(\otimes_{n=1}^{\infty}\tau)$-*a.e.* γ, *we have* $Leb_2(\hat{\mathbb{C}} \setminus F_\gamma) = 0$ *(*Leb_2 *denotes the Lebesgue meas. on* $\hat{\mathbb{C}}$*).*

Theorem 5. *Let* f, Y, A, F_γ *be as in Theorem 4.*
For each $\tau \in \mathfrak{M}_{1,c}(Y)$, *let* $\mathrm{Min}(\tau)$ *be the set of all minimal sets of* τ, *and we set* $\chi(\tau, x_i) := \int_Y \log|h'(x_i)|\, d\tau(h) \quad (i = 1, 2)$.
Then, *there exist non-empty open subsets* $A_1, A_2, \ldots, A_5$ *of* A *with* $A = \coprod_{i=1}^{5} A_i$ *(disjoint union) such that all of the following (1)–(5) hold.*

(1) *Let* $\tau \in A_1$. *Then we have the following (i)–(iv).*

(i) $\mathrm{Min}(\tau) = \{\{x_1\}, \{x_2\}, \{\infty\}, L_\tau\}$, *where* L_τ *is an "attracting minimal set" of* τ *with* $L_\tau \subset \mathbb{C} \setminus \{x_1, x_2\}$.

(ii) *For each* $i = 1, 2$ *and each* $h \in supp\,\tau$, *we have* $|h'(x_i)| > 1$ *and* $D_\tau = \{x_1, x_2\}$.

(iii) *For all but countably many* $z \in \hat{\mathbb{C}}$, *for* $(\otimes_{n=1}^{\infty}\tau)$ -*a.e.* $\gamma = (\gamma_1, \gamma_2, \ldots) \in Y^{\mathbb{N}}$, *we have*

$$d(\gamma_n \circ \cdots \circ \gamma_1(z), L_\tau \cup \{\infty\}) \to 0 \text{ as } n \to \infty.$$

(iv) *For each* $\gamma = (\gamma_1, \gamma_2, \ldots) \in (supp\,\tau)^{\mathbb{N}}$, *we have* $L_\tau \cup \{\infty\} \subset F_\gamma$ *and* $\{x_1, x_2\} \subset \hat{\mathbb{C}} \setminus F_\gamma$.

(2) *Let* $\tau \in A_2$. *Then we have the following (i)–(v) .*

(i) $\mathrm{Min}(\tau) = \{\{x_1\}, \{x_2\}, \{\infty\}\}$.

(ii) *For each* $i = 1, 2$, *we have* $\chi(\tau, x_i) > 0$.

(iii) $D_\tau = \{x_1, x_2\}$.

(iv) *For all but countably many* $z \in \hat{\mathbb{C}}$, *for* $(\otimes_{n=1}^{\infty}\tau)$ -*a.e.* $\gamma = (\gamma_1, \gamma_2, \ldots) \in Y^{\mathbb{N}}$, *we have*

$$\gamma_n \circ \cdots \circ \gamma_1(z) \to \infty \text{ as } n \to \infty.$$

(v) *For* $(\otimes_{n=1}^{\infty}\tau)$*-a.e.* $\gamma = (\gamma_1, \gamma_2, \ldots) \in Y^{\mathbb{N}}$, *we have*

$$\infty \in F_\gamma \quad \text{and} \quad \{x_1, x_2\} \subset \hat{\mathbb{C}} \setminus F_\gamma.$$

(3) *Let* $\tau \in A_3$*. Then we have the following* (i)–(v).

(i) $\mathrm{Min}(\tau) = \{\{x_1\}, \{x_2\}, \{\infty\}\}$.

(ii) $\chi(\tau, x_1) < 0$ *and* $\chi(\tau, x_2) > 0$.

(iii) *Let* $A_{3,a} := \{\tau \in A_3 \mid D_\tau = \{x_2\}\}$ *and* $A_{3,b} = \{\tau \in A_3 \mid D_\tau = \{x_1, x_2\}\}$. *Then* $A_{3,a}, A_{3,b}$ *are non-empty open subsets of* A_3 *and*

$$A_3 = A_{3,a} \coprod A_{3,b} \ (\text{disjoint union}).$$

(iv) *For all but countably many* $z \in \hat{\mathbb{C}}$, *for* $(\otimes_{n=1}^{\infty}\tau)$ *-a.e.* $\gamma = (\gamma_1, \gamma_2, \ldots) \in Y^{\mathbb{N}}$, *we have*

$$d(\gamma_n \circ \cdots \circ \gamma_1(z), \{x_1, \infty\}) \to 0 \ \text{as}\ n \to \infty.$$

(v) *For* $(\otimes_{n=1}^{\infty}\tau)$*-a.e.* $\gamma \in Y^{\mathbb{N}}$, *we have* $\{x_1, \infty\} \subset F_\gamma$ *and* $x_2 \in \hat{\mathbb{C}} \setminus F_\gamma$.

(4) *Let* $\tau \in A_4$*. Then we have the following* (i)–(v).

(i) $\mathrm{Min}(\tau) = \{\{x_1\}, \{x_2\}, \{\infty\}\}$.

(ii) $\chi(\tau, x_1) > 0$ *and* $\chi(\tau, x_2) < 0$.

(iii) *Let* $A_{4,a} := \{\tau \in A_4 \mid D_\tau = \{x_1\}\}$ *and* $A_{4,b} = \{\tau \in A_4 \mid D_\tau = \{x_1, x_2\}\}$. *Then* $A_{4,a}, A_{4,b}$ *are non-empty open subsets of* A_4 *and*

$$A_4 = A_{4,a} \coprod A_{4,b} \ (\text{disjoint union}).$$

(iv) *For all but countably many* $z \in \hat{\mathbb{C}}$, *for* $(\otimes_{n=1}^{\infty}\tau)$ *-a.e.* $\gamma = (\gamma_1, \gamma_2, \ldots) \in Y^{\mathbb{N}}$, *we have*

$$d(\gamma_n \circ \cdots \circ \gamma_1(z), \{x_2, \infty\}) \to 0 \ \text{as}\ n \to \infty.$$

(v) *For* $(\otimes_{n=1}^{\infty}\tau)$-*a.e.* $\gamma \in Y^{\mathbb{N}}$, *we have* $\{x_2, \infty\} \subset F_\gamma$ *and* $x_1 \in \hat{\mathbb{C}} \setminus F_\gamma$.

(5) *Let* $\tau \in A_5$. *Then we have the following* (i)–(vi).

(i) $\mathrm{Min}(\tau) = \{\{x_1\}, \{x_2\}, \{\infty\}\}$.

(ii) *For each* $i = 1, 2$, *we have* $\chi(\tau, x_i) < 0$.

(iii) $D_\tau = \{x_1, x_2\}$.

(iv) *For* **each** $z \in \hat{\mathbb{C}}$,
for $(\otimes_{n=1}^{\infty}\tau)$ -*a.e.* $\gamma = (\gamma_1, \gamma_2, \ldots) \in Y^{\mathbb{N}}$, *we have*

$$d(\gamma_n \circ \cdots \circ \gamma_1(z), \{x_1, x_2, \infty\}) \to 0 \text{ as } n \to \infty.$$

(v) *For* $(\otimes_{n=1}^{\infty}\tau)$-*a.e.* $\gamma \in Y^{\mathbb{N}}$, *we have that* $\{x_1, x_2, \infty\} \subset F_\gamma$.

(vi) *For* $(\otimes_{n=1}^{\infty}\tau)$-*a.e.* $\gamma \in Y^{\mathbb{N}}$, *for each* $i = 1, 2$, *for each point* z *in the connected component* U_i *of* F_γ *with* $x_i \in U_i$, *we have*

$$\gamma_n \circ \cdots \circ \gamma_1(z) \to x_i \quad \text{as } n \to \infty.$$

Remark. *For any deterministic iteration dynamics of a single quadratic map* f, *we CANNOT have a phenomenon such as* (vi). *In fact, we CANNOT have two attracting minimal sets of* f *in* $\mathbb{C}$.

Remark 6. Statements of Theorems 2, 3, 4, 5 cannot hold for deterministic dynamics of a single $f \in \mathrm{Rat}$ with $\deg(f) \geq 2$.
In fact, in the Julia set $J(f)$ of f, we have a chaotic phenomenon.
See Mañé's paper (1988)[1] etc.

Thus Theorems 2, 3, 4, 5 describe randomness-induced phenomena (new phenomena in random dynamical systems which cannot hold for deterministic dynamical systems).

Idea of Proofs of Theorems 2, 3, 4, 5.

(1) We use complex analysis, Montel's theorem (a family of uniformly bounded holomorphic functions on a domain is equicontinuous on the domain), hyperbolic metric.

(2) We classify minimal sets and analyze the bifurcation of minimal sets. etc. By using these, enlarging the support of the original τ a little bit, we destroy non-attracting minimal sets which do not meet D_τ.

Summary

(1) We introduce the notion of weak mean stability in i.i.d. random (holomorphic) 1-dimensional dynamical systems.

(2) If a random holomorphic dynamical system on $\hat{\mathbb{C}}$ is weakly mean stable and satisfies some mild assumtions, then for all but countably many $z \in \hat{\mathbb{C}}$, for a.e. orbit starting with z, the Lyapunov exponent is negative. Note that this statement cannot hold for deterministic dynamics of a single holo. map f on $\hat{\mathbb{C}}$ with $\deg(f) \geq 2$.

(3) Given an analytic family Y of rational maps (with some mild conditions), generic random holomorphic dynamical systems (with multiplicative noise) of elements of Y are weakly mean stable. Also, we can classify such generic random holomorphic dynamical systems of elements of Y in terms of averaged behavior and quenched dynamics.

References:

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[2] H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, Proc. London Math. Soc. (2011) 102(1), pp 50–112.

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[4] H. Sumi, *Negativity of Lyapunov Exponents and Convergence of Generic Random Polynomial Dynamical Systems and Random Relaxed Newton's Methods*, 61 pages, https://arxiv.org/abs/1608.05230.
Some contents of this talk are included in this paper.