ON THE CONTINUITY OF LYAPUNOV EXPONENTS FOR SYSTEMS WITH ADDITIVE NOISE

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Abstract. In this short essay I show a small result on continuity of Lyapunov exponents for systems with additive noise.

1. Introduction

Noise induced phenomena are an important research theme in mathematics, applied mathematics and physics: Noise Induced Order [5], Stochastich Resonance and Anomalous Diffusion [1, 6] are some of the observed phenomena, both in models and in experiments.

In [2] we presented a computer assisted proof of Noise Induced Order for the Matsumoto-Tsuda model of the Belosouv-Zhabotinsky reaction, and we proved some continuity results for the Lyapunov exponent of the system.

In this paper I would like to present a self-contained proof of the Lipschitz continuity of the Lyapunov exponent when the annealed transfer operator is mixing; this is a generalization of the arguments in [2].

This proof is quite general and the only hypothesis are the regularity of the noise kernel and the fact that for some noise size the system is mixing; another condition for the proof to work is that the weak norm has to be L^1, but this is pretty natural. I will present it for one-dimensional systems and Bounded Variation noise but the proof can be easily extended to higher dimension and more regular kernels.

2. Statement of the results

Let $T : [0,1] \to [0,1]$, I want to study the "Lyapunov exponent" of the system where I replace T^n by $(T + \xi_n) \circ (T + \xi_{n-1}) \circ \ldots \circ (T + \xi_1)$, where $(\xi_n, \ldots, \xi_1)$ are chosen in the interval $[-\epsilon, \epsilon]$ using a probability distribution with density function ρ_ϵ.

The noise may take orbits outside the interval $[0,1]$; this may reflect on the behaviour of the annealed operator of the random dynamical system therefore I suppose that we have **periodic boundary conditions**, i.e., as in [4], if $f \in L^1([0,1])$, letting $\hat{f}$ be its periodic extension, i.e.,

$$\hat{f}(x+k) = f(x) \quad k \in \mathbb{Z}$$

I denote by abuse of notation, for $x \in [0,1]$

$$\rho_\epsilon * f(x) := \rho_\epsilon * \hat{f}(x).$$

Let m be the Lebesgue measure on $[0,1]$; for each ϵ the behaviour of the system with noise can be described by a Markov chain, such that for each Borel subset E of $[0,1]$ we have that the probability of x entering E is

$$P_\epsilon(x, E) = \int_E \rho_\epsilon(y - T(x)) dm(y). \tag{1}$$

From here on, suppose that $\log(|T'|)$ is in $L^1(m)$.

Definition 1. *If T is a nonsingular Borel measurable dynamical system on X then it induces a linear operator on the space of signed Borel measures $L : SM(X) \to SM(X)$, defined on $\mu \in SM(X)$ by*

$$L\mu(A) = \mu(T^{-1}A),$$

*the **transfer operator**.*

Remark 1. It is worth observing that $L(\delta_x) = \delta_{T(x)}$; we can think of the transfer operator as the operator that embodies how mass is transferred around by the dynamics on X.

Definition 2. *Let ρ be a BV density function and*

$$\rho_\epsilon(x) = \frac{1}{\epsilon}\rho\left(\frac{x}{\epsilon}\right)$$

then, we call

$$L_\epsilon = \rho_\epsilon * L,$$

*the **annealed transfer operator** associated to the random dynamical system with noise size ϵ. The annealed transfer operator is the operator that encodes how densities evolve under the action of the Markov chain* (1).

Remark 2. The annealed transfer operator gives us information on the behaviour of the system averaged over all possible noise realizations. We can observe this by duality, let $\phi \in L^\infty([0,1])$

$$< \phi, L_\epsilon \delta_x >= \int \phi(T(x) - \xi) d\rho_\epsilon(\xi).$$

We are taking the average (weighted by ρ_ϵ) of ϕ over all the possible images of the point x.

If ρ is in BV the (annealed) random dynamical system with noise of size ϵ has (at least) one stationary measure with density in BV (Lemma 1). For the moment being, suppose that for each ϵ the operator L_ϵ has a unique fixed point f_ϵ. The object of study of this essay is the well definedness and continuity of the following function.

Definition 3. *The function*

$$\lambda(\epsilon) = \int_{-1}^{1} \log(|T'(x)|) f_\epsilon dm.$$

is the "Lyapunov exponent in function of the noise".

Definition 4. *In the following we will denote by V_0 the space of average* 0 *signed measures. By abuse of notation we also denote by V_0 the intersection of average* 0 *signed measures with more regular Banach spaces as L_1 and BV (this will be clear from context).*

I will show the following result.

Theorem 1. *Let L_{ϵ_0} be the annealed transfer operator associated to the dynamical system T with additive noise with kernel ρ_{ϵ_0}, for a fixed noise size ϵ_0. Suppose*

$$||L^n_{\epsilon_0}|_{V_0}||_{BV} \leq C_0 \eta_0^n,$$

with $\eta_0 < 1$, $C_0 > 0$.

Then, for all $\epsilon > \epsilon_0$ there exist C_ϵ and η_ϵ such that

$$||L^n_{\epsilon}|_{V_0}||_{BV} \leq C_\epsilon \eta_\epsilon^n.$$

This implies that the stationary measure f_ϵ is unique for $\epsilon > \epsilon_0$; moreover, the stationary measure f_ϵ varies continuously (in BV norm) with respect to the noise *size, which in turn implies that the function $\lambda(\epsilon)$ is well-defined and locally Lipschitz continuous for all $\epsilon > \epsilon_0$.*

3. Well definedness and continuity of $\lambda(\epsilon)$

This is a simple inequality, where the regularization properties of convolution have strong consequences on the regularity of the fixed points of the operator L_ϵ. The same inequality works for any "regularity" seminorm, i.e., if the noise is of regularity C^k (or even analytic), the fixed point of the annealed transfer operator has the same regularity.

In the following section I show how one of the consequences of the uniform correlation decay in BV norm is that the operator L_ϵ has a unique stationary measure. This shows that the function $\lambda(\epsilon)$ is well defined.

3.1. **Regularity of the stationary measure.**

We state this results with respect to the variation seminorm Var(.); this lemma easily generalizes to higher regularity.

Lemma 1. *Suppose $||\rho_\epsilon||_{BV} < \infty$. Then L_ϵ preserves BV and if f_ϵ is a stationary density in L^1, then f_ϵ is in BV and*

$$||f_\epsilon||_{BV} \leq ||\rho_\epsilon||_{BV}$$

Proof. Both statements follows from the properties of convolution, for any $g \in L^1$ we have that:

$$||L_\epsilon g||_{BV} \leq ||\rho_\epsilon||_{BV}||g||_{L^1},$$

i.e. $||L_\epsilon||_{L^1 \to BV} \leq ||\rho_\epsilon||_{BV}$ which in turn implies that

$$||L_\epsilon||_{BV} \leq ||\rho_\epsilon||_{BV}.$$

If f_ϵ is a fixed point then

$$||f_\epsilon||_{BV} = ||L_\epsilon f_\epsilon||_{BV} \leq ||\rho_\epsilon||_{BV}||f_\epsilon||_{L^1} \leq ||\rho_\epsilon||_{BV}.$$

□

Remark 3. Indeed, a stronger result is true, i.e., if $\mathcal{M}$ is the space of signed measures endowed with the Wasserstein norm [2], we have that L_ϵ is a continuous operator from $\mathcal{M}$ to BV, i.e., any stationary measure is in BV and has a density.

3.2. **Decay of correlations in L^1 and BV for L_ϵ.**

Lemma 2. *Suppose L_ϵ has exponential decay of correlations with respect to bounded variation observables. Then, there exists N such that*

$$||L_\epsilon^N|_{V_0}||_{L^1} \leq \frac{1}{2}.$$

Proof. Decay of correlations for bounded variations observables guarantees that

$$||L_\epsilon^n|_{V_0}||_{BV} \leq C\theta^n.$$

Observing that $||L_\epsilon^n f||_{BV} \leq ||\rho_\epsilon||_{BV}||f||_{L^1}$ we have that

$$||L_\epsilon^{n+1}|_{V_0}||_{L^1} \leq \left(\frac{C \cdot \mathrm{Var}(\rho_\epsilon)}{\theta}\right)\theta^{n+1},$$

by choosing $N > (-1 - \log_2(C) - \log_2(\mathrm{Var}(\rho_\epsilon)))/\log_2(\theta)$ we have the thesis. □

The following Lemma is a generalization of a result in [2]: if for some noise amplitude the operator contracts L^1, then for all bigger amplitudes the annealed operators also contracts L^1.

Lemma 3. *Let* $\rho_\epsilon = \epsilon^{-1}\rho(x/\epsilon)$ *and* N_ϵ *the associated noise operator. If* $\epsilon < \hat{\epsilon}$ *and* $||(N_\epsilon L)^i||_{V_0 \to L^1} \leq C_i < 1$*, then*

$$||(N_{\hat{\epsilon}} L)^i||_{V_0 \to L^1} \leq C_i(\epsilon/\hat{\epsilon})^i + \left[1 - (\epsilon/\hat{\epsilon})^i\right] < 1. \tag{2}$$

Proof. We observe that

$$\frac{1}{\epsilon+\epsilon}\rho\left(\frac{x}{\epsilon+\epsilon}\right) = \frac{\epsilon}{\epsilon+\epsilon}\rho\left(\frac{x}{\epsilon}\right) + \frac{\epsilon}{\epsilon+\epsilon}\frac{1}{\epsilon}\left(\rho\left(\frac{x}{\epsilon+\epsilon}\right) - \rho\left(\frac{x}{\epsilon}\right)\right);$$

moreover, the convolution operator

$$Mf = \frac{1}{\epsilon}\left(\rho\left(\frac{x}{\epsilon+\epsilon}\right) - \rho\left(\frac{x}{\epsilon}\right)\right) * f$$

is a Markov operator.

Therefore

$$N_{\hat{\epsilon}} Lf = \frac{\epsilon}{\hat{\epsilon}} N_\epsilon Lf + \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right) MLf,$$

and

$$(N_{\hat{\epsilon}} L)^i f = \left(\frac{\epsilon}{\hat{\epsilon}}\right)^i (N_\epsilon L)^i f + \left(1 - \left(\frac{\epsilon}{\hat{\epsilon}}\right)^i\right) Qf$$

for a suitable Markov operator Q. Taking the L^1 norm, the result follows. □

This result is strong, but it has even stronger consequences, proved in the next lemma, i.e., if we see L^1 contraction for a noise amplitude ϵ we have BV contraction for all bigger noise amplitudes.

Lemma 4. *Suppose* $||L_\epsilon^n|_{V_0}||_{L^1} \leq C\theta^n$*; then*

$$||L_\epsilon^n|_{V_0}||_{BV} \leq \left(\frac{C||\rho_\epsilon||_{BV}}{\theta}\right)\theta^n.$$

Proof. Let $f \in V_0$, the proof is a chain of inequalities,

$$\begin{aligned} ||L_\epsilon^{n+1} f||_{BV} &\leq ||\rho_\epsilon||_{BV}||L||_{L^1}||L_\epsilon^n f||_{L^1} \\ &\leq ||\rho_\epsilon||_{BV} C\theta^n ||f||_{L^1} \\ &\leq ||\rho_\epsilon||_{BV} C\theta^n ||f||_{BV}. \end{aligned}$$

□

3.3. **Unicity of the stationary measure and well-definedness of** $\lambda(\epsilon)$**.** From the results on the decay of correlation, we obtain the unicity of the stationary measure and therefore the fact that $\lambda(\epsilon)$ is well-defined.

Corollary 1. *Let* $T : [0,1] \to [0,1]$ *be a dynamical system such that there exists* ϵ_0 *such that for all* $\epsilon \in [0, \epsilon_0)$ *the operators* L_ϵ *have a uniform bound on the decay of correlations in* BV*, then the function* $\lambda(\epsilon)$ *is well defined.*

Proof. For all ϵ any stationary measure has density f_ϵ in $BV \subset L^1$. Due to the lemmas above, for all ϵ there exists an N such that $||L_\epsilon^N|_{V_0}||_{L^1} < 1/2$. By contradiction, suppose there are two stationary densities, f_ϵ and g_ϵ, then

$$||f_\epsilon - g_\epsilon||_{L^1} = ||L_\epsilon^N(f_\epsilon - g_\epsilon)||_{L^1} \leq \frac{1}{2}||f_\epsilon - g_\epsilon||_{L^1},$$

which implies that f_ϵ coincides with g_ϵ. □

3.4. **Continuity in BV of the stationary measure with respect to the noise size.** In this subsection we prove the continuity of the stationary measure with respect to the noise size at a fixed noise size $\epsilon > 0$.

Corollary 2. *Let $\rho_\epsilon = \epsilon^{-1}\rho(x/\epsilon)$ and N_ϵ the associated noise operator. Then, for $\hat{\epsilon} > \epsilon > 0$ we have that*

$$||L_{\hat{\epsilon}} - L_\epsilon||_{L^1} \leq 2\left(1 - \frac{\epsilon}{\hat{\epsilon}}\right)$$

and that

$$||L_{\hat{\epsilon}} - L_\epsilon||_{L^1\to BV} \leq 3\left(1 - \frac{\epsilon}{\hat{\epsilon}}\right)||\rho_\epsilon||_{BV}$$

Proof. From the proof of Lemma 3 we have that

$$L_{\hat{\epsilon}}f - \frac{\epsilon}{\hat{\epsilon}}L_\epsilon f = \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right)MLf,$$

where M is a Markov operator. Then

$$||L_{\hat{\epsilon}} - L_\epsilon||_{L^1} \leq ||L_\epsilon - \frac{\epsilon}{\hat{\epsilon}}L_\epsilon||_{L^1} + ||L_{\hat{\epsilon}} - \frac{\epsilon}{\hat{\epsilon}}L_\epsilon||_{L^1} \leq 2\left(1 - \frac{\epsilon}{\hat{\epsilon}}\right).$$

Similarly

$$\begin{aligned}||L_{\hat{\epsilon}} - L_\epsilon||_{L^1\to BV} &\leq ||L_\epsilon - \frac{\epsilon}{\hat{\epsilon}}L_\epsilon||_{L^1\to BV} + ||L_{\hat{\epsilon}} - \frac{\epsilon}{\hat{\epsilon}}L_\epsilon||_{L^1\to BV}\\ &\leq 3\left(1 - \frac{\epsilon}{\hat{\epsilon}}\right)||\rho_\epsilon||_{BV},\end{aligned}$$

since

$$\frac{1}{\epsilon}||\rho\left(\frac{x}{\hat{\epsilon}}\right) - \rho\left(\frac{x}{\epsilon}\right)||_{BV} \leq 2||\rho_\epsilon||_{BV}.$$

□

Now, we want to estimate the distance between the fixed points of L_ϵ and $L_{\hat{\epsilon}}$. This proofs follows the line of Theorem 1 in [3].

Theorem 2. *Let L_ϵ and $L_{\hat{\epsilon}}$ be the annealed operators for the noise of size ϵ and $\hat{\epsilon}$ respectively. Suppose moreover that $||L_\epsilon^N|_{V_0}||_{BV} \leq C\theta_\epsilon^N$ for $\theta_\epsilon < 1$. Suppose f_ϵ and $f_{\hat{\epsilon}}$ are the unique fixed points of L_ϵ and $L_{\hat{\epsilon}}$ respectively. Then*

$$||f_{\hat{\epsilon}} - f_\epsilon||_{BV} \leq \frac{C}{1-\theta_\epsilon}3\left(1 - \frac{\epsilon}{\hat{\epsilon}}\right)||\rho_\epsilon||_{BV}.$$

Proof. Let N be such that $C\theta_\epsilon^N < 1/2$, then

$$\begin{aligned}||f_\epsilon - f_{\hat{\epsilon}}||_{BV} &= ||L_\epsilon^N f_\epsilon - L_{\hat{\epsilon}}^N f_{\hat{\epsilon}}||_{BV}\\ &\leq ||L_\epsilon^N f_\epsilon - L_\epsilon^N f_{\hat{\epsilon}}||_{BV} + ||L_\epsilon^N f_{\hat{\epsilon}} - L_{\hat{\epsilon}}^N f_{\hat{\epsilon}}||_{BV},\end{aligned}$$

which implies that

$$\begin{aligned}||f_\epsilon - f_{\hat{\epsilon}}||_{BV} &\leq 2||L_\epsilon^N f_{\hat{\epsilon}} - L_{\hat{\epsilon}}^N f_{\hat{\epsilon}}||_{BV}\\ &\leq \sum_{k=0}^{N-1}||L_\epsilon^k|_{V_0}||_{BV}||L_\epsilon - L_{\hat{\epsilon}}||_{L^1\to BV}||L^{N-k-1}f_{\hat{\epsilon}}||_{L^1}\\ &\leq \sum_{k=0}^{N-1}C\theta_\epsilon^k\cdot 3\left(1 - \frac{\epsilon}{\hat{\epsilon}}\right)||\rho_\epsilon||_{BV},\end{aligned}$$

which implies the thesis since $||f_{\hat{\epsilon}}||_{L^1} = 1$. □

4. Proof of Theorem 1

Proof. Let $\epsilon, \hat{\epsilon} > \epsilon_0$ then, by Theorem 2 we have

$$
\begin{aligned}
|\lambda(\epsilon) - \lambda(\hat{\epsilon})| &= \int \log(|T'|)(f_\epsilon - f_{\hat{\epsilon}})dm \\
&\leq || \log(|T'|)||_{L^1} ||f_\epsilon - f_{\hat{\epsilon}}||_{BV} \\
&\leq || \log(|T'|)||_{L^1} \frac{C}{1-\theta_\epsilon} 3 \left(1 - \frac{\epsilon}{\hat{\epsilon}}\right) ||\rho_\epsilon||_{BV}.
\end{aligned}
$$

□

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