The Laplacian on some self-conformal fractals and Weyl's asymptotics for its eigenvalues: A survey of the ergodic-theoretic aspects

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Abstract

This short survey is aimed at sketching the ergodic-theoretic aspects of the author's recent studies on Weyl's eigenvalue asymptotics for a "geometrically canonical" Laplacian defined by the author on some self-conformal circle packing fractals including the classical *Apollonian gasket*. The main result being surveyed is obtained by applying Kesten's renewal theorem [*Ann. Probab.* **2** (1974), 355–386, Theorem 2] for functionals of Markov chains on general state spaces and provides an alternative probabilistic proof of the result by Oh and Shah [*Invent. Math.* **187** (2012), 1–35, Corollary 1.8] on the asymptotic distribution of the circles in the Apollonian gasket.

1 Introduction

This short survey concerns the author's recent studies in [5, 6, 7, 8] on Weyl's eigenvalue asymptotics for a "geometrically canonical" Laplacian defined by the author on circle packing fractals which are invariant with respect to certain Kleinian groups (i.e., discrete groups of Möbius transformations on $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$), including the classical *Apollonian gasket* (Figure 1). Here we focus on sketching the ergodic-theoretic aspects of the proof of the eigenvalue asymptotics, which in fact serves as an alternative proof, though limited to the cases of certain specific Kleinian groups, of the result by Oh and Shah in [13] on the asymptotic distribution of the circles in circle packing fractals invariant with respect to a very large class of Kleinian groups; see [9] for a survey of the analytic aspects of the author's "geometrically canonical" Laplacian. Also we restrict our attention to the case of the Apollonian gasket for simplicity of the presentation.

Notation. (1) We adopt the convention that $\mathbb{N} := \{n \in \mathbb{Z} \mid n > 0\}$, i.e., $0 \notin \mathbb{N}$.
(2) The cardinality (the number of elements) of a set A is denoted by $\#A$.
(3) The closure and boundary of $A \subset \mathbb{C}$ in $\mathbb{C}$ are denoted by $\overline{A}$ and ∂A, respectively.

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(4) Let $n \in \mathbb{N}$. The Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$ and the operator norm of a real $n \times n$ matrix M with respect to $|\cdot|$ is denoted by $\|M\|$.

2 The Apollonian gasket and its fractal geometry

In this section, we introduce the Apollonian gasket and state its geometric properties needed for our purpose. The following definition and proposition form the basis of the construction and further detailed studies of the Apollonian gasket.

Definition 2.1 (tangential disk triple, ideal triangle)**.** (0) We set $S := \{1, 2, 3\}$.

(1) Let $D_1, D_2, D_3 \subset \mathbb{C}$ be either three open disks or two open disks and an open half-plane. The triple $\mathcal{D} := (D_1, D_2, D_3)$ of such open subsets of $\mathbb{C}$ is called a *tangential disk triple* if and only if $\#(\overline{D_j} \cap \overline{D_k}) = 1$ (i.e., D_j and D_k are *externally* tangent) for any $j, k \in S$ with $j \neq k$.

(2) Let $\mathcal{D} = (D_1, D_2, D_3)$ be a tangential disk triple. The open subset $\mathbb{C} \setminus \bigcup_{j\in S} \overline{D_j}$ of $\mathbb{C}$ is then easily seen to have a unique bounded connected component, which is denoted by $T(\mathcal{D})$ and called the *ideal triangle* associated with $\mathcal{D}$.

(3) A tangential disk triple $\mathcal{D} = (D_1, D_2, D_3)$ is called *positively oriented* if and only if its associated ideal triangle $T(\mathcal{D})$ is to the left of $\partial T(\mathcal{D})$ when $\partial T(\mathcal{D})$ is oriented so as to have $\{q_j(\mathcal{D})\}_{j=1}^3$ in this order, where $\{q_j(\mathcal{D})\} := \overline{D_k} \cap \overline{D_l}$ for $\{j, k, l\} = S$.

Finally, we define $\mathsf{TDT}^+ := \{\mathcal{D} \mid \mathcal{D}$ is a positively oriented tangential disk triple$\}$ and $\mathsf{TDT}^{\oplus} := \{\mathcal{D} \mid \mathcal{D} = (D_1, D_2, D_3) \in \mathsf{TDT}^+, D_1, D_2, D_3$ are disks$\}$.

The following proposition is classical and can be shown by some elementary (though lengthy) Euclidean-geometric arguments. We set $\mathrm{rad}(D) := r$ and $\mathrm{curv}(D) := r^{-1}$ for an open disk $D \subset \mathbb{C}$ of radius r and $\mathrm{curv}(D) := 0$ for an open half-plane $D \subset \mathbb{C}$.

Proposition 2.2. *Let $\mathcal{D} = (D_1, D_2, D_3) \in \mathsf{TDT}^+$ and set $\alpha := \mathrm{curv}(D_1)$, $\beta := \mathrm{curv}(D_2)$, $\gamma := \mathrm{curv}(D_3)$ and $\kappa := \kappa(\mathcal{D}) := \sqrt{\beta\gamma + \gamma\alpha + \alpha\beta}$.*

(1) *Let $D_{\mathrm{cir}}(\mathcal{D}) \subset \mathbb{C}$ denote the* circumscribed disk *of $T(\mathcal{D})$, i.e., the unique open disk with $\{q_1(\mathcal{D}), q_2(\mathcal{D}), q_3(\mathcal{D})\} \subset \partial D_{\mathrm{cir}}(\mathcal{D})$. Then $\partial D_{\mathrm{cir}}(\mathcal{D})$ is perpendicular to ∂D_j for any $j \in S$, $\overline{T(\mathcal{D})} \setminus \{q_1(\mathcal{D}), q_2(\mathcal{D}), q_3(\mathcal{D})\} \subset D_{\mathrm{cir}}(\mathcal{D})$, and $\mathrm{curv}(D_{\mathrm{cir}}(\mathcal{D})) = \kappa$.*

(2) *There exists a unique* inscribed disk *$D_{\mathrm{in}}(\mathcal{D})$ of $T(\mathcal{D})$, i.e., a unique open disk $D_{\mathrm{in}}(\mathcal{D}) \subset \mathbb{C}$ such that $D_{\mathrm{in}}(\mathcal{D}) \subset T(\mathcal{D})$ and $\#(\overline{D_{\mathrm{in}}(\mathcal{D})} \cap \overline{D_j}) = 1$ for any $j \in S$. Moreover, $\mathrm{curv}(D_{\mathrm{in}}(\mathcal{D})) = \alpha + \beta + \gamma + 2\kappa$.*

The following notation is standard in studying self-similar sets.

Definition 2.3. (1) We set $W_0 := \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, $W_m := S^m$ for $m \in \mathbb{N}$ and $W_* := \bigcup_{m\in\mathbb{N}\cup\{0\}} W_m$. For $w \in W_*$, the unique $m \in \mathbb{N} \cup \{0\}$ satisfying $w \in W_m$ is denoted by $|w|$ and called the *length* of w.

(2) Let $w, v \in W_*$, $w = w_1 \dots w_m$, $v = v_1 \dots v_n$. We define $wv \in W_*$ by $wv := w_1 \dots w_m v_1 \dots v_n$ ($w\emptyset := w$, $\emptyset v := v$). We also define $w^1 \dots w^k$ for $k \geq 3$ and $w^1, \dots, w^k \in W_*$ inductively by $w^1 \dots w^k := (w^1 \dots w^{k-1})w^k$. For $w \in W_*$ and $n \in \mathbb{N} \cup \{0\}$ we set $w^n := w \dots w \in W_{n|w|}$. We write $w \leq v$ if and only if $w = v\tau$ for some $\tau \in W_*$, and write $w \not\leq v$ if and only if neither $w \leq v$ nor $v \leq w$ holds.

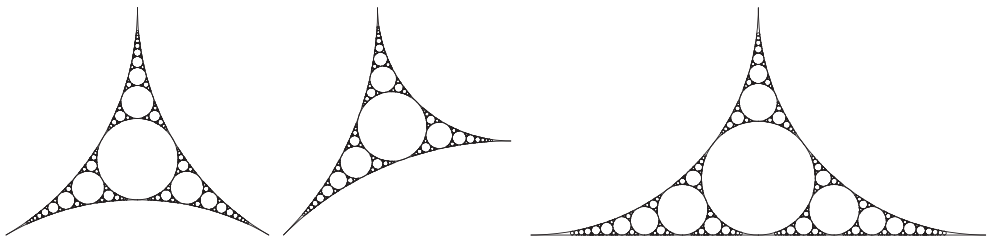

Figure 1: Apollonian gasket

Proposition 2.2-(2) enables us to define natural "contraction maps" $\Phi_w : \mathsf{TDT}^+ \to \mathsf{TDT}^+$ for each $w \in W_*$, which in turn is used to define the Apollonian gasket $K(\mathcal{D})$ associated with $\mathcal{D} \in \mathsf{TDT}^+$, as follows.

Definition 2.4. We define $\Phi_1, \Phi_2, \Phi_3 : \mathsf{TDT}^+ \to \mathsf{TDT}^+$ by $\Phi_1(\mathcal{D}) := (D_{\mathrm{in}}(\mathcal{D}), D_2, D_3)$, $\Phi_2(\mathcal{D}) := (D_1, D_{\mathrm{in}}(\mathcal{D}), D_3)$ and $\Phi_3(\mathcal{D}) := (D_1, D_2, D_{\mathrm{in}}(\mathcal{D}))$. We also set $\Phi_w := \Phi_{w_m} \circ \cdots \circ \Phi_{w_1}$ ($\Phi_\emptyset := \mathrm{id}_{\mathsf{TDT}^+}$) and $\mathcal{D}_w := \Phi_w(\mathcal{D})$ for $w = w_1 \dots w_m \in W_*$ and $\mathcal{D} \in \mathsf{TDT}^+$.

Definition 2.5 (Apollonian gasket)**.** Let $\mathcal{D} \in \mathsf{TDT}^+$. We define the *Apollonian gasket* $K(\mathcal{D})$ associated with $\mathcal{D}$ (see Figure 1) by

$$K(\mathcal{D}) := \overline{T(\mathcal{D})} \setminus \bigcup\nolimits_{w \in W_*} D_{\mathrm{in}}(\mathcal{D}_w) = \bigcap\nolimits_{m \in \mathbb{N}} \bigcup\nolimits_{w \in W_m} \overline{T(\mathcal{D}_w)}. \tag{2.1}$$

The curvatures of the disks involved in (2.1) admit the following simple expression.

Definition 2.6. We define 4×4 real matrices M_1, M_2, M_3 by

$$M_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad M_3 := \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix} \tag{2.2}$$

and set $M_w := M_{w_1} \cdots M_{w_m}$ for each $w = w_1 \dots w_m \in W_*$ ($M_\emptyset := \mathrm{id}_{4\times 4}$). Note that then for any $n \in \mathbb{N} \cup \{0\}$ we easily obtain

$$M_{1^n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ n^2 & 1 & 0 & n \\ n^2 & 0 & 1 & n \\ 2n & 0 & 0 & 1 \end{pmatrix}, \quad M_{2^n} = \begin{pmatrix} 1 & n^2 & 0 & n \\ 0 & 1 & 0 & 0 \\ 0 & n^2 & 1 & n \\ 0 & 2n & 0 & 1 \end{pmatrix}, \quad M_{3^n} = \begin{pmatrix} 1 & 0 & n^2 & n \\ 0 & 1 & n^2 & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2n & 1 \end{pmatrix}, \tag{2.3}$$

$$\begin{aligned} M_{2^n 3} &= \begin{pmatrix} 1 & n^2 & (n+1)^2 & n^2+n+1 \\ 0 & 1 & 1 & 1 \\ 0 & n^2 & (n+1)^2 & n(n+1) \\ 0 & 2n & 2(n+1) & 2n+1 \end{pmatrix}, \quad M_{3^n 2} = \begin{pmatrix} 1 & (n+1)^2 & n^2 & n^2+n+1 \\ 0 & (n+1)^2 & n^2 & n(n+1) \\ 0 & 1 & 1 & 1 \\ 0 & 2(n+1) & 2n & 2n+1 \end{pmatrix}, \\ M_{3^n 1} &= \begin{pmatrix} (n+1)^2 & 0 & n^2 & n(n+1) \\ (n+1)^2 & 1 & n^2 & n^2+n+1 \\ 1 & 0 & 1 & 1 \\ 2(n+1) & 0 & 2n & 2n+1 \end{pmatrix}, \quad M_{1^n 3} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ n^2 & 1 & (n+1)^2 & n^2+n+1 \\ n^2 & 0 & (n+1)^2 & n(n+1) \\ 2n & 0 & 2(n+1) & 2n+1 \end{pmatrix}, \\ M_{1^n 2} &= \begin{pmatrix} 1 & 1 & 0 & 1 \\ n^2 & (n+1)^2 & 0 & n(n+1) \\ n^2 & (n+1)^2 & 1 & n^2+n+1 \\ 2n & 2(n+1) & 0 & 2n+1 \end{pmatrix}, \quad M_{2^n 1} = \begin{pmatrix} (n+1)^2 & n^2 & 0 & n(n+1) \\ 1 & 1 & 0 & 1 \\ (n+1)^2 & n^2 & 1 & n^2+n+1 \\ 2(n+1) & 2n & 0 & 2n+1 \end{pmatrix}. \end{aligned} \tag{2.4}$$

Proposition 2.7. *Let $\mathcal{D} = (D_1, D_2, D_3) \in \mathsf{TDT}^+$, let $\alpha, \beta, \gamma, \kappa$ be as in Proposition 2.2, let $w \in W_*$ and set $(D_{w,1}, D_{w,2}, D_{w,3}) := \mathcal{D}_w$. Then*

$$\big(\mathrm{curv}(D_{w,1}), \mathrm{curv}(D_{w,2}), \mathrm{curv}(D_{w,3}), \kappa(\mathcal{D}_w)\big) = (\alpha, \beta, \gamma, \kappa) M_w. \tag{2.5}$$

Proof. This follows by induction in $|w|$ using Proposition 2.2-(2) and Definition 2.4. □

We next collect basic facts regarding the Hausdorff dimension and measure of $K(\mathcal{D})$. For each $s \in (0, +\infty)$ let $\mathcal{H}^s : 2^{\mathbb{C}} \to [0, +\infty]$ denote the s-dimensional Hausdorff (outer) measure on $\mathbb{C}$ with respect to the Euclidean metric, and for each $A \subset \mathbb{C}$ let $\dim_{\mathrm{H}} A := \sup\{s \in (0, +\infty) \mid \mathcal{H}^s(A) = +\infty\} = \inf\{s \in (0, +\infty) \mid \mathcal{H}^s(A) = 0\}$ denote its Hausdorff dimension. As is well known, it easily follows from the definition of $\mathcal{H}^s$ that the image $f(A)$ of $A \subset \mathbb{C}$ by a Lipschitz continuous map $f : A \to \mathbb{C}$ with Lipschitz constant $C \in [0, +\infty)$ satisfies $\mathcal{H}^s(f(A)) \le C^s \mathcal{H}^s(A)$ for any $s \in (0, +\infty)$. On the basis of this observation, we easily get the following lemma.

Lemma 2.8. *Let $\mathcal{D}, \mathcal{D}' \in \mathsf{TDT}^+$. Then there exists $c \in (0, +\infty)$ such that $\mathcal{H}^s(K(\mathcal{D})) \le c^s \mathcal{H}^s(K(\mathcal{D}'))$ for any $s \in (0, +\infty)$. In particular, $\dim_{\mathrm{H}} K(\mathcal{D}) = \dim_{\mathrm{H}} K(\mathcal{D}')$.*

Proof. Let $f_{\mathcal{D}',\mathcal{D}}$ denote the unique orientation-preserving Möbius transformation on the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ such that $f_{\mathcal{D}',\mathcal{D}}(q_j(\mathcal{D}')) = q_j(\mathcal{D})$ for any $j \in S$. Then $f_{\mathcal{D}',\mathcal{D}}(K(\mathcal{D}')) = K(\mathcal{D})$, since Möbius transformations map any open disk in $\widehat{\mathbb{C}}$ onto another. Now the assertion follows from the Lipschitz continuity of $f_{\mathcal{D}',\mathcal{D}}|_{\overline{D_{\mathrm{cir}}(\mathcal{D}')}}$. □

Definition 2.9. Noting Lemma 2.8 and choosing $\mathcal{D} \in \mathsf{TDT}^+$ arbitrarily, we define

$$d := \dim_{\mathrm{H}} K(\mathcal{D}). \tag{2.6}$$

Theorem 2.10 (Boyd [1]; see also [4, 11, 12])**.** *$1 < d < 2$. In fact,*

$$1.300197 < d < 1.314534. \tag{2.7}$$

Moreover, for the d-dimensional Hausdorff measure $\mathcal{H}^d(K(\mathcal{D}))$ of $K(\mathcal{D})$ we have the following theorem, which was proved first by Sullivan [14] through considerations on the isometric action of Möbius transformations on the three-dimensional hyperbolic space, and later by Mauldin and Urbański [11] through purely two-dimensional arguments.

Theorem 2.11 (Sullivan [14, Theorem 2], Mauldin and Urbański [11, Theorem 2.6])**.** *For any $\mathcal{D} \in \mathsf{TDT}^+$,*

$$0 < \mathcal{H}^d(K(\mathcal{D})) < +\infty. \tag{2.8}$$

Note that for each $\mathcal{D} = (D_1, D_2, D_3), \mathcal{D}' = (D_1', D_2', D_3') \in \mathsf{TDT}^+$, the Möbius transformation $f_{\mathcal{D}',\mathcal{D}}$ as in the proof of Lemma 2.8 is a Euclidean isometry if and only if $\mathrm{curv}(D_j) = \mathrm{curv}(D_j')$ for any $j \in S$, and in particular that the value of $\mathcal{H}^d(K(\mathcal{D}))$ for $\mathcal{D} = (D_1, D_2, D_3) \in \mathsf{TDT}^+$ is determined solely by $(\mathrm{curv}(D_j))_{j \in S}$. Thus we can use the triple $(\mathrm{curv}(D_j))_{j \in S}$ as a parameter to represent the particular Euclidean geometry of *each* Apollonian gasket $K(\mathcal{D})$, and the following definition introduces the space of such parameters, regarded as that of all possible Euclidean geometries of $K(\mathcal{D})$, $\mathcal{D} \in \mathsf{TDT}^+$.

Definition 2.12. (0) We set $\kappa(g) := \sqrt{\beta\gamma + \gamma\alpha + \alpha\beta}$ for $g = (\alpha, \beta, \gamma) \in [0, +\infty)^3$.
(1) We define $\Gamma \subset [0, +\infty)^4$ by $\Gamma := \big\{(g, \kappa(g)) \;\big|\; g \in [0, +\infty)^3,\ \kappa(g) > 0\big\}$ and equip Γ with the metric $\rho_\Gamma : \Gamma \times \Gamma \to [0, +\infty)$ given by $\rho_\Gamma\big((g_1, \kappa(g_1)), (g_2, \kappa(g_2))\big) := |g_1 - g_2|$. We also set $\Gamma^\circ := \Gamma \cap (0, +\infty)^4$, which is an open subset of Γ.
(2) For each $\varepsilon \in (0, 1)$, we define a compact subset Γ_ε of (Γ, ρ_Γ) by

$$\Gamma_\varepsilon := \big\{(g, \kappa(g)) \;\big|\; g = (\alpha, \beta, \gamma) \in [0, \varepsilon^{-1}]^3,\ \min\{\beta + \gamma, \gamma + \alpha, \alpha + \beta\} \geq \varepsilon\big\}. \quad (2.9)$$

In Definition 2.12, the coordinate $\kappa(g)$ is attached for the sake of the simplicity in applying Proposition 2.7. It is of course redundant for parametrizing the geometries of $K(\mathcal{D})$ and each $(g, \kappa(g)) \in \Gamma$ should (and will) be identified with g, which is why the metric ρ_Γ has been defined as above. Clearly $\big(\mathrm{curv}(D_1), \mathrm{curv}(D_2), \mathrm{curv}(D_3), \kappa(\mathcal{D})\big) \in \Gamma$ for any $\mathcal{D} = (D_1, D_2, D_3) \in \mathsf{TDT}^+$, and conversely it is not difficult to see that any $(g, \kappa(g)) \in \Gamma$ is of this form for some $\mathcal{D} = (D_1, D_2, D_3) \in \mathsf{TDT}^+$. Thus the value of the Hausdorff measure $\mathcal{H}^d(K(\mathcal{D}))$ of $K(\mathcal{D})$ for $\mathcal{D} \in \mathsf{TDT}^+$ induces a function $\mathcal{H}_\Gamma : \Gamma \to (0, +\infty)$ on the parameter space Γ defined as follows.

Definition 2.13. Recalling Theorem 2.11, we define $\mathcal{H}_\Gamma : \Gamma \to (0, +\infty)$ by

$$\mathcal{H}_\Gamma(g) := \mathcal{H}_\Gamma(\alpha, \beta, \gamma) := \mathcal{H}^d(K(\mathcal{D})) \qquad \text{for each } g = (\alpha, \beta, \gamma, \kappa) \in \Gamma, \quad (2.10)$$

where we take any $\mathcal{D} = (D_1, D_2, D_3) \in \mathsf{TDT}^+$ with $\big(\mathrm{curv}(D_1), \mathrm{curv}(D_2), \mathrm{curv}(D_3)\big) = (\alpha, \beta, \gamma)$. Note that it is easy to see that for any $(\alpha, \beta, \gamma, \kappa) \in \Gamma$ and any $s \in (0, +\infty)$,

$$\mathcal{H}_\Gamma(\alpha, \beta, \gamma) = \mathcal{H}_\Gamma(\beta, \gamma, \alpha) = \mathcal{H}_\Gamma(\alpha, \gamma, \beta) = s^d \mathcal{H}_\Gamma(s\alpha, s\beta, s\gamma). \quad (2.11)$$

For our purpose we will need some uniform continuity of $\mathcal{H}_\Gamma$. In fact, we can prove the following theorem by showing that the Möbius transformation $f_{\mathcal{D}',\mathcal{D}}$ as in the proof of Lemma 2.8 is close to a Euclidean isometry uniformly on $\overline{D_{\mathrm{cir}}(\mathcal{D}')}$ if $\mathcal{D}, \mathcal{D}' \in \mathsf{TDT}^+$ are close in the parameter space Γ. Recall Proposition 2.7, which yields $gM_w \in \Gamma$ and $\mathcal{H}_\Gamma(gM_w) = \mathcal{H}^d(K_w(\mathcal{D}))$ for $g, \mathcal{D}$ as in (2.10) and any $w \in W_*$.

Theorem 2.14. *There exist $c_1, c_2 \in (0, +\infty)$ such that for each $\varepsilon \in (0, 1)$,*

$$\big|\mathcal{H}_\Gamma(g_1 M_w) - \mathcal{H}_\Gamma(g_2 M_w)\big| \leq c_{1,\varepsilon} \mathcal{H}_\Gamma(g_1 M_w) \rho_\Gamma(g_1, g_2) \quad (2.12)$$

$$\text{and} \qquad \mathcal{H}_\Gamma(g_2 M_w) \leq c_{2,\varepsilon} \mathcal{H}_\Gamma(g_1 M_w) \quad (2.13)$$

for any $g_1, g_2 \in \Gamma_\varepsilon$ and any $w \in W_*$, where $c_{1,\varepsilon} := c_1 \varepsilon^{1-30d}$ and $c_{2,\varepsilon} := c_2 \varepsilon^{-15d}$.

Theorem 2.14 easily leads to the following corollary. Note that for each $w \in W_*$, $g \mapsto \mathcal{H}_\Gamma(gM_w)^{1/d} gM_w$ defines a map from $\{g \in \Gamma \mid \mathcal{H}_\Gamma(g) = 1\}$ to itself by virtue of (2.11), and (2.14) below gives an upper bound on its Lipschitz constant.

Corollary 2.15. *There exist $c_3, c_4 \in (0, +\infty)$ such that for each $\varepsilon \in (0, 1)$,*

$$\rho_\Gamma\big(\mathcal{H}_\Gamma(g_1 M_w)^{1/d} g_1 M_w, \mathcal{H}_\Gamma(g_2 M_w)^{1/d} g_2 M_w\big) \leq c_{3,\varepsilon} \mathcal{H}_\Gamma\big(\tfrac{g_0 M_w}{\|M_w\|}\big)^{1/d} \rho_\Gamma(g_1, g_2) \quad (2.14)$$

$$\text{and} \qquad \big|\log \mathcal{H}_\Gamma(g_1 M_w) - \log \mathcal{H}_\Gamma(g_2 M_w)\big| \leq c_{4,\varepsilon} \rho_\Gamma(g_1, g_2) \quad (2.15)$$

for any $g_0, g_1, g_2 \in \Gamma_\varepsilon$ and any $w \in W_*$, where $c_{3,\varepsilon} := c_3 \varepsilon^{-45d}$ and $c_{4,\varepsilon} := c_4 \varepsilon^{1-45d}$.

The factor $\mathcal{H}_\Gamma\bigl(\frac{g_0M_w}{\|M_w\|}\bigr)^{1/d}$ in the right-hand side of (2.14) cannot be bounded by a constant independent of w; in fact, it can be shown that $\inf_{n\in\mathbb{N}}\mathcal{H}_\Gamma\bigl(\frac{g_0M_{j^n}}{\|M_{j^n}\|}\bigr)/n > 0$ for any $g_0 \in \Gamma$ and any $j \in S$. This factor in (2.14), however, can be replaced by a constant for the words $w \in W_*$ ending with j^nk for some $j, k \in S$ with $j \neq k$ and $n \in \mathbb{N}$, and consequently we obtain Proposition 2.17 and Corollary 2.18 below. The idea of working with this class of words to study dynamics on the Apollonian gasket is originally due to Mauldin and Urbański [11] and plays crucial roles also in our study.

Definition 2.16. We define $I \subset W_* \setminus \{\emptyset\}$ by $I := \{j^nk \mid j, k \in S,\ j \neq k,\ n \in \mathbb{N}\}$, so that $\tau \not\asymp \upsilon$ for any $\tau, \upsilon \in I$ with $\tau \neq \upsilon$ and $K(\mathcal{D})\setminus V_0(\mathcal{D}) = \bigcup_{\tau\in I} K_\tau(\mathcal{D})$ for $\mathcal{D} \in \mathsf{TDT}^+$.

Proposition 2.17. *There exist $c_5, c_6 \in (0, +\infty)$ and $\varepsilon_0 \in (0, 1)$ such that for any $g \in \Gamma$ and any $\tau \in I$,*

$$c_5|gM_\tau|^{-d} \le \mathcal{H}_\Gamma(gM_\tau) \le c_6|gM_\tau|^{-d} \tag{2.16}$$

$$\text{and} \qquad \mathcal{H}_\Gamma(gM_\tau)^{1/d}gM_\tau \in \Gamma_{\varepsilon_0}. \tag{2.17}$$

Corollary 2.18. *There exists $c_7 \in (0, +\infty)$ such that for each $\varepsilon \in (0, 1)$,*

$$\rho_\Gamma\bigl(\mathcal{H}_\Gamma(g_1M_{w\tau})^{1/d}g_1M_{w\tau}, \mathcal{H}_\Gamma(g_2M_{w\tau})^{1/d}g_2M_{w\tau}\bigr) \le c_{7,\varepsilon}\rho_\Gamma(g_1, g_2) \tag{2.18}$$

for any $g_1, g_2 \in \Gamma_\varepsilon$, any $w \in W_*$ and any $\tau \in I$, where $c_{7,\varepsilon} := c_7\varepsilon^{-45d}$.

3 The asymptotic formula for counting functions

The following theorem is a corollary of the general result by Oh and Shah [13, Theorem 1.4] for circle packing fractals invariant with respect to a large class of Kleinian groups.

Theorem 3.1 (Oh and Shah [13, Corollary 1.8])**.** *There exists $c_8 \in (0, +\infty)$ such that for any $\mathcal{D} \in \mathsf{TDT}^+$,*

$$\lim_{\lambda\to+\infty}\lambda^{-d}\#\{w \in W_* \mid \mathrm{curv}(D_{\mathrm{in}}(\mathcal{D}_w)) \le \lambda\} = c_8\mathcal{H}^d(K(\mathcal{D})). \tag{3.1}$$

Theorem 3.1 can be deduced from the following theorem, which is applicable to more general counting functions including that for the eigenvalues of the "geometrically canonical" Laplacian on the Apollonian gasket introduced by Teplyaev in [15, Theorem 5.17] and studied extensively by the author in [6].

Theorem 3.2 ([6])**.** *Let Γ' denote either of Γ and Γ°, and for each $n \in \mathbb{N}$ let $\lambda_n : \Gamma' \to (0, +\infty)$ be continuous and satisfy $\lambda_n(sg) = s\lambda_n(g)$ for any $(g, s) \in \Gamma' \times (0, \infty)$. Suppose that $\lambda_1(g) = \min_{n\in\mathbb{N}}\lambda_n(g)$ and $\lim_{n\to\infty}\lambda_n(g) = +\infty$ for any $g \in \Gamma'$, set $\mathcal{N}(g, \lambda) := \#\{n \in \mathbb{N} \mid \lambda_n(g) \le \lambda\}$ for each $(g, \lambda) \in \Gamma' \times [0, +\infty)$, and suppose that there exist $\eta \in [0, d)$ and $c \in (0, +\infty)$ such that for any $(g, \lambda) \in (\Gamma' \cap \Gamma_{\varepsilon_0}) \times [0, +\infty)$,*

$$\sum\nolimits_{\tau\in I}\mathcal{N}(gM_\tau, \lambda) \le \mathcal{N}(g, \lambda) \le \sum\nolimits_{\tau\in I}\mathcal{N}(gM_\tau, \lambda) + c\lambda^\eta + c, \tag{3.2}$$

where ε_0 is as in (2.17). Then there exists $c_0 \in (0, +\infty)$ such that for any $g \in \Gamma' \cap \Gamma_{\varepsilon_0}$,

$$\lim_{\lambda\to+\infty}\lambda^{-d}\mathcal{N}(g, \lambda) = c_0\mathcal{H}_\Gamma(g). \tag{3.3}$$

Moreover, additionally if for each $g \in \Gamma'$ there exists $c_g \in (0, +\infty)$ such that (3.2) with c_g in place of c holds for any $\lambda \in [0, +\infty)$, then (3.3) holds for any $g \in \Gamma'$.

Theorem 3.1 is an easy corollary of Theorem 3.2. Indeed, choose a bijection $\mathbb{N} \ni n \mapsto w(n) \in W_*$ with $w(1) = \emptyset$ and define $\lambda_n(g) := gM_{w(n)}\begin{pmatrix}1\\1\\1\\2\end{pmatrix}$ for each $(g,n) \in \Gamma \times \mathbb{N}$. Then $\#\{w \in W_* \mid \mathrm{curv}(D_{\mathrm{in}}(\mathcal{D}_w)) \le \lambda\} = \#\{n \in \mathbb{N} \mid \lambda_n(g(\mathcal{D})) \le \lambda\} =: \mathcal{N}(g(\mathcal{D}), \lambda)$ for any $\lambda \in [0, +\infty)$ for each $\mathcal{D} = (D_1, D_2, D_3) \in \mathsf{TDT}^+$ by Proposition 2.7 and Proposition 2.2-(2), where $g(\mathcal{D}) := \big(\mathrm{curv}(D_1), \mathrm{curv}(D_2), \mathrm{curv}(D_3), \kappa(\mathcal{D})\big) \in \Gamma$, and clearly $\lambda_1(g) = \min_{n\in\mathbb{N}} \lambda_n(g)$ and $\lim_{n\to\infty} \lambda_n(g) = +\infty$ for any $g \in \Gamma$ by Definition 2.6. Furthermore for each $g = (\alpha, \beta, \gamma, \kappa) \in \Gamma$ and for any $\lambda \in [0, +\infty)$, taking $\mathcal{D} \in \mathsf{TDT}^+$ with $g(\mathcal{D}) = g$ and noting that $W_* = \{j^n \mid j \in S,\, n \in \mathbb{N} \cup \{0\}\} \cup \bigcup_{\tau\in I}\{\tau w \mid w \in W_*\}$ with the union disjoint, from (2.3) we easily obtain

$$\begin{aligned}\mathcal{N}(g,\lambda) &= \sum\nolimits_{\tau\in I} \mathcal{N}(gM_\tau, \lambda) + \#\left\{ j^n \;\middle|\; j \in S,\, n \in \mathbb{N}\cup\{0\},\, gM_{j^n}\begin{pmatrix}1\\1\\1\\2\end{pmatrix} \le \lambda \right\}\\ &\le \sum\nolimits_{\tau\in I} \mathcal{N}(gM_\tau, \lambda) + 3\big(\min\{\beta+\gamma, \gamma+\alpha, \alpha+\beta\}\big)^{-1/2}\lambda^{1/2}.\end{aligned}$$

Thus Theorem 3.2 is applicable to the above choice of $\lambda_n(g)$ and implies that (3.3) holds for any $g \in \Gamma$ for some $c_0 \in (0, +\infty)$, which is nothing but Theorem 3.1.

The rest of this article is devoted to a brief sketch of the proof of Theorem 3.2. The key idea is to apply Kesten's renewal theorem [10, Theorem 2] to the Markov chain on $\{g \in \Gamma \mid \mathcal{H}_\Gamma(g) = 1\}$ defined as follows.

Definition 3.3. (1) We set $W_0^I := \{\emptyset\}$, where $\emptyset$ denotes the empty word, $W_m^I := I^m = \{\omega_1 \dots \omega_m \mid \omega_k \in I \text{ for any } k \in \{1, \dots, m\}\}$ for $m \in \mathbb{N}$ and $W_*^I := \bigcup_{m\in\mathbb{N}\cup\{0\}} W_m^I$, which are regarded as subsets of W_* in the natural manner.

(2) We set $\Sigma^I := I^{\mathbb{N}} = \{\omega_1\omega_2\omega_3\dots \mid \omega_k \in I \text{ for any } k \in \mathbb{N}\}$, which is equipped with the product topology of the discrete topology on I, and define the *shift map* $\sigma^I : \Sigma^I \to \Sigma^I$ by $\sigma^I(\omega_1\omega_2\omega_3\dots) := \omega_2\omega_3\omega_4\dots$. For $w \in W_*^I$ we define $\sigma_w^I : \Sigma^I \to \Sigma^I$ by $\sigma_w^I(\omega_1\omega_2\omega_3\dots) := w\omega_1\omega_2\omega_3\dots$. For $\omega = \omega_1\omega_2\omega_3\dots \in \Sigma^I$ and $m \in \mathbb{N}\cup\{0\}$, we set $[\omega]_m^I := \omega_1\dots\omega_m \in W_m^I$.

Definition 3.4. Set $\widetilde{\Gamma} := \{g \in \Gamma \mid \mathcal{H}_\Gamma(g) = 1\}$, $[g]_\Gamma := \mathcal{H}_\Gamma(g)^{1/d} g \in \widetilde{\Gamma}$ for each $g \in \Gamma$, $\Omega := \widetilde{\Gamma} \times \Sigma^I$, which is equipped with the product topology, and let $\mathcal{F}$ denote the Borel σ-field of Ω. We define a continuous map $\theta : \Omega \to \Omega$ by $\theta(g,\omega) := \big([gM_{[\omega]_1^I}]_\Gamma, \sigma^I(\omega)\big)$, and for each $n \in \mathbb{N}\cup\{0\}$ we define random variables $X_n : \Omega \to \widetilde{\Gamma}$ and $u_n, \mathcal{V}_n : \Omega \to \mathbb{R}$ on $(\Omega, \mathcal{F})$ by $X_n(g,\omega) := [gM_{[\omega]_n^I}]_\Gamma$, $u_n(g,\omega) := d^{-1}\log\big(\mathcal{H}_\Gamma\big(gM_{[\omega]_n^I}\big)/\mathcal{H}_\Gamma\big(gM_{[\omega]_{n+1}^I}\big)\big)$ and $\mathcal{V}_n(g,\omega) := -d^{-1}\log\mathcal{H}_\Gamma\big(gM_{[\omega]_n^I}\big) = \sum_{k=0}^{n-1} u_k$. Also for each $g \in \widetilde{\Gamma}$, we define $\mathbb{P}_g$ as the unique probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}_g[\{g\} \times \sigma_w^I(\Sigma^I)] = \mathcal{H}_\Gamma(gM_w)$ for any $w \in W_*^I$, and the expectation with respect to $\mathbb{P}_g$ is denoted by $\mathbb{E}_g[(\cdot)]$.

Clearly $\big(\Omega, \mathcal{F}, \{X_n\}_{n\in\mathbb{N}\cup\{0\}}, \{\mathbb{P}_g\}_{g\in\widetilde{\Gamma}}\big)$ is a time-homogeneous Markov chain on $\widetilde{\Gamma}$ with transition function $\mathcal{P}(g,\cdot) := \mathbb{P}_g[X_1 \in \cdot] = \sum_{\tau\in I} \mathcal{H}_\Gamma(gM_\tau)\delta_{[gM_\tau]_\Gamma}$, where δ_g for $g \in \widetilde{\Gamma}$ denotes the (unique) Borel probability measure on $\widetilde{\Gamma}$ such that $\delta_g(\{g\}) = 1$. It is also easy to see that for each $g \in \widetilde{\Gamma}$ and each $n \in \mathbb{N}\cup\{0\}$, *the conditional law* $\mathbb{P}_g[u_n \in \cdot \mid \{X_k\}_{k\in\mathbb{N}\cup\{0\}}, \{u_k\}_{k\in(\mathbb{N}\cup\{0\})\setminus\{n\}}]$ *of* u_n *is determined solely by* X_n, X_{n+1}; see [10, (1.1)] for the precise formulation of this property. Thus the stochastic processes of Definition 3.4 fall within the general framework of Kesten's renewal theory in [10].

The idea of considering these stochastic processes is hinted by (3.2). Indeed, assume the setting of Theorem 3.2, set $\widetilde{\Gamma}'_{\varepsilon_0} := \widetilde{\Gamma} \cap \Gamma' \cap \Gamma_{\varepsilon_0}$ and define $\widetilde{\mathcal{N}}, \mathcal{R}_0 : \widetilde{\Gamma}'_{\varepsilon_0} \times \mathbb{R} \to [0, +\infty)$ by $\widetilde{\mathcal{N}}(g,s) := e^{-ds}\mathcal{N}(g, e^s)$ and $\mathcal{R}_0(g,s) := e^{-ds}\bigl(\mathcal{N}(g,e^s) - \sum_{\tau \in I} \mathcal{N}(gM_\tau, e^s)\bigr)$. Then it easily follows from the assumptions of Theorem 3.2 that $0 \le \mathcal{R}_0(g,s) \le c_9 e^{-(d-\eta)|s|}$ for any $(g,s) \in \widetilde{\Gamma}'_{\varepsilon_0} \times \mathbb{R}$ for some $c_9 \in (0, +\infty)$ and that for any $(g,s) \in \widetilde{\Gamma}'_{\varepsilon_0} \times \mathbb{R}$,

$$\begin{aligned}\widetilde{\mathcal{N}}(g,s) - \mathcal{R}_0(g,s) &= \sum\nolimits_{\tau \in I} \mathcal{H}_\Gamma(gM_\tau) e^{-d(s + d^{-1}\log \mathcal{H}_\Gamma(gM_\tau))} \mathcal{N}\bigl([gM_\tau]_\Gamma, e^{s + d^{-1}\log \mathcal{H}_\Gamma(gM_\tau)}\bigr) \\ &= \sum\nolimits_{\tau \in I} \mathcal{H}_\Gamma(gM_\tau) \widetilde{\mathcal{N}}\bigl([gM_\tau]_\Gamma, s + d^{-1}\log \mathcal{H}_\Gamma(gM_\tau)\bigr) = \mathbb{E}_g\bigl[\widetilde{\mathcal{N}}(X_1, s - \mathcal{V}_1)\bigr]. \quad (3.4)\end{aligned}$$

A repetitive use of (3.4) further shows that

$$\widetilde{\mathcal{N}}(g,s) = \mathbb{E}_g\Bigl[\sum\nolimits_{n \in \mathbb{N} \cup \{0\}} \mathcal{R}_0(X_n, s - \mathcal{V}_n)\Bigr] \qquad \text{for any } (g,s) \in \widetilde{\Gamma}'_{\varepsilon_0} \times \mathbb{R}, \quad (3.5)$$

which together with $\mathcal{R}_0(g,s) \le c_9 e^{-(d-\eta)|s|}$ implies that $\sup_{(g,s) \in \widetilde{\Gamma}'_{\varepsilon_0} \times \mathbb{R}} \widetilde{\mathcal{N}}(g,s) < +\infty$.

Thus the proof of Theorem 3.2 is reduced to proving that a function on $\widetilde{\Gamma}'_{\varepsilon_0} \times \mathbb{R}$ of the form (3.5) converges to c_8 as $s \to +\infty$ for any $g \in \widetilde{\Gamma}'_{\varepsilon_0}$ for some $c_8 \in (0, +\infty)$. This is exactly what Kesten's renewal theorem [10, Theorem 2] asserts under a reasonable set of assumptions on $\bigl(\Omega, \mathcal{F}, \{(X_n, u_n)\}_{n \in \mathbb{N} \cup \{0\}}, \{\mathbb{P}_g\}_{g \in \widetilde{\Gamma}}\bigr)$ and $\mathcal{R}_0$, among which the following *unique ergodicity* of $\{(X_n, u_n)\}_{n \in \mathbb{N} \cup \{0\}}$ is the most important:

Proposition 3.5. (1) *There exists a unique Borel probability measure ν on $\widetilde{\Gamma}$ such that $\nu(A) = \int_{\widetilde{\Gamma}} \mathcal{P}(g, A)\, d\nu(g)$ for any Borel subset A of $\widetilde{\Gamma}$. Moreover, $\nu(\widetilde{\Gamma} \cap \Gamma^\circ \cap \Gamma_{\varepsilon_0}) = 1$.*

(2) *The probability measure $\mathbb{P}_\nu$ on $(\Omega, \mathcal{F})$ defined by $\mathbb{P}_\nu[A] := \int_{\widetilde{\Gamma}} \mathbb{P}_g[A]\, d\nu(g)$ for each $A \in \mathcal{F}$ is invariant and ergodic with respect to θ.*

Proof. (1) Since $\mathcal{P}(g, \widetilde{\Gamma} \cap \Gamma_{\varepsilon_0}) = 1$ for any $g \in \widetilde{\Gamma}$ by (2.17), any Borel probability measure ν on $\widetilde{\Gamma}$ with the property $\nu = \int_{\widetilde{\Gamma}} \mathcal{P}(g, \cdot)\, d\nu(g)$ must satisfy $\nu(\widetilde{\Gamma} \cap \Gamma_{\varepsilon_0}) = 1$, and the Markov chain $\{X_n\}_{n \in \mathbb{N} \cup \{0\}}$ can be restricted to the *compact* set $\widetilde{\Gamma} \cap \Gamma_{\varepsilon_0}$. Then we easily see from Corollary 2.18 that its transition function $\mathcal{P}_0 := \mathcal{P}|_{\widetilde{\Gamma} \cap \Gamma_{\varepsilon_0}}$ has the property that $\{\mathcal{P}_0^n f\}_{n \in \mathbb{N} \cup \{0\}}$ is uniformly equicontinuous on $\widetilde{\Gamma} \cap \Gamma_{\varepsilon_0}$ for any continuous function $f : \widetilde{\Gamma} \cap \Gamma_{\varepsilon_0} \to \mathbb{R}$. Now we can easily conclude the existence of ν from the classical theorem of Krylov and Bogolioubov (see [3, Theorem 1.10]), its uniqueness from [16, Theorem 4.1.11], and $\nu(\widetilde{\Gamma} \cap \Gamma^\circ) = 1$ from the uniqueness of ν and the fact that $\mathcal{P}(g, \widetilde{\Gamma} \cap \Gamma^\circ) = 1$ for any $g \in \widetilde{\Gamma} \cap \Gamma^\circ$ by (2.4).

(2) This can be deduced from (1) by standard arguments; see [6, Section 8] for details. □

Now it is not difficult to verify [10, Conditions I.1–I.4] on the basis of Proposition 3.5, (2.13), (2.15) and (2.18), which allows us to apply [10, Theorem 2] to the present case and thereby to get the following theorem.

Theorem 3.6 ([10, Theorem 2] applied to the present case)**.** *Let Γ' denote either of Γ and Γ°, set $\widetilde{\Gamma}'_{\varepsilon_0} := \widetilde{\Gamma} \cap \Gamma' \cap \Gamma_{\varepsilon_0}$, and let $f : \widetilde{\Gamma}'_{\varepsilon_0} \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $|f(g,s)| \le c e^{-|s|/c}$ for any $(g,s) \in \widetilde{\Gamma}'_{\varepsilon_0} \times \mathbb{R}$ for some $c \in (0, +\infty)$. Then for any $g \in \widetilde{\Gamma}'_{\varepsilon_0}$,*

$$\lim_{s \to +\infty} \mathbb{E}_g\Bigl[\sum\nolimits_{n \in \mathbb{N} \cup \{0\}} f(X_n, s - \mathcal{V}_n)\Bigr] = \Bigl(\int_{\widetilde{\Gamma}} \mathbb{E}_y[u_0]\, d\nu(y)\Bigr)^{-1} \int_{\widetilde{\Gamma}'_{\varepsilon_0}} \int_{\mathbb{R}} f(y,s)\, ds\, d\nu(y). \quad (3.6)$$

Finally, we can easily deduce Theorem 3.2 from Theorem 3.6. The function $\mathcal{R}_0$ as in (3.5) is obviously not continuous, but this problem can be avoided by considering the function $\mathcal{Z}(g,t) := \sum_{n\in\mathbb{N}} e^{-\lambda_n(g)t}$ instead of $\mathcal{N}(g,\lambda)$. Indeed, the boundedness of $\widetilde{\mathcal{N}}$ implies that of the function $\widetilde{\mathcal{Z}}(g,s) := e^{-ds}\mathcal{Z}(g,e^{-s})$, and $\widetilde{\mathcal{Z}}$ and the continuous function $\mathcal{R}(g,s) := e^{-ds}\big(\mathcal{Z}(g,e^{-s}) - \sum_{\tau\in I}\mathcal{Z}(gM_\tau,e^{-s})\big)$ on $\widetilde{\Gamma}'_{\varepsilon_0}\times\mathbb{R}$ are easily seen to satisfy the same properties as those for $\widetilde{\mathcal{N}},\mathcal{R}_0$ mentioned above. It follows that Theorem 3.6 is applicable to $f=\mathcal{R}$ and implies that $\lim_{s\to+\infty}\widetilde{\mathcal{Z}}(g,s) = c'_0$ for any $g\in\widetilde{\Gamma}'_{\varepsilon_0}$ for some $c'_0\in[0,+\infty)$, but $c'_0=0$ would mean $\mathcal{R}|_{\mathrm{supp}[\nu]\times\mathbb{R}}=0$ by (3.6) and, together with (3.4) and (3.5) for $\widetilde{\mathcal{Z}},\mathcal{R}$, would easily result in the contradiction that $\widetilde{\mathcal{Z}}|_{\mathrm{supp}[\nu]\times\mathbb{R}}=0$, where $\mathrm{supp}[\nu]$ denotes the smallest closed subset of $\widetilde{\Gamma}'_{\varepsilon_0}$ with $\nu(\mathrm{supp}[\nu])=1$. Thus $c'_0\in(0,+\infty)$, and now Theorem 3.2 follows by an application of Karamata's Tauberian theorem [2, p. 445, Theorem 2] to $\mathcal{N}(g,\cdot)$ for each $g\in\widetilde{\Gamma}'_{\varepsilon_0}$.

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