Estimate of martingale dimension revisited

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Abstract

The concept of martingale dimension is defined for symmetric diffusion processes and is interpreted as the multiplicity of filtration. However, if the underlying space is a fractal-like set, then estimating the martingale dimension quantitatively is a difficult problem. To date, the only known nontrivial estimates have been those for canonical diffusions on a class of self-similar fractals. This paper surveys existing results and discusses more-general situations.

1 Introduction

To date, various concepts of dimensionality have been introduced in diverse fields of analysis. The Hausdorff dimension d_H is the most familiar and is related strongly to the geometry of the underlying space. The spectral dimension d_s is a more analytic concept and appears in on-diagonal estimates of the fundamental solutions of the heat equations. The martingale dimension d_m is associated with diffusion processes and indicates the multiplicity of filtration. We begin by explaining the dimensions d_s and d_m more precisely in the framework of Dirichlet forms.

Let K be a locally compact separable metric space and let μ be a σ-finite Borel measure on K with full support. Let $C_c(K)$ denote the set of all real-valued functions on K with compact support. This is regarded as a normed space with the supremum norm. Suppose that we are given a strongly local regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$. In other words, $\mathcal{F}$ is a dense subspace of $L^2(K,\mu)$, and $\mathcal{E}\colon \mathcal{F}\times\mathcal{F}\to\mathbb{R}$ is a non-negative definite symmetric bilinear form that satisfies the following:

- Closedness: If a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{F}$ and $f\in L^2(K,\mu)$ satisfy

$$\lim_{N\to\infty}\sup_{m,n\ge N}\mathcal{E}(f_m-f_n,f_m-f_n)=0 \text{ and } \lim_{n\to\infty}\|f_n-f\|_{L^2(K,\mu)}=0,$$

  then it holds that $f\in\mathcal{F}$ and $\lim_{n\to\infty}\mathcal{E}(f_n-f,f_n-f)=0$.

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- Markov property: For any $f \in \mathcal{F}$, $\hat{f} := \max\{0, \min\{1, f\}\}$ belongs to $\mathcal{F}$ and satisfies $\mathcal{E}(\hat{f}, \hat{f}) \le \mathcal{E}(f, f)$.
- Regularity: The space $\mathcal{F} \cap C_c(K)$ is dense in both $\mathcal{F}$ and $C_c(K)$. Here, the topology of $\mathcal{F}$ is induced by the norm $\|f\|_{\mathcal{F}} := (\mathcal{E}(f, f) + \|f\|^2_{L^2(K,\mu)})^{1/2}$.
- Strong locality: If $f, g \in \mathcal{F}$ and $a \in \mathbb{R}$ satisfy $f \cdot (g - a) = 0$ μ-a.e., then $\mathcal{E}(f, g) = 0$.

Then, there exists uniquely a non-positive self-adjoint operator L on $L^2(K, \mu)$ such that the domain of $\sqrt{-L}$ is equal to $\mathcal{F}$ and

$$\mathcal{E}(f, g) = \int_K (\sqrt{-L} f)(\sqrt{-L} g)\, d\mu \quad \text{for every } f, g \in \mathcal{F}.$$

By letting $T_t = e^{tL}$ for $t \ge 0$, $\{T_t\}_{t\ge 0}$ forms a strongly continuous contraction semigroup on $L^2(K, \mu)$. This extends to a semigroup on $L^\infty(K, \mu)$ in the natural way, which is denoted using the same symbol. The Markov property of $(\mathcal{E}, \mathcal{F})$ induces that of $\{T_t\}_{t\ge 0}$; that is, $0 \le f \le 1$ μ-a.e. implies that $0 \le T_t f \le 1$ μ-a.e. for every $t \ge 0$.

For a subset A of K, we define the 1-capacity $\mathrm{Cap}_1(A)$ of A by

$$\mathrm{Cap}_1(A) = \inf\{\mathcal{E}(f, f) + \|f\|^2_{L^2(K,\mu)} \mid f \in \mathcal{F},\ f \ge 1\ \mu\text{-a.e. on a neighborhood of } A\}.$$

A function f of K is called quasi-continuous if there exists for every $\varepsilon > 0$ an open set U of K such that $\mathrm{Cap}_1(U) < \varepsilon$ and $f|_{K\setminus U}$ is continuous. A set $A \subset K$ with $\mathrm{Cap}_1(A) = 0$ is called an exceptional set. A statement depending on each point x of K is said to hold quasi-everywhere (q.e.) if there exists an exceptional set N such that the statement holds for all $x \in K \setminus N$.

From the general theory of Dirichlet forms [6], $(\mathcal{E}, \mathcal{F})$ induces a diffusion process $\{X_t\}_{t\ge 0}$ on K with no killing inside. More precisely, $\{X_t\}_{t\ge 0}$ is defined on a filtered probability space $(\Omega, \mathcal{F}_\infty, P, \{P_x\}_{x\in K_\Delta}, \{\mathcal{F}_t\}_{t\ge 0})$. Here, $K_\Delta := K \cup \{\Delta\}$ is the one-point compactification of K and $\{\mathcal{F}_t\}_{t\ge 0}$ is the minimum complete admissible filtration of the process $\{X_t\}_{t\ge 0}$. For any $t > 0$ and a bounded Borel function f on K, it holds that $E_x[f(X_t)]$ is a quasi-continuous modification of $T_t f(x)$. Here, E_x denotes the integration with respect to P_x.

If there exists an integral density (called the transition density) $p_t(\cdot, \cdot)$ of T_t with respect to μ and, for some $d_s > 0$ and $c > 0$,

$$c^{-1} t^{-d_s/2} \le p_t(x, x) \le c t^{-d_s/2}, \quad x \in K,\ t \in (0, 1],$$

then we call d_s the spectral dimension associated with $(\mathcal{E}, \mathcal{F})$ or $\{X_t\}_{t\ge 0}$.

In the following, we may assume without loss of generality that there exist shift operators $\theta_t \colon \Omega \to \Omega$ for $t \ge 0$ that satisfy $X_s \circ \theta_t = X_{s+t}$ for all $s \ge 0$. The lifetime of $\{X_t(\omega)\}_{t\ge 0}$ is denoted by $\zeta(\omega)$.

A $[-\infty, +\infty]$-valued function $A_t(\omega)$ $(t \ge 0,\ \omega \in \Omega)$ is called an additive functional if

- for each $t \ge 0$, A_t is $\mathcal{F}_t$-measurable; and

- there exist a set $\Lambda \in \mathcal{F}_\infty$ and an exceptional set $N \subset K$ such that $P_x(\Lambda) = 1$ for all $x \in K \setminus N$ and $\theta_t\Lambda \subset \Lambda$ for all $t > 0$; moreover, for each $\omega \in \Lambda$, $A_\cdot(\omega)$ is right continuous and has the left limit on $[0, \zeta(\omega))$, $A_0(\omega) = 0$, $A_\cdot(\omega) \in \mathbb{R}$ on $[0, \zeta(\omega))$, $A_\cdot(\omega) = A_{\zeta(\omega)}(\omega)$ on $[\zeta(\omega), \infty)$, and

$$A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s\omega) \quad \text{for } t, s \geq 0.$$

The aforementioned set Λ is called a defining set of A. Two additive functionals A and A' are identified if, for any $t > 0$, $P_x(A_t = A'_t) = 1$ for q.e. x.

Let $\mathring{\mathcal{M}}$ denote the space of all martingale additive functionals with finite energy. That is, $\mathring{\mathcal{M}}$ is the totality of additive functionals $M = \{M_t\}_{t\geq 0}$ such that

- M is a real-valued additive functional;
- $M_\cdot(\omega)$ is right continuous and has a left limit on $[0, \infty)$ for ω in a defining set of M;
- $E_x[M_t^2] < \infty$ and $E_x[M_t] = 0$ for all $t > 0$ and q.e. $x \in K$; and
- the total energy $e(M)$ of M, namely

$$e(M) = \sup_{t>0} \frac{1}{2t} \int_K E_x[M_t^2]\, \mu(dx),$$

is finite.

Then, the martingale dimension[1] d_m (with respect to $(\mathcal{E}, \mathcal{F})$) is defined in [10] as the smallest number D such that there exist $M^{(1)}, \ldots, M^{(D)} \in \mathring{\mathcal{M}}$ such that for every $M \in \mathring{\mathcal{M}}$ there exist $\varphi_s^{(j)} \in L^2(K, \mu)$ satisfying

$$M_t = \sum_{j=1}^{D} \left(\varphi^{(j)} \bullet M^{(j)}\right)_t, \quad t \geq 0. \tag{1.1}$$

Here, $\varphi \bullet M$ is the stochastic integral in the sense of martingale additive functionals; see [6, Section 5.6] for its precise definition. Here we mention only that if $\varphi \in C_c(K)$, then it is given by the standard stochastic integral

$$(\varphi \bullet M)_t = \int_0^t \varphi(X_s)\, dM_s.$$

If there are no integers D satisfying the above, then d_m is defined as $+\infty$.

Other than the case where the Dirichlet form is given by the L^2-inner product of the "gradient of functions" with respect to a "Riemannian metric" with explicit information, determining the value of d_m is a difficult problem in general. Martingale dimensions can be interpreted analytically as the "maximal effective dimensions of the virtual (co-)tangent spaces of K;" see [12, 13] for further details.

[1] Precisely speaking, this is called the "AF-martingale dimension" in [10], where AF represents "additive functional." The concept of martingale dimension can be defined for general (not necessarily symmetric) diffusion processes in a similar but slightly different manner (cf. [21]), which we do not discuss here.

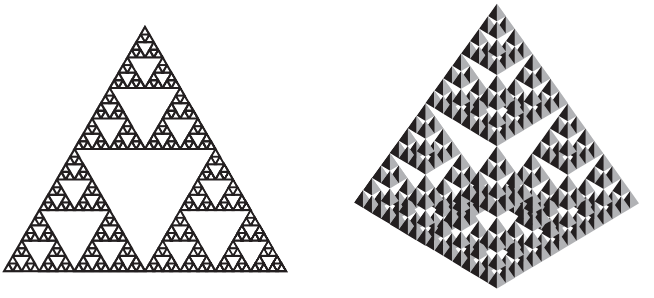

Figure 1: d-dimensional Sierpinski gaskets ($d = 2, 3$)

2 Survey of previous results

In this section, we survey some known results for the dimensions in typical examples.

Example 1 (Euclidean spaces)**.** Let $K = \mathbb{R}^d$ and μ be the d-dimensional Lebesgue measure. Define

$$\mathcal{E}(f, g) := \frac{1}{2}\int_{\mathbb{R}^d} (\nabla f, \nabla g)\, d\mu, \quad f, g \in \mathcal{F} := H^1(\mathbb{R}^d),$$

where $H^1(\mathbb{R}^d)$ denotes the first-order L^2-Sobolev space on $\mathbb{R}^d$. The diffusion process $\{X_t\}_{t\ge 0}$ associated with $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d, \mu)$ is nothing but d-dimensional Brownian motion. The transition density $p_t(x, y)$ is expressed explicitly as

$$p_t(x, y) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

In this case, $d_H = d_s = d$, and furthermore $d_m = d$. Indeed, we can take d as D and the jth component of $X_t - X_0$ as $M_t^{(j)}$ for $j = 1, \dots, d$ in (1.1).

Example 2 (Sierpinski gaskets)**.** For $d \ge 2$, the d-dimensional Sierpinski gasket K (Figure 1) is defined as the unique nonempty compact subset of $\mathbb{R}^d$ such that

$$K = \bigcup_{j=1}^{d+1} \psi_j(K),$$

where $\psi_j \colon \mathbb{R}^d \to \mathbb{R}^d$ ($j = 1, \dots, d+1$) is given by $\psi_j(x) = (x + a_j)/2$ and $a_1, \dots, a_{d+1} \in \mathbb{R}^d$ are given points that are affinely independent. The Hausdorff dimension d_H is equal to $\log(d+1)/\log 2$. There exists a canonical diffusion process ("Brownian motion") $\{X_t\}_{t\ge 0}$ [7, 17, 5] and the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$, where μ is the normalized Hausdorff measure on K. Also, the continuous transition density $p_t(x, y)$ exists and satisfies the sub-Gaussian estimate [5]:

$$p_t(x, y) \asymp \frac{c}{t^{d_s/2}} \exp\left(-\left(\frac{|x-y|^{2d_H/d_s}}{ct}\right)^{\frac{1}{(2d_H/d_s)-1}}\right), \quad t \in (0, 1], \tag{2.1}$$

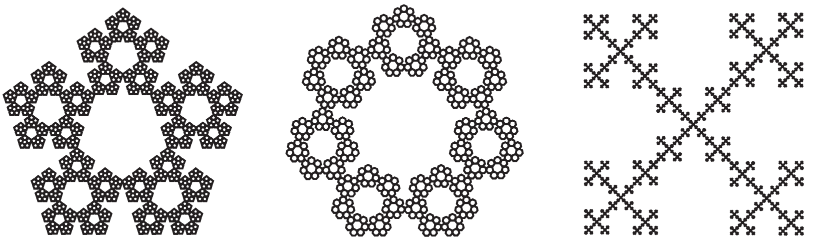

Figure 2: Examples of nested fractals

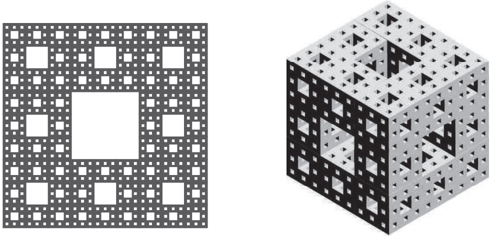

Figure 3: d-dimensional standard Sierpinski carpets ($d = 2, 3$)

where $d_s = 2\log(d+3)/\log(d+1) \in (1, \min\{2, d_H\})$. In particular, the inequality $d_s < 2$ implies that the process $\{X_t\}_{t\geq 0}$ is point recurrent. The martingale dimension d_m was proved to be 1 in [18], which is the first nontrivial result in the problem of determining d_m.

Example 3 (Nested fractals)**.** Nested fractals [20] are self-similar sets in Euclidean spaces with some good symmetries. Sierpinski gaskets are typical examples of nested fractals. See Figure 2 for other examples. In particular, they are finitely ramified, that is, they become disconnected by deleting appropriate finite points. The Hausdorff dimension d_H is calculated easily from the general theory. As in Example 2, Brownian motion [20] and the associated Dirichlet form exist, and transition density exists and satisfies the quantitative estimate (2.1) with different constant $d_s \in (1, \min\{2, d_H\})$ ([16], see also [1]). The martingale dimension d_m has been proved to be 1 in [9].

Example 4 (Sierpinski carpets)**.** Sierpinski carpets are typical examples of self-similar fractals that are not finitely ramified, that is, infinitely ramified. See Figure 3. As in the previous examples, the Hausdorff dimension d_H is calculated easily. Brownian motion exists [2, 19, 3, 4] and its transition density satisfies the estimate (2.1) with different constant $d_s \in (1, d_H)$, although the exact value of d_s is unknown [2, 3]. It was proved in [11] that the martingale dimension d_m satisfies the inequality

$$1 \leq d_m \leq d_s. \tag{2.2}$$

In particular, if $d_s < 2$ (that is, the process is point recurrent), then $d_m = 1$ because d_m is an integer or $+\infty$.

Note that the estimate (2.2) of d_m is valid also in Examples 2 and 3. So far, nontrivial estimates of d_m have been shown for only self-similar Dirichlet forms on self-similar sets as in the examples above. In the next section, we provide a nontrivial result about d_m for more-general (not necessarily self-similar) spaces.

3 Main result

As before, let K be a locally compact separable metric space and let μ be a σ-finite Borel measure on K with full support. Suppose that we are given a strongly local regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$. We introduce some more concepts associated with $(\mathcal{E},\mathcal{F})$. For an open set $U \subset K$ and $h \in \mathcal{F}$, we say that h is harmonic on U if

$$\mathcal{E}(h,h) = \inf\{\mathcal{E}(f,f) \mid f \in \mathcal{F},\ f = h\ \mu\text{-a.e. on } U\}.$$

For a Borel set V and an open set U in K with $V \subset U$, we define the relative capacity $\mathrm{Cap}(V,U)$ by

$$\mathrm{Cap}(V,U) = \inf\left\{\mathcal{E}(g,g) \;\middle|\; \begin{array}{l} g \in \mathcal{F},\ g = 1\ \mu\text{-a.e. on a neighborhood of } V, \\ \text{and } g = 0\ \mu\text{-a.e. on } K \setminus U \end{array}\right\}.$$

For $f \in \mathcal{F}$, we define the energy measure ν_f of f as follows [6, Section 3.2]. If f is bounded, then ν_f is a positive finite Borel measure on K that is characterized by

$$\int_K \varphi\, d\nu_f = 2\mathcal{E}(f\varphi, f) - \mathcal{E}(\varphi, f^2), \quad \varphi \in \mathcal{F} \cap C_c(K).$$

For general $f \in \mathcal{F}$, the measure ν_f is defined as $\nu_f(A) := \lim_{n\to\infty} \nu_{f_n}(A)$ for Borel sets A of K, where $f_n = \max\{-n, \min\{f, n\}\}$. A Borel measure ν on K is called a minimal energy-dominant measure [10] if

(i) for every $f \in \mathcal{F}$, $\nu_f \ll \nu$;

(ii) if another σ-finite Borel measure ν' on K satisfies condition (i) with ν replaced by ν', then $\nu \ll \nu'$.

Such a measure always exists [10, Proposition 2.7] and we assume it is fixed. We introduce the following assumption.

Assumption 5. (i) There exists a family of open subsets $\{U_k^{(n)}\}_{k\in\mathbb{N},\, n\in\mathbb{N}}$ of K such that the following hold.

- For each n, $\{U_k^{(n)}\}_{k\in\mathbb{N}}$ are disjoint and $(\mu+\nu)\left(K \setminus \bigsqcup_{k\in\mathbb{N}} U_k^{(n)}\right) = 0$.
- For each n, the family $\{U_k^{(n+1)}\}_{k\in\mathbb{N}}$ is an essential subdivision of $\{U_k^{(n)}\}_{k\in\mathbb{N}}$ in the sense that, for each k, $U_k^{(n+1)} \subset U_{k'}^{(n)}$ for some k'.
- The σ-field generated by $\{U_k^{(n)};\ k \in \mathbb{N},\ n \in \mathbb{N}\} \cup \{\text{all } (\mu+\nu)\text{-null sets}\}$ includes the Borel σ-field of K.

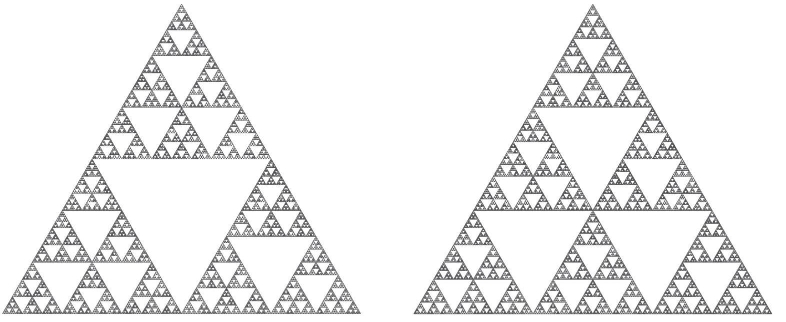

Figure 4: Examples of random recursive Sierpinski gaskets

(ii) There exist a positive constant C and a compact subset $V_k^{(n)}$ of $U_k^{(n)}$ for each n and k, such that, for every n and k,

- $\nu_h(U_k^{(n)}) \le C\nu_h(V_k^{(n)})$ for every $h \in \mathcal{F}$ that is harmonic on $U_k^{(n)}$;
- for every $f \in \mathcal{F}$ with $f = 0$ on $K \setminus V_k^{(n)}$,

$$\|f\|_{L^\infty(K,\mu)}^2 \le C\,\mathrm{Cap}(V_k^{(n)}, U_k^{(n)})^{-1}\mathcal{E}(f,f). \tag{3.1}$$

Theorem 6 ([14])**.** Under Assumption 5, $d_m = 1$.

The following are examples that satisfy Assumption 5.

(i) Dirichlet forms associated with regular harmonic structures on post-critically finite self-similar sets [15], in particular, on nested fractals. Thus, Theorem 6 includes the corresponding result in Example 3.

(ii) Canonical Dirichlet forms on random recursive Sierpinski gaskets [8] (Figure 4). This is an example in which the underlying space is a fractal set but not a self-similar one.

We give two remarks on this theorem.

- Inequality (3.1) corresponds to the case $d_s < 2$. Thus, the result is consistent with (2.2).
- When $d_s > 2$, we conjecture that the inequality (2.2) holds under Assumption 5 with "$L^\infty(K,\mu)$" in (3.1) replaced by "$L^{\frac{2d_s}{d_s-2}}(K, \mu/\mu(U_k^{(n)}))$" (possibly with suitable extra assumptions). Currently, we face some technical obstacles to handling the case $d_s \ge 2$.

References

[1] M. T. Barlow, Diffusions on fractals, in *Lectures on probability theory and statistics* (Saint-Flour, 1995), 1–121, Lecture Notes in Math. **1690**, Springer, Berlin, 1998.

[2] M. T. Barlow and R. F. Bass, The construction of Brownian motion on the Sierpiński carpet, Ann. Inst. H. Poincaré Probab. Statist. **25** (1989), 225–257.

[3] M. T. Barlow and R. F. Bass, Brownian motion and harmonic analysis on Sierpinski carpets, Canad. J. Math. **51** (1999), 673–744.

[4] M. T. Barlow, R. F. Bass, and T. Kumagai, Stability of parabolic Harnack inequalities on metric measure spaces, J. Math. Soc. Japan **58** (2006), 485–519.

[5] M. T. Barlow and E. A. Perkins, Brownian motion on the Sierpiński gasket, Probab. Theory Related Fields **79** (1988), 543–623.

[6] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*. Second revised and extended edition, de Gruyter Studies in Mathematics **19**, Walter de Gruyter, Berlin, 2011.

[7] S. Goldstein, Random walks and diffusions on fractals, in *Percolation theory and ergodic theory of infinite particle systems* (Minneapolis, Minn., 1984–1985), 121–129, IMA Vol. Math. Appl. **8**, Springer, 1987.

[8] B. M. Hambly, Brownian motion on a random recursive Sierpinski gasket, Ann. Probab. **25** (1997), 1059–1102.

[9] M. Hino, Martingale dimensions for fractals, Ann. Probab. **36** (2008), 971–991.

[10] M. Hino, Energy measures and indices of Dirichlet forms, with applications to derivatives on some fractals, Proc. Lond. Math. Soc. (3) **100** (2010), 269–302.

[11] M. Hino, Upper estimate of martingale dimension for self-similar fractals, Probab. Theory Related Fields **156** (2013), 739–793.

[12] M. Hino, Measurable Riemannian structures associated with strong local Dirichlet forms, Math. Nachr. **286** (2013), 1466–1478.

[13] M. Hino, Indices of Dirichlet forms, Sugaku Expositions **30** (2017), 187–205.

[14] M. Hino, in preparation.

[15] J. Kigami, *Analysis on fractals*, Cambridge Tracts in Mathematics **143**, Cambridge University Press, Cambridge, 2001.

[16] T. Kumagai, Estimates of transition densities for Brownian motion on nested fractals, Probab. Theory Related Fields **96** (1993), 205–224.

[17] S. Kusuoka, A diffusion process on a fractal, in *Probabilistic methods in mathematical physics* (Katata/Kyoto, 1985), 251–274, Academic Press, 1987.

[18] S. Kusuoka, Dirichlet forms on fractals and products of random matrices, Publ. Res. Inst. Math. Sci. **25** (1989), 659–680.

[19] S. Kusuoka and X. Y. Zhou, Dirichlet forms on fractals: Poincaré constant and resistance, Probab. Theory Related Fields **93** (1992), 169–196.

[20] T. Lindstrøm, *Brownian motion on nested fractals*, Mem. Amer. Math. Soc. **83** (1990), no. 420.

[21] M. Motoo and S. Watanabe, On a class of additive functionals of Markov processes, J. Math. Kyoto Univ. **4** (1964/1965), 429–469.

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