On the existence of absolutely continuous σ-finite invariant measures for random dynamical systems

Hisayoshi Toyokawa

Department of Mathematics, Hokkaido University
toyokawa@math.sci.hokudai.ac.jp

1 Introduction

Finding an absolutely continuous finite or σ-finite infinite invariant measure (σ-finite acim, for short) for a given system (described by a transformation or a Markov process) is one of the classical problems in ergodic theory. Thus, there are lots of previous researches for this problem (see for example [A97, DS66, Fog69, HK64, In12, In20, Sch95, Th80] and references therein). However, necessary and sufficient conditions for the existence of a σ-finite acim have still not been well-known. In this paper, for a given Markov operator over a probability space, we give some equivalent conditions for the existence of a σ-finite acim with certain support property. One of the equivalent conditions is weak almost periodicity of the jump operator with respect to some sweep-out set (which implies the Jacobs-de Leeuw-Glicksberg splitting theorem [E06]). Here the method of jump operators is generalization of the method of jump transformations established in [Sch95, Th80]. Because we consider general Markov operators, we can apply our setting not only to deterministic systems but also to random dynamical systems represented by null-preserving transition probabilities. Our result is applicable to certain one-dimensional random dynamical system arising from intermittent Markov maps with uniformly contractive part.

To be more precise, we consider a probability space $(X, \mathscr{F}, m)$ and a *Markov operator* P defined on $L^1 = L^1(X, \mathscr{F}, m)$ into itself, i.e, P satisfies $Pf \geq 0$ and $\|Pf\|_1 = \|f\|_1$ whenever $f \in L^1_+ = \{g \in L^1 : g \geq 0\}$. The adjoint operator of P is denoted by P^* which is defined on L^∞. Then a finite (resp. σ-finite) measure μ on $(X, \mathscr{F})$ is said to be a *finite* (resp. *σ-finite*) *acim* if μ is absolutely continuous w.r.t. m ($\mu \ll m$) and the Radon-Nikodym derivative $d\mu/dm$ is a (not identically zero) fixed point of P. Notice that the domain of P can be extended to the set of all non-negative measurable functions and the definition of a σ-finite acim makes sense even if μ is a σ-finite infinite measure. When we have a null-preserving transition probability $\mathbb{P}(x, A)$ for $x \in X$ and $A \in \mathscr{F}$ (i.e., $\mathbb{P}(x, N) = 0$ for a.e. x if $m(N) = 0$), which describes our system, the corresponding

Markov operator P is given by,

$$\int_A Pf dm = \int_X f(x)\mathbb{P}(x,A)dm(x)$$

for each $f \in L^1$ and $A \in \mathscr{F}$. If we consider a deterministic system, given by a non-singular transformation $T : X \to X$ (i.e., $m \circ T^{-1} \ll m$), the corresponding transition probability is $\mathbb{P}(x,A) = 1_{T^{-1}A}(x)$. Then the Markov operator associated to a given non-singular transformation is called the Perron-Frobenius operator P given by

$$\int_X Pf \cdot g dm = \int_X f \cdot g \circ T dm$$

for $f \in L^1$ and $g \in L^\infty$. We will show the existence of a σ-finite acim for a Markov operator in the next section. That is, we can apply our results to both non-singular transformations and null-preserving transition probabilities.

2 Main Result

In this section, we present our main results. Our results Theorem 2.2 and Theorem 2.6 give equivalent conditions for the existence of a finite or σ-finite acim with the maximal support condition for a given Markov operator. Here, "the maximal support condition" means the support of the invariant measure contains a proper sweep-out set (see Definition 2.1). That is, almost all trajectories under the process will eventually concentrate on the support of the invariant measure. Throughout this section suppμ denotes the support of μ, i.e., supp$\mu = \{\frac{d\mu}{dm} > 0\}$.

In order to state Theorem 2.2, we need the following definition of a sweep-out set.

Definition 2.1 (A sweep-out set)**.** *For a Markov operator over $L^1(X,\mathscr{F},m)$, a set $E \in \mathscr{F}$ is called a (P-)* sweep-out set *(w.r.t. m) if $\lim_{n\to\infty} (P^* I_{E^c})^n 1_X(x) = 0$ m-a.e. $x \in X$ where I_{E^c} denotes the restriction operator on E^c.*

Recall that a Markov operator P is called *weakly almost periodic* if for any $f \in L^1$ the sequence of functions $\{P^n f\}$ is weakly precompact. In the following Theorem 2.2, weak almost periodicity of a Markov operator plays a key role as an equivalent condition for the existence of a finite acim with the maximal support condition.

Theorem 2.2 ([T])**.** *Let P be a Markov operator over a probability space $(X,\mathscr{F},m)$. Then the followings are equivalent.*

1. *There exists a finite acim μ for P s.t.* suppμ *is a sweep-out set;*
2. *$\{P^n 1_X\}_n$ is weakly precompact;*
3. *P is weakly almost periodic.*

Remark 2.3. *(1) The condition 1 in Theorem 2.2 can be paraphrased:*

1'. There exists a finite acim μ for P s.t. $\lim_{n\to\infty} P^{*n} 1_{\mathrm{supp}\mu}(x) = 1$ *m-a.e.* $x \in X$.

(2) The condition in 1 " suppμ *is a sweep-out set" is a necessary condition of ergodicity of (P, m), where (P, m) is called ergodic if $E \in \mathscr{F}$ with $P^* 1_E = 1_E$ implies $E = \emptyset$ or X* (mod *m*).

We prepare the methods of inducing and jump to state Theorem 2.6, equivalent conditions for the existence of a σ-finite acim. The following definition of the induced operator or the jump operator is the generalization of the induced transformation or the jump transformation (see [A97, Fog69, In20, Sch95, Th80, T] for details).

Definition 2.4 (The induced operator/The jump operator)**.** *For a Markov operator P with a sweep-out set E, the* induced operator *P_E is defined by*

$$P_E = I_E P \sum_{n \geq 0} (I_{E^c} P)^n ,$$

and the jump operator *$\hat{P}_E$ is defined by*

$$\hat{P}_E = P I_E \sum_{n \geq 0} (P I_{E^c})^n .$$

Remark 2.5. *(1) The induced operator P_E and the jump operator $\hat{P}_E$ are also Markov operators over $L^1(X, \mathscr{F}, m)$ as long as E is sweep-out.*

(2) When P is the Perron-Frobenius operator for some non-singular transformation, the restricted induced operator $P_E I_E$ (defined on $L^1(E, \mathscr{F} \cap E, m|_E)$) and the jump operator $\hat{P}_E$ are the Perron-Frobenius operators corresponding to the induced transformation and the jump transformation, respectively.

The following theorem give equivalent conditions for the existence of a σ-finite (it might be infinite) acim with the maximal support condition. Equivalent conditions are characterized by the methods of induced operator and jump operator respectively.

Theorem 2.6 ([T])**.** *Let P be a Markov operator over a probability space $(X, \mathscr{F}, m)$. Then the followings are equivalent.*

1. *There exists a σ-finite acim μ for P s.t.* suppμ *contains a P-sweep-out set A w.r.t. m with $\mu(A) < \infty$;*
2. *There exists a sweep-out set E s.t. the induced operator P_E admits a finite acim μ_E with* supp$\mu_E = E$ (mod *m*);
3. *There exists a sweep-out set E s.t. the jump operator $\hat{P}_E$ is weakly almost periodic.*

Remark 2.7. *We can apply Theorem 2.2 to the condition 3 in Theorem 2.6. That is, we only have to check weak precompactness of $\{\hat{P}_E^n 1_X\}_n$.*

3 Example of Random Dynamical System

In this section, we apply Theorem 2.6 to certain one-dimensional random dynamical system. Our random dynamical system is random iteration of non-uniformly expanding maps which have uniformly contractive part on average. Throughout this section, our phase space $(X, \mathscr{B}(X), \lambda)$ is the unit interval with the Lebesgue measure.

Let I be an at most countable non-empty subset of $\mathbb{N}$ and for each $i \in I$, J_i be also an at most countable non-empty subset of $\mathbb{N}$. We consider $\{T_i : i \in I\}$ a family of piecewise linear Markov maps on the unit interval $X = [0, 1]$ with the Lebesgue measure λ satisfying:

(a-1) $T_i |_{X_n}: X_n \to X_{n-1}$ for $n \geq 2$ and $i \in I$, given by

$$T_i |_{X_n} (x) = \frac{n+1}{n-1}x - \frac{1}{n(n-1)};$$

(a-2) $T_i |_{X_1}: X_1 \to \bigcup_{k \in J_i} X_k$, a surjective and monotonically increasing map which is piecewise linear in the sense

$$T_i |'_{X_1} = \frac{\sum_{k \in J_i} \lambda(X_k)}{\lambda(X_1)}$$

whenever the derivative can be defined, for each $i \in I$

where $X_n = \left(\frac{1}{n+1}, \frac{1}{n}\right]$ is the cylinder of rank one for $n \geq 1$. Then the point 0 is the common fixed point for all T_i where the derivative of all T_i is one. We call a family of transformations $\{T_i : i \in I\}$ with the above conditions (a-1) and (a-2) *piecewise linear intermittent Markov maps* (with index $\{J_i\}_{i \in I}$).

By using piecewise linear intermittent Markov maps $\{T_i : i \in I\}$, we define our random dynamical system. For a given probability vector $\{p_i\}_{i \in I}$ (i.e., $p_i \geq 0$ and $\sum_{i \in I} p_i = 1$), *the random iteration of piecewise linear intermittent Markov maps of* $\{T_i, p_i : i \in I\}$ is given by the following transition probability:

$$\mathbb{P}(x, E) = \sum_{i \in I} p_i 1_E(T_i x) \quad (x \in X,\ E \in \mathscr{B}(X)) . \tag{3.1}$$

That is, each transformation T_i will be chosen with probability p_i and the selected transformation will be applied to the system. For the transition probability given by (3.1), we can define the Markov operator P on $L^1 = L^1(X, \mathscr{B}, \lambda)$ since each transformation is non-singular w.r.t. λ, which is determined by

$$\int_A P f d\lambda = \int_X f(x)\mathbb{P}(x, A) dm(x) \quad (A \in \mathscr{B}(X),\ f \in L^1).$$

Equivalently, the Markov operator associated to this random dynamical system is given by $P = \sum_{i \in I} p_i P_i$ where each P_i is the Perron-Frobenius operator corresponding to

T_i. We remark that the random iteration of piecewise linear intermittent Markov maps $\{T_i, p_i : i \in I\}$ may not satisfy expanding property on average in the following sense:

$$\sup_{X \setminus U_0} \sum_{i \in I} \frac{p_i}{|T_i'|}(x) < 1 \tag{3.2}$$

where U_0 is a small neighborhood of the indifferent fixed point 0. Indeed, if $1 \notin J_i$ for any $i \in I$ then the average of derivatives of $\{T_i, p_i : i \in I\}$ is obviously strictly less than one on X_1.

Then by Theorem 2.6 we can show the existence of a σ-finite acim for random iterations of piecewise linear intermittent Markov maps:

Proposition 3.1. *Any random iteration of piecewise linear intermittent Markov maps* $\{T_i, p_i : i \in I\}$ *admits a* σ*-finite acim.*

Remark 3.2. *If* $\{T_i, p_i : i \in I\}$ *satisfies expanding property on average in (3.2) sense, then the existence of a* σ*-finite acim was already shown in [In20] via the method of inducing. However, for random iterations of non-uniformly expanding maps which do not satisfy expanding property on average, statistical properties including the existence of a* σ*-finite acim are not well-studied. Therefore, our result could be interpreted as the first step toward the direction of non-uniformly expanding random maps with uniformly contractive part.*

References

[A97] J. Aaronson, *An introduction to infinite ergodic theory*, Mathematical Surveys and Monographs, 50. American Mathematical Society, (1997).

[DS66] D. W. Dean, and L. Sucheston, On invariant measures for operators, *Z. Wahrscheinlichkeitstheorie verw. Geb.*, **6** (1966), 1–9.

[E06] E. Y. Emel'yanov, *Non-spectral Asymptotic Analysis of One-Parameter Operator Semigroups*, Operator Theory Advances and Applications, Vol.173 (2006).

[Fog69] S. R. Foguel, *The ergodic theory of Markov processes*, Van Nostrand Mathematical Studies, No. 21. Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, (1969).

[HK64] A. Hajian, and S. Kakutani, Weakly wandering sets and invariant measures, *Trans. Amer. Math. Soc.*, **110** (1964), 136–151.

[In12] T. Inoue, Invariant measures for position dependent random maps with continuous random parameters, *Studia Math.*, **208** (2012), 11–29.

[In20] T. Inoue, First return maps of random map and invariant measures, *Nonlinearity*, **33** (2020), 249–275.

[Sch95] F. Schweiger, *Ergodic theory of fibred systems and metric number theory*, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, (1995).

[Th80] M. Thaler, Estimates of the invariant densities of endomorphisms with indifferent fixed points, *Israel J. Math.*, **37** (1980), 303–314.

[T] H. Toyokawa, σ-finite invariant densities for eventually conservative Markov operators, *Discrete Continuous Dynam. Systems - A*, to appear.