# Sample-wise central limit theorem with deterministic centering for nonsingular random dynamical systems

Takehiko Morita[1]
Department of Mathematics,
Graduate School of Science, Osaka University

## 1. Introduction

Let $(M, \mathcal{B}, m)$ be a probability space. Consider a family $\{\tau_s\}_{s\in S}$ of $m$-nonsingular transformations on $(M, \mathcal{B}, m)$ indexed by a measurable space $(S, \mathcal{B}(S))$ such that the map $S \times M \ni (s, x) \mapsto \tau_s x \in M$ is $(\mathcal{B}(S) \times \mathcal{B})/\mathcal{B}$-measurable. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\sigma : \Omega \to \Omega$ a $P$-preserving transformation. Take an $S$-valued random variable $\xi$ on $(\Omega, \mathcal{F}, P)$ and define an $S$-valued strictly stationary process $\{\xi_n\}_{n=0}^{\infty}$ by $\xi_n = \xi \circ \sigma^n$ $(n \geq 0)$. The family $\mathcal{X} = \{X_n\}$ of random maps $X_n : M \to M$ is called *the random dynamical system given by* $(\{\tau_s\}_{s\in S}, \sigma, \xi)$ if the maps in $\mathcal{X}$ are defined by

$$X_0(\omega)x = x, \;\; X_{n+1}(\omega)x = \tau_{\xi_n(\omega)} X_n(\omega)x \text{ for } (x, \omega) \in M \times \Omega, \quad (n \geq 0)$$

For a random dynamical system $\mathcal{X}$ given by $(\{\tau_s\}_{s\in S}, \sigma, \xi)$, we introduce its (*direct*) *product* as the random dynamical system given by $(\{\tau_s \times \tau_s\}_{s\in S}, \sigma, \xi)$ and we denote it by $\mathcal{X} \times \mathcal{X}$. Following Kakutani [8], we introduce the skew product transformations $T_1 : M \times \Omega \to M \times \Omega$ and $T_2 : M^2 \times \Omega \to M^2 \times \Omega$ corresponding to $\mathcal{X}$ and $\mathcal{X} \times \mathcal{X}$ by

$$T_1(x, \omega) = (X_1(\omega)x, \sigma\omega) \quad \text{for } (x, \omega) \in M \times \Omega$$

and

$$T_2(x, y, \omega) = (X_1(\omega)x, X_1(\omega)y, \sigma\omega) \quad \text{for } (x, y, \omega) \in M^2 \times \Omega,$$

respectively.

In this article we consider the case where there exists a unique $m \times P$-absolutely continuous probability measure $Q_1$ with density $H_1$ such that the measure-preserving dynamical system $(T_1, Q_1)$ is exact, i.e. $\bigcap_{n=0}^{\infty} T_1^{-n}(\mathcal{B} \times \mathcal{F})$ is trivial with respect to $Q_1$ (see [6] Chapter 10 for details). To make things much simpler we assume that the transfer operator $\mathcal{L}_{T_1} = \mathcal{L}_{T_1, m\times P}$ of $T_1$ with respect to $m \times P$ is asymptotically stable. (Furthermore, we will need to assume the asymptotic stability of $\mathcal{L}_{T_2} = \mathcal{L}_{T_2, m^2\times P}$ later). Then the $m$-absolutely continuous probability measure $\rho \cdot m$ on $\mathcal{B}$ with density $\rho(\cdot) = \displaystyle\int_{\Omega} H(\cdot, \omega)\, P(d\omega)$ can be regarded as the physical measure for $P$-almost every sample $\omega$. In fact, for any observable $f$ in $L^{\infty}(m)$ there exists $\Gamma \in \mathcal{F}$ with $P(\Gamma) = 1$ such that $\omega \in \Gamma$ yields that $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_n(\omega)x) = \int_X f \rho \, dm$ $\;m$-a.e. $x$. So it is not too much to say that 'almost

[1]Partially supported by the Grant-in-Aid for Scientific Research (B) 22340034, Japan Society for the Promotion of Science.

sure sample-wise or quenched strong law of large numbers' holds. Therefore it is only natural to consider the central limit theorem (abbr. CLT) for the sequence $\{f(X_n(\omega)x)\}_{n\geq 0}$ as the next problem.

For an observable $f$ we put $S_n(T_1)f(x,\omega) = \sum_{k=0}^{n-1} f(T_1^n(x,\omega)) = \sum_{k=0}^{n-1} f(X_n(\omega)x)$ for $(x,\omega) \in M \times \Omega$. Our main concern is the sample-wise (quenched) central limit problem for the sequence $\{X_n(\omega)\}_{n\geq 0}$ of random variables on the space $(X, \mathcal{B}, m)$ with fixed $\omega \in \Omega$. More practically, we consider the problem allowing an exceptional $P$-null set, i.e. the study of the condition for that there exists $\Gamma \in \mathcal{F}$ with probability one such that for any $\omega \in \Gamma$, the random variable on the probability space $(M, \mathcal{B}, m)$ obtained by making an appropriate centering on $S_n(T_1)f(\cdot,\omega)$ converges in law to a normal distribution as $n \to \infty$.

From general theory of weakly dependent stationary sequence of random variables, the sample-averaged (annealed) CLT with deterministic centering holds true for a large class of observables $f$, i.e., $Z_n(T_1)f = (1/\sqrt{n})(S_n(T_1)f - n\int_M f\,\rho\,dm)$ converges in law to an appropriate normal distribution $N(0, v(f))$ with respect to $m \times P$ as $n \to \infty$ (see [5] and [7]). Moreover, we can show that such a sample-averaged CLT implies that for any $u \in C_0(\mathbb{R})$, the sequence $\int_M u(Z_n(T_1)f)\,dm$ in $L^1(P)$ converges to $\int_{\mathbb{R}} u(t)\,N(0, v(f))(dt)$ weakly in $L^1(P)$ as $n \to \infty$ (see [11] and cf.Theorem 3.1 below), where $C_0(\mathbb{R})$ denotes the totality of continuous functions on $\mathbb{R}$ with compact support. Therefore this fact may lead us to wishful thinking that the sample-wise CLT with the same deterministic centering also holds true. But the results in [12] (see also [9]) make us recognize that the noise dependent centering $S_n(T_1)f(\cdot,\omega) - \sum_{k=0}^{n-1}\int_M f(X_k(\omega)y)\,H(y,\omega)\,m(dy)$ seems natural and even proper in view of the martingale approximation for stationary processes. In fact, a counter-example given in Section 4.4 of [1] (see also Example 4.2 in Section 4) shows that the deterministic centering does not work in general, even in the case where the $\{\xi_n\}_{n\geq 0}$ is independent and identically distributed if we consider 'almost sure sample-wise CLT'. Therefore, it turns out that some of observations and assertions in [11] have difficulties and fail to hold.

The purpose of the article is to announce the author's recent results on the condition for that the sample-averaged CLT for $S_n(T_1)f/\sqrt{n}$ implies the a.s.sample-wise CLT with the same deterministic centering. To be more precise, the following results are stated in Section 3. First we give the conditions for the sample-averaged CLT under the asymptotic stability of the transfer operator for the skew-product transformation $T_1$. One may notice that Theorem 3.1 is a sort of corrected version of Proposition 3.2 in [11]. Next we discuss about the conditions which guarantee that the sequence of random variables

$\int_M u(S_n(T_1)f/\sqrt{n})\,dm$ in the above converges strongly to $\int_{\mathbb{R}} u\,dN(0,v(f))$ in $L^1(P)$ (Theorem 3.2). Finally, we shall announce an a.s.sample-wise CLT with deterministic centering under some additional conditions (Theorem 3.4). Section 4 is devoted to examples. On account of limited space we have to restrict ourselves just give the statements of theses results. So proofs and details will be given elsewhere.

## 2. Asymptotic stability of transfer operator

In order to state the present results, we need to introduce the notion of asymptotic stability of the transfer operator. Let $(M,\mathcal{B},m)$ be a probability space and $\tau$ an $m$-nonsingular transformation. Then *the transfer operator*(*the Perron-Frobenius operator*) $\mathcal{L}_\tau=\mathcal{L}_{\tau,m}$ of $\tau$ with respect to $m$ is defined to be the operator on $L^1(m)$ satisfying

$$\int_M f\cdot\mathcal{L}_\tau g\,dm=\int_M (f\circ\tau)g\,dm \quad \text{for } f\in L^\infty(m) \text{ and } g\in L^1(m).$$

For the sake of convenience we put

$$\Delta(\tau,g,n;h)=\mathcal{L}_\tau^n g-\left(\int_M g\,dm\right)h \qquad \text{for } g,\,h\in L^1(m).$$

The transfer operator $\mathcal{L}_\tau$ is said to be *asymptotically stable with respect to* $h\in L^1(m)$ if for each $g\in L^1(m)$ $\|\Delta(\tau,g,n;h)\|_{1,m}\to 0\,(n\to\infty)$ holds. Note that such an $h$ in the condition turns out to be a unique $\tau$-invariant density and the measure-theoretic dynamical system $(\tau,hm)$ is exact i.e. $\bigcap_{n=0}^\infty \tau^{-n}\mathcal{B}$ is trivial with respect to the $m$-absolutely continuous $\tau$-invariant measure $hm$ with density $h$.

Consider the random dynamical system $\mathcal{X}$ given by $(\{\tau_s\}_{s\in S},\sigma,\xi)$ and its product $\mathcal{X}\times\mathcal{X}$. Since $T_1$ and $T_2$ are $m\times P$-nonsingular and $m^2\times P$-nonsingular, respectively, we can consider the transfer operators $\mathcal{L}_{T_1}=\mathcal{L}_{T_1,m\times P}$ on $L^1(m\times P)$ and $\mathcal{L}_{T_2}=\mathcal{L}_{T_2,m^2\times P}$ on $L^1(m^2\times P)$, respectively. It is not hard to see that if $\mathcal{L}_{T_2}$ is asymptotically stable with respect to $H_2\in L^1(m^2\times P)$, so is $\mathcal{L}_{T_1}$ with respect to $H_1\in L^1(m\times P)$, where $H_1(x,\omega)=\int_M H_2(x,y,\omega)\,m(dy) \quad ((x,\omega)\in M\times\Omega)$.

## 3. Results

First we state that the sample-averaged CLT is equivalent to a sort of weak $L^1$-version of sample-wise CLT if $\mathcal{L}_{T_1}$ is asymptotically stable. Precisely, we have the following.

Theorem 3.1. *Consider the random dynamical system given by* $(\{\tau_s\}_{s\in S},\sigma,\xi)$ *such that the transfer operator* $\mathcal{L}_{T_1}$ *on* $L^1(m\times P)$ *is asymptotically stable. Let* $N(0,v)$ *denote the normal distribution with mean* $0$ *and variance* $v\geq 0$, *where* $N(0,0)$ *is regarded as the point mass* $\delta_0$ *at* $0\in\mathbb{R}$. *Let* $f$ *be a real-valued element in* $L^\infty(m)$ *satisfying the deterministic centering condition* $\int_m f\rho\,dm=0$, *where* $\rho$ *is defined as* $\rho(\cdot)=\int_\Omega H_1(\cdot,\omega)P(d\omega)$ *by using* *the unique invariant density* $H_1$ *for* $T_1$ *with respect to* $m \times P$. *Then the following are equivalent.*

(1) $S_n(T_1)f/\sqrt{n}$ *converges in law to* $N(0, v)$ *with respect to some* $m \times P$*-absolutely continuous probability measure.*

(2) $S_n(T_1)f/\sqrt{n}$ *converges in law to* $N(0, v)$ *with respect to any* $m \times P$*-absolutely continuous probability measure.*

(3) *There exists a probability density* $g \in L^1(m)$ *such that for any bounded continuous function* $u$ *on* $\mathbb{R}$, $\int_M u(S_n(T_1)f(x, \cdot)/\sqrt{n})g(x)\, dm$ *converges weakly to* $\int_{\mathbb{R}} u(t)\, N(0, v)(dt)$ *in* $L^1(P)$ *as* $n \to \infty$.

(4) *For any bounded continuous function* $u$ *on* $\mathbb{R}$ *and any probability density* $g \in L^1(m)$, $\int_M u(S_n(T_1)f(x, \cdot)/\sqrt{n})g(x)\, dm$ *converges to* $\int_{\mathbb{R}} u(t)\, N(0, v)(dt)$ *weakly in* $L^1(P)$ *as* $n \to \infty$.

(5) *There exists a probability density* $g \in L^1(m)$ *such that for any* $t \in \mathbb{R}$, $\int_M e^{\sqrt{-1}t(S_n(T_1)f(x,\cdot)/\sqrt{n})}g(x)\, dm$ *converges to* $e^{-vt^2/2}$ *weakly in* $L^1(P)$ *as* $n \to \infty$.

(6) *For any* $t \in \mathbb{R}$ *and any probability density* $g \in L^1(m)$, $\int_M e^{\sqrt{-1}t(S_n(T_1)f(x,\cdot)/\sqrt{n})}g(x)\, dm$ *converges to* $e^{-vt^2/2}$ *weakly in* $L^1(P)$ *as* $n \to \infty$.

For $f \in L^1(m)$, we denote by $F_f$ and $\tilde{f}$ the elements in $L^1(m^2)$ defined by $F_f(x, y) = f(x) - f(y)$ and $\tilde{f}(x, y) = f(x)f(y)$, respectively. For $\mathcal{B}$-measurable function $f$ and $\mathcal{B}^2$-measurable function $F$, we abuse the notation $S_n f(x, \omega)$ and $S_n F(x, y, \omega)$ to denote $S_n(T_1)f(x, \omega)$ and $S_n(T_2)F(x, y, \omega)$, respectively if there is no fear of confusion. Note that if $f \in L^\infty(m)$ satisfies the deterministic centering condition $\int_M f\rho\, dm = 0$ with respect to $T$, then $\tilde{f}$ also satisfies the deterministic centering condition $\int_{M^2} \tilde{f}\rho_2\, dm^2 = 0$ with respect to $T_2$, where $\rho_2(x, y) = \int_\Omega H_2(x, y, \omega)\, P(d\omega)$. Indeed,

$$\begin{aligned}\int_{M^2} \tilde{f}\rho_2\, dm^2 &= \int_{M^2} \left( (f(x) - f(y)) \int_\Omega H_2(x, y, \omega)\, dP \right) dm^2 \\ &= \int_{M^2 \times \Omega} (f(x) - f(y)) H_2(x, y, \omega)\, d(m^2 \times P) = 0.\end{aligned}$$

holds true since $H_2$ is symmetric.

From now on we impose the stronger condition that $\mathcal{L}_{T_2}$ is asymptotically stable on our system. Recall that this yields that $\mathcal{L}_{T_1}$ is asymptotically stable. For $\Phi \in L^1(m \times P)$,

$\Psi \in L^1(m^2 \times P)$, and nonnegative integer $n$, we put

$$\Delta(T_1, \Phi, n) = \mathcal{L}_{T_1}^n \Phi - \int_{M\times\Omega} \Phi \, d(m \times P) \cdot H_1, \text{ and}$$

(3.1)

$$\Delta(T_2, \Psi, n) = \mathcal{L}_{T_2}^n \Psi - \int_{M^2\times\Omega} \Psi \, d(m^2 \times P) \cdot H_2.$$

Next for a real-valued element $f \in L^\infty(m)$, we consider autocorrelations

$$C(T_1, f, n) = \int_{M\times\Omega} (f \circ T_1^n) f H_1 \, d(m \times P) - \left( \int_{M\times\Omega} f H_1 \, d(m \times \Omega) \right)^2 \text{ and}$$

(3.2)

$$C(T_2, F_f, n) = \int_{M^2\times\Omega} (F_f \circ T_2^n) f H_2 \, d(m^2 \times P) - \left( \int_{M^2\times\Omega} F_f H_2 \, d(m^2 \times \Omega) \right)^2.$$

Under the condition

$$\text{(3.3)} \qquad \sum_{n=0}^{\infty} \|\Delta(T_1, fH_1, n)\|_{1,m\times P} < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \|\Delta(T_2, F_f H_2, n)\|_{1,m^2\times P} < \infty,$$

the limit variances are given by the following absolutely convergent series.

$$\text{(3.4)} \quad v(f) = C(T_1, f, 0) + 2\sum_{n=1}^{\infty} C(T_1, f, n) \text{ and } v(F_f) = C(T_2, F_f, 0) + 2\sum_{n=1}^{\infty} C(T_2, F_f, n).$$

Now we are in a position to state the next result.

THEOREM 3.2. *Assume that the transfer operator of the slew product transformation $T_2$ with respect to $m^2 \times P$ is asymptotically stable with respect to $H_2 \in L^1(m^2 \times P)$. Let $f$ be a real-valued element in $L^\infty(m)$ satisfying the deterministic centering condition $\int_M f\rho \, dm = 0$. Furthermore, we assume that $f$ and $F_f$ satisfy the conditions in* (3.3). *Then the following are equivalent.*

(1) *There exists a probability density $g \in L^1(m)$ such that $S_n F_f/\sqrt{n}$ converges in law to $N(0, 2v(f))$ with respect to $m^2 \times P$-absolutely continuous probability measure with density $\tilde{g}$.*

(2) *For any probability density $g \in L^1(m)$, $S_n F_f/\sqrt{n}$ converges in law to $N(0, 2v(f))$ with respect to $m^2 \times P$-absolutely continuous probability measure with density $\tilde{g}$.*

(3) *There exists a probability density $g \in L^1(m)$ such that for any bounded continuous function $u$ on $\mathbb{R}$, $\int_M u(S_n/\sqrt{n}) g \, dm$ converges to $\int_{\mathbb{R}} u \, dN(0, v)$ strongly in $L^1(P)$ as $(n \to \infty)$.*

(4) *For any probability density $g \in L^1(m)$ and for any bounded continuous function $u$ on $\mathbb{R}$, $\int_M u(S_n/\sqrt{n}) g \, dm$ converges to $\int_{\mathbb{R}} u \, dN(0, v)$ strongly in $L^1(P)$ as $(n \to \infty)$.*

(5) *There exists a probability density $g \in L^1(m)$ such that for any $t \in \mathbb{R}$, $\int_M e^{\sqrt{-1}t(S_n/\sqrt{n})} g\, dm$ converges to $e^{-vt^2/2}$ strongly in $L^1(P)$ as $(n \to \infty)$.*

(6) *For any probability density $g \in L^1(m)$ and for any $t \in \mathbb{R}$, $\int_M e^{\sqrt{-1}t(S_n/\sqrt{n})} g\, dm$ converges to $e^{-vt^2/2}$ strongly in $L^1(P)$ as $(n \to \infty)$.*

(7) $v(F_f) = 2v(f)$.

(8) $$\int_{M^2\times\Omega} \tilde{f} H_2\, d(m^2 \times P) + 2\sum_{n=1}^{\infty} \int_{M^2\times\Omega} f(x) f(X_n(\omega)y) H_2\, d(m^2 \times P) = 0.$$

(9) $$\lim_{n\to\infty} \frac{1}{n} \int_{M^2\times\Omega} S_n f(x,\omega) S_n f(y,\omega) H_2\, d(m^2 \times P) = 0.$$

Remark 3.3. In the assertion (9) in Theorem 3.2, the single point set $\{H_2\}$ can be replaced by the totality of probability densities in $L^1(m^2 \times P)$ for which $\|\Delta(T_2, F, n)\|_{1,m^2\times P}$ decays sufficiently fast. For example, under the same conditions in Theorem 3.2, (9) is equivalent to the following.

(9)′ $\lim_{n\to\infty} \frac{1}{n} \int_{M^2\Omega} S_n f(x,\omega) S_n f(y,\omega) F(x,y,\omega)\, d(m^2 \times P) = 0$ holds for any probability density $F \in L^1(m^2 \times P)$ satisfying $\lim_{n\to\infty} n^5 \|\Delta(T_2, F, n)\|_{1,m^2\times P} = 0$.

We note that the condition '$\lim_{n\to\infty} n^5 \|\Delta(T_2, F, n)\|_{1,m^2\times P} = 0$' is rather technical and it is not optimal.

Finally, we give the almost sure sample-wise CLT with deterministic centering. In [1], [2], and [3] a kind of quenched CLT(called almost sure sample-wise CLT in this article) with deterministic centering condition is obtained by using an averaged large deviation estimate. Here we give a slightly abstract result by using the estimate of fourth moment rather than the large deviation estimate. Precisely, we have the following.

Theorem 3.4. *Assume that the transfer operator of the slew product transformation $T_2$ with respect to $m^2 \times P$ is asymptotically stable. Let $f$ be a real-valued element in $L^\infty(m)$ satisfying the deterministic centering condition $\int_M f \rho\, dm = 0$ such that $f$ and $F_f$ satisfy the conditions in (3.3) and $v(T_2, F_f) = 2v(T_1, f)$ holds. Let $g \in L^1(m)$ be a probability density. We assume that there exist constants $C > 0$, $\alpha > 0$, and $\beta > 0$ such that the following conditions are fulfilled.*

(a) $$\left| \int_{M\times\Omega} e^{\sqrt{-1}tS_n f/\sqrt{n}} H_1\, d(m \times P) - e^{-vt^2/2} \right| \le C\,(1 + |t|^\alpha)\, \frac{1}{n^\beta}. \text{ and}$$
$$\left| \int_{M^2\times\Omega} e^{\sqrt{-1}tS_n F_f/\sqrt{n}} H_2\, d(m^2 \times P) - e^{-vt^2} \right| \le C\,(1 + |t|^\alpha)\, \frac{1}{n^\beta}, \text{ where } v = v(f).$$

(b) $\sup_n \int_{M\times\Omega} \frac{(S_n f)^4}{n^2} H_1 \, d(m\times P) \leq C.$

(c) $\|\Delta(T_1, g, n)\|_{1,m\times P} \leq \frac{C}{n^2}$ *and* $\|\Delta(T_2, \tilde{g}, n)\|_{1,m^2\times P} \leq \frac{C}{n^2}.$

*Then for $P$-a.s.$\omega$, $S_n f(\cdot,\omega)/\sqrt{n}$ converges in law to the normal distribution $N(0,v)$ with respect to the $m$-absolutely continuous probability measure $gm$, where $\tilde{g} \in L^1(m^2)$ is defined by $\tilde{g}(x,y) = g(x)g(y)$ for $(x,y) \in M^2$.*

## 4. Examples

Throughout this section, $(M,\mathcal{B},m)$ is the unit interval with the usual Lebesgue space structure, $S = \{0,1\}$ with the discrete topology, and $(\Omega,\mathcal{F}) = (S^{\mathbb{Z}_+}, \mathcal{B}(S^{\mathbb{Z}_+}))$, where $Z_+$ is the totality of nonnegative integers and $\mathcal{B}(S^{\mathbb{Z}_+})$ is the topological Borel field of $S^{\mathbb{Z}_+}$ endowed with the product topology of $S$. Each element $\omega \in \Omega$ can be expressed as $\omega = (\omega(0)\omega(1)\omega(2)\cdots)$. Thus the shift $\sigma : \Omega \to \Omega$ satisfies $(\sigma\omega)(k) = \omega(k)$ for $k \in \mathbb{Z}_+$. Choose $p$ and $q$ satisfying $0 < p,\ q < 1$ and $p+q = 1$ and define a locally constant function $U$ so that

$$U(\omega) = \begin{cases} \log p & \text{if } \omega(0) = \omega(1), \\ \log q & \text{if } \omega(0) \neq \omega(1). \end{cases}$$

Let $P = P_U$ be the Gibbs measure corresponding to the potential $U$, i.e. the unique $\sigma$-invariant probability measure satisfying $\int_\Omega (f\circ\sigma) g \, dP = \int_\Omega f \mathcal{L}_U g \, dP$ for continuous functions $f$ and $g$, where $\mathcal{L}_U$ is the transfer operator defined by $\mathcal{L}_U g(\omega) = e^{U(0\omega)} g(0\omega) + e^{U(1\omega)} g(1\omega)$ (see [4]). As an $S$-valued random variable $\xi : \Omega \to S$, we employ $\xi(\omega) = \omega(0)$. Clearly, $P$ makes $\{\xi_n = \xi\circ\sigma^n,\ n \geq 0\}$ a symmetric stationary Markov chain.

Put $I(0) = [0,1/2]$ and $I(1) = [1/2,1]$. Let $\mathcal{T}$ be the totality of piecewise $C^2$ uniformly expanding maps $\tau : M \to M$ such that $\text{ess.inf}|d\tau/dx| > 1$ and $\tau|_{\text{int}I(j)} : \text{int}I(j) \to M$ is a $C^2$-diffeomorphism which can be extended to a $C^2$ function on $I(j)$ for $j = 0,\ 1$. It is well known that for each $\tau \in \mathcal{T}$ there exists a unique invariant density $h_\tau \in L^1(m)$ with Lipschitz continuous version and the transfer operator $\mathcal{L}_{\tau,m}$ is asymptotically stable with respect to $h_\tau$. Choose $\tau_0,\ \tau_1 \in \mathcal{T}$ and consider a random dynamical system $\mathcal{X}$ given by $(\{\tau_0,\ \tau_1\}, \sigma, \xi)$. Write as $h_0 = h_{\tau_0}$ and $h_1 = h_{\tau_1}$ for simplicity. Note that the transfer operators of the skew product transformations $T_1$ and $T_2$ with respect to $m\times P$ and $m^2 \times P$ are given by

$$\begin{aligned} \mathcal{L}_{T_1}\Phi(x,\omega) &= e^{U(0\cdot\omega)}\mathcal{L}_{\tau_0}(\Phi(\cdot,0\omega))(x) + e^{U(1\cdot\omega)}\mathcal{L}_{\tau_1}(\Phi(\cdot,1\omega))(x) \\ \mathcal{L}_{T_2}\Psi(x,y,\omega) &= e^{U(0\cdot\omega)}\mathcal{L}_{\tau_0\times\tau_0}(\Psi(\cdot,\cdot,0\omega))(x,y) + e^{U(1\cdot\omega)}\mathcal{L}_{\tau_1\times\tau_1}(\Psi(\cdot,\cdot,1\omega))(x,y) \end{aligned} \tag{4.1}$$

for $\Phi \in L^1(m\times P)$ and $\Psi \in L^1(m^2\times P)$, where $\mathcal{L}_{\tau_j} = \mathcal{L}_{\tau_j,m}$ is the transfer operator of $\tau_j$ with respect to the Lebesgue measure $m$ for $j = 0,\ 1$. We have the following.

PROPOSITION 4.1. *Let $(M, \mathcal{B}, m)$, $(\Omega, \mathcal{F}, P, \sigma)$, and $\{\tau_0, \tau_1\} \subset \mathcal{T}$ be as above. Consider the random dynamical system $\mathcal{X}$ given by $(\{\tau_0,\ \tau_1\}, \sigma, \xi)$ and its direct product $\mathcal{X} \times \mathcal{X}$. Then there exist a probability densities $H_1 \in L^1(m \times P)$ and $H_2 \in L^1(m^2 \times P)$ such that the transfer operators $\mathcal{L}_{T_1}$ and $\mathcal{L}_{T_2}$ are asymptotically stable with respect to $H_1$ and $H_2$, respectively. Moreover, $H_1$ and $H_2$ have continuous versions and for any Lipschitz continuous real-valued function $f$ on $M$, the autocorrelations given by* (3.2) *decay exponentially fast as $n \to \infty$.*

Note that if $p = q = 1/2$, then $\{\xi_n\}_{n=0}^{\infty}$ is independent and densities $H_1$ and $H_2$ have deterministic versions by virtue of the result in [10]. On the contrary, if $h_0 \neq h_1$ we can show by using the formula (4.1) that $p \neq q$ implies that $H_1$ and $H_2$ can not have deterministic versions.

First we give an example for which the deterministic centering does not work.

EXAMPLE 4.2. (cf. [1]) Choose $\tau_0$ and $\tau_1$ so that

$$\tau_0 x = \begin{cases} 2x, & \text{if } x \in I(0) \\ 2x-1, & \text{otherwise} \end{cases}, \quad \tau_1 x = \begin{cases} x^2 + (3/2)x, & \text{if } x \in I(0) \\ x^2 + (1/2)x - 1/2, & \text{otherwise} \end{cases}.$$

Clearly $h_0 = 1$. By $\mathcal{L}_{\tau_1} h_1 = h_1$, it is not hard to see that $h_1(0) = 2h_1(1/2)$ and $h_1(1) = (2/3)h_1(1/2)$. Therefore $h_0 \neq h_1$. Consider the observable $f = h_0 - h_1$. Then we have $\displaystyle\int_M f h_0\, dm - \int_M f h_1\, dm = \int_M (h_0 - h_1)^2\, dm > 0$. Thus the symmetricity of the Markov chain $\{\xi_n\}_{n=0}^{\infty}$ enable us to prove $v(F_f) < 2v(f)$ in the same way as in [1]. Note that we also show that $v(f) > 0$.

It is obvious that if $h_0 = h_1$ the deterministic centering trivially works. Therefore we are interested in a nontrivial example for which the deterministic centering works.

EXAMPLE 4.3. Let $\iota : M \to M$ be the involution $\iota x = 1 - x$. Choose $\tau_j \in \mathcal{T}$ such that $\iota \circ \tau_j = \tau_j \circ \iota$ holds for $j = 0, 1$. Then it is not hard to see that the invariant densities $H_1$ and $H_2$ satisfy $H_1(\iota x, \omega) = H_1(x, \omega)$ and $H_2(\iota x, y, \omega) = H_2(x, \iota y, \omega) = H_2(\iota x, \iota y, \omega) = H_2(x, y, \omega)$. Therefore if $f$ satisfies $f \circ \iota = -f$, we have $\displaystyle\int_M f(x) H_2(x, y, \omega)\, m(dx) = 0$. This yields the condition (8) in Theorem 3.2. To be more concrete choose $\tau_0$ and $\tau_1$ so that

$$\tau_0 x = \begin{cases} 2x, & \text{if } x \in I(0) \\ 2x-1, & \text{otherwise} \end{cases}, \quad \tau_1 x = \begin{cases} x^2 + (3/2)x, & \text{if } x \in I(0) \\ -x^2 + (7/2)x - 3/2, & \text{otherwise} \end{cases}.$$

Consider the observable $f(x) = \cos(k\pi x)$ with $k$ odd. Obviously $f \circ \iota = -f$. Moreover, we can show that Theorem 3.4 is applicable to $f$ with $v(f) > 0$.

## References

[1] M. Abdelkader and R. Aimino, *On the quenched central limit theorem for random dynamical systems*, J. Phys. A : Math. Theor. **49** (2016), 1–13.

[2] A. Ayyer, C. Liverani, and M. Stenlund, *Quenched CLT for random toral automorphism*, Discrete Contin. Dyn. Syst **24** (2009), 331–348.

[3] R. Aimino, M. Nicol, and S. Vaienti *Annealed and quenched limit theorems for random expanding dynamical systems*, Probab. Theory Relat. Fields **162** (2015), 233–274.

[4] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Lecture Note in Math. **470**, Springer Verlag, Berlin-Hedelberg-New York 1975.

[5] R. C. Bradley, Introduction to strong mixing conditions, Vol II, Kendrick Press, Heber City 2007.

[6] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai Ergodic theory, Springer, Berlin-Hedelberg-New York 1980.

[7] I. A. Ibragimov and Yu. V. Linnik, Independent and stationary sequences of random variables , Wolters-Noordhoff, Groningen 1971.

[8] S. Kakutani, *Random ergodic theorem and Markoff processes with a stable distribution*, Proc. 2nd. Berkeley (1957), 241–261.

[9] Y. Kifer, *Limit theorems for random transformations* and processes in random environments, Trans. Amer. Math. Soc. **350** (1998), 1481–1518.

[10] T. Morita, *Deterministic version lemmas in ergodic theory of random dynamical systems*, Hiroshima Math. J. **18** (1988) 15–29.

[11] T. Morita, *Asymptotic behavior of one-dimensional random dynamical systems— revisit*, RIMS Kôkyûroku **1942** (2015) 31–43.

[12] B.-Z. Rubshtein, *A central limit theorem for conditional distributions* Convergence in ergodic theory and probability ed. Bergelson, March, and Rosenblatt, 373–380, Ohio State Univ. Math. Res. Inst. Publ. **5**, de Gruyter, Berlin, 1996.

Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka, Osaka 560-0043
JAPAN

大阪大学・大学院理学研究科　盛田 健彦