Infinite Invariant Measure in a Heterogeneous Accumulation Process

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Abstract

We provide an exact form of the infinite invariant density in a semi-Markov process. In the semi-Markov process, the state almost surely converges to zero in the long-time limit, i.e., accumulation to the origin, and the infinite invariant density characterizes the accumulation process. We demonstrate two distributional behaviors for time averages of the absolute value of the state, where the shape of the distribution depends on whether the observable is integrable with respect to the infinite invariant density. Therefore, the infinite invariant density plays an important role in characterizing the non-stationary accumulation process.

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I. INTRODUCTION

Chaos plays a fundamental role in providing a stochastic description from a deterministic dynamical system [1]. This stochastic approach enables us to use the micro-canonical ensemble in a Hamiltonian system. In mathematics, this fundamental property can be formulated by ergodicity, which guarantees the equivalence between a time-averaged quantity and the corresponding ensemble average. In particular, time averages starting from different initial points converge to a unique constant and the constant is given by the average with respect to the invariant density. This uniqueness results from the uniqueness of the invariant density (the invariant measure absolutely continuous with respect to the Lebesgue measure is unique). For Hamiltonian systems, the invariant measure is the Lebesgue measure, which implies the micro-canonical ensemble.

Mathematically, a concept of ergodicity can be generalized to the infinite measure space where the invariant density cannot be normalized (*the infinite invariant density*) [2]. A significant feature in infinite ergodic theory is distributional behaviors in time-averaged observables [2–6]. For ordinary ergodic systems, a time-averaged observable converges to a constant, which means that the distribution of the time average is the delta function. On the other hand, time-averaged observables are still random variables even in the long-time limit when dynamical systems are ergodic in the infinite measure space. These distributional limit theorems in infinite ergodic theory are deeply connected to those in probability theory. In particular, the Mittag-Leffler distribution and the generalized arcsine distribution, which appears in infinite ergodic theory, are well-known in occupation time statistics for stochastic processes [7, 8].

While this generalization of ergodicity is independently developed from physics, relevant physical phenomena has been unveiled recently, e.g., intensity of fluorescence in quantum dots [9], diffusion coefficient of a diffusing biomolecule in living cells [10–12], and interface fluctuations in Kardar-Parisi-Zhang universality class [13], where distributional behaviors of time averages are observed, which are similar to infinite ergodic theory. In infinite ergodic theory, integrability of the observable with respect to the invariant density gives a condition that discriminates distributional limit theorems. Therefore, it is important to know an exact form of the infinite invariant density, which is not known explicitly in many cases. Here, we provide an exact form of the infinite invariant density and demonstrate the role of the infinite invariant density in an accumulation process.

II. CONTINUOUS SEMI-MARKOV PROCESS

We consider a continuous semi-Markov process (CSMP), where the state may have any value. In CSMPs, the state keeps constant until it changes (see Fig. 1) and an inter-event time of two successive points of state changes is an independent and identically distributed (IID) random variable. Here, we assume that the state value is determined by the inter-event time [14]. As a specific coupling, the absolute values $|v_n|$ of the nth state value and the corresponding inter-event time τ_n are connected by

$$|v_n| = \tau_n^{\nu-1}, \tag{1}$$

or equivalently via

$$\tau_n = |v_n|^{\frac{1}{\nu-1}}. \tag{2}$$

The sign of the state value becomes + or - with probability 1/2. A typical physical example of this process is the velocity of the generalized Lévy walk (GLW) [15, 16]. In the GLW a walker moves with constant velocities v_n over time segments of lengths τ_n between turning points occurring at times t_n, i.e, $\tau_n = t_n - t_{n-1}$, where flight duration τ_n is an IID random variable. Standard Lévy walk corresponds to case $\nu = 1$, implying the velocity does not depend on the flight duration. In what follows we focus on case $0 < \nu < 1$.

A CSMP consists of a sequence of elementary events (v_n, τ_n). We note that τ_n $(n = 1, \cdots)$ is IID random variable. Thus the CSMP is fully characterized by the joint PDF of v and τ in an elementary event:

$$\phi(v, \tau) = \left\langle \delta\left(v - v_i\right) \delta\left(\tau - \tau_i\right)\right\rangle, \tag{3}$$

which is given by

$$\phi(v, \tau) = \frac{1}{2}\left[\delta\left(v - \tau^{\nu-1}\right) + \delta\left(v + \tau^{\nu-1}\right)\right] \ \psi(\tau) \tag{4}$$

PDF $\psi(\tau)$ of τ is defined through the marginal density of the joint PDF $\psi(v, \tau)$:

$$\psi(\tau) = \int_{-\infty}^{+\infty} \phi(v, \tau) dv = \left\langle \delta\left(\tau - T_i\right)\right\rangle. \tag{5}$$

Similarly PDF $\phi(v)$ of v is given by

$$\chi(v) = \int_{0}^{+\infty} \phi(v, \tau)\, d\tau = \left\langle \delta\left(v - v_i\right)\right\rangle. \tag{6}$$

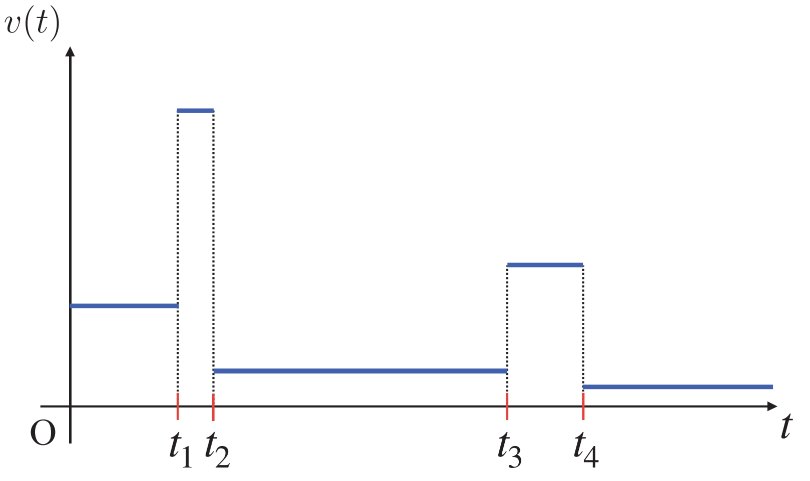

FIG. 1. Realization of a semi-Markov process. The state changes discontinuously at renewal points, i.e., t_1, t_2, t_3, and t_4.

We assume that $\psi(\tau)$ follows a power-law form:

$$\psi(\tau) \sim \frac{c}{|\Gamma(-\gamma)|}\tau^{-1-\gamma} \quad (\tau \to \infty), \tag{7}$$

where the parameter $\gamma > 0$ characterizes the algebraic decay and c is a scale parameter. In particular, we are interested in the regime $0 < \gamma < 1$.

III. RESULTS

A. Infinite invariant density

The propagator of the state variable, denoted by $p(v,t)$, is given by

$$p(v,t) = \chi(v)\ \left[\langle N(t)\rangle - \langle N(t - t_c(v))\rangle\right], \tag{8}$$

where $N(t)$ is the number of state changes with $N(0) = 1$ and $N(t) = 0$ for $t < 0$ and $t_c(v) \equiv |v|^{\frac{1}{\nu-1}}$ [14]. For $\gamma > 1$, the propagator converges to a stationary distribution in the long-time limit:

$$p(v,t) \to p_{\rm eq}(v) = \frac{\chi(v)|v|^{\frac{1}{\nu-1}}}{\langle\tau\rangle} \quad (t \to \infty). \tag{9}$$

On the other hand, there is no stationary distribution in the case of $\gamma \leq 1$. In fact, the propagator explicitly depends on t even in the long-time limit:

$$p(v,t) \sim \frac{t^{\gamma}|v|^{-1+\frac{\gamma}{1-\nu}}}{2(1-\nu)|\Gamma(-\gamma)|\Gamma(1+\gamma)} \tag{10}$$

for $v < v_c(t)$ and

$$p(v,t) \sim \chi(v)\,|v|^{\frac{1}{\nu-1}}\,\frac{t^{\gamma-1}}{c\Gamma(\gamma)} \tag{11}$$

for $v_c(t) < v$. We note that the supports of Eqs. (10) and (11) depends on t. In particular, the support of Eq. (10) vanishes as time goes on; i.e., the state accumulates into the origin. For $v_c(t) < v$, a formal steady state $I_\infty(v)$ can be obtained by

$$I_\infty(v) \equiv \lim_{t\to\infty} t^{1-\gamma} p(v,t) = \frac{\chi(v)\,|v|^{\frac{1}{\nu-1}}}{c\Gamma(\gamma)}, \tag{12}$$

which does not depend on t but cannot be normalized, i.e., infinite invariant density. In particular, the asymptotic form of the infinite density for $v \ll 1$ can be represented as

$$I_\infty(v) \sim \frac{1}{2(1-\nu)|\Gamma(-\gamma)|\Gamma(\gamma)}|v|^{-1-\frac{1-\gamma}{1-\nu}}. \tag{13}$$

B. Distributional limit theorems

Time average of the absolute value of the state, i.e., $f(v) = |v|$ shows trajectory-to-trajectory fluctuations. The distribution depends on whether the observable is integrable with respect to the infinite invariant measure or not. When $f(v)$ is integrable with respect to the infinite invariant measure, i.e.,

$$\langle f(v)\rangle_{\rm inf} = \int_0^\infty f(v) I_\infty(v) dv < \infty, \tag{14}$$

the distribution follows the Mittag-Leffler distribution, which does not depend on ν [14]. On the other hand, when $f(v)$ is not integrable with respect to the infinite invariant measure, the distribution depends on both ν and γ [14]. Thus, there are two distributional limit theorems in the semi-Markov process. Distributional limit theorems for time averages of $f(v) = |v|$ are summarized in Fig. 2.

IV. CONCLUSION

In a semi-Markov process, there exists a formal steady state that cannot be normalized, i.e., the infinite invariant density. The form of the propagator for the state value outside the

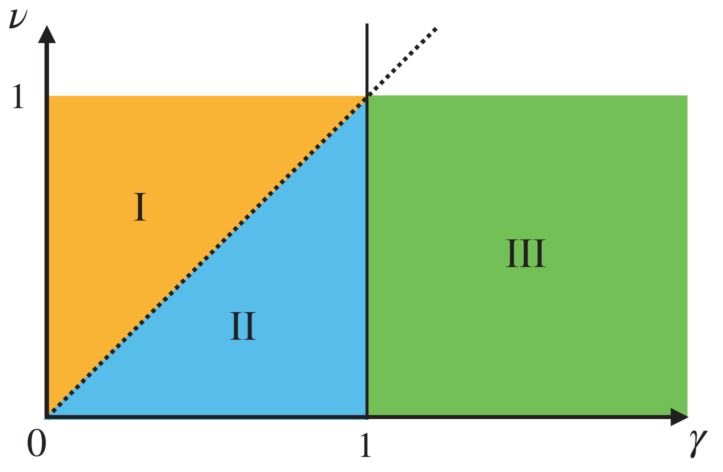

FIG. 2. Phase diagram for time average of $f(v) = |v|$. The invariant density becomes the infinite one for $\gamma \le 1$. The dashed line represents the boundary that the observable $f(v)$ is integrable with respect to the infinite/probability measure; i.e., $\int f(v) I_\infty(v) dv = \infty$ and $\gamma < 1$ in region I, $\int f(v) I_\infty(v) dv < \infty$ and $\gamma < 1$ in region II, and $\int f(v) p_{\rm eq}(v) dv < \infty$ and $\gamma > 1$ in region III. The form of PDF for the time average is different in regions I and II.

origin is described by the infinite invariant density whereas the probability that the state is outside the origin becomes zero in the long-time limit. Therefore, the state in the semi-Markov process almost surely converges to zero in the long-time limit, i.e., accumulation into the origin. The infinite invariant density discriminates distributional limit theorems in time-averaged observables. Therefore, infinite invariant density plays an important role in characterizing the accumulation process in the semi-Markov process.

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