RECONSTRUCTIONS OF ONE-SIDED DYNAMICAL SYSTEMS FROM THE ANALYSIS OF EXPERIMENTAL TIME SERIES

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1. Introduction

Throughout this article, all spaces are separable metric spaces and maps are continuous functions. Let $\mathbb{N}$ be the set of all nonnegative integers, i.e., $\mathbb{N} = \{0, 1, 2, ...\}$ and let $\mathbb{Z}$ be the set of all integers and $\mathbb{R}$ the real line.

A map $h : X \to Y$ is an *embedding* if $h : X \to h(X)$ is a homeomorphism. A pair (X, T) is called a *one-sided dynamical system* (abbreviated as *dynamical system*) if X is a separable metric space and $T : X \to X$ is any map. Moreover, if $T : X \to X$ is a homeomorphism, i.e., invertible, then (X, T) is called a *two-sided dynamical system.* Also if $T : X \to X$ is not a homeomorphism, (X, T) called a *non-invertible dynamical system.*

Reconstruction of dynamical systems from a scalar time series is a topic that has been extensively studied. The theoretical basis for methods of recovering dynamical systems on compact manifolds from one-dimensional data was studied by Takens [Tak81, Tak02]. In 1981, Takens [Tak81], by use of Whitney's embedding theorem, proved that under some conditions of (two-sided) diffeomorphisms on a manifold, the dynamical system can be reconstructed from the observations made with generic functions.

Theorem 1.1. (Takens' reconstruction theorem for diffeomorphisms [Tak81] and [Noa91]) *Suppose that M is a compact smooth manifold of dimension d. Let $D^r(M)$ be the space of all C^r-diffeomorphisms on M and $C^r(M, \mathbb{R})$ the set of all C^r-functions ($r \geq 1$) to $\mathbb{R}$. If E is the set of all pairs $(T, f) \in D^r(M) \times C^r(M, \mathbb{R})$ such that the delay observation map $I_{T,f}^{(0,1,2,..,2d)} : M \to \mathbb{R}^{2d+1}$ defined by*

$$x \mapsto (fT^j(x))_{j=0}^{2d}$$

is an embedding, then E is open and dense in $D^r(M) \times C^r(M, \mathbb{R})$.

Moreover, in 2002 Takens [Tak02], extended his theorem for endomorphisms on compact smooth manifolds as follows.

Theorem 1.2. (Takens' reconstruction theorem for endomorphisms [Tak02]) *Suppose that M is a compact smooth manifold of dimension d. Then there is an open dense subset $\mathcal{U} \subset \mathrm{End}^1(M) \times C^1(M, \mathbb{R})$, where $\mathrm{End}^1(M)$ denotes* *the space of all C^1-endomorphisms on M, such that, whenever $(T, f) \in \mathcal{U}$, there is a map $\pi : I_{T,f}^{(0,1,..,2d)}(M) \to M$ with $\pi \cdot I_{T,f}^{(0,1,..,2d)} = T^{2d}$.*

Embeddings of two-sided dynamical systems in the two-sided shift $(\mathbb{R}^{\mathbb{Z}}, \sigma)$ have been studied by many authors (e.g. see [AAM18, Coo15, Gut15, Gut16, GQS18, GT14, Jaw74, Lin99, LW00, Ner91, SYC91, Tak81]).

In [Kat20], we studied embeddings of one-sided (= non-invertible) dynamical systems in the one-sided shift $(\mathbb{R}^{\mathbb{N}}, \sigma)$. In this article, by use of the topological methods introduced in the paper [Kat20], we extend the above Takens' reconstruction theorems of dynamical systems on compact manifolds to theorems of "non-invertible" dynamical systems for a large class of compact metric spaces (see [Kat21] for the proofs of results of this article). In this article, we do not assume injectivity of T and so the proofs of our results cannot any longer rely on the embedding theorems of Whitney and Menger-Nöbeling [Eng95]. Instead, an essential role is played by the notion defined in Definition 2.1.

2. Definitions and notations

For a space X, $\dim X$ means the topological (covering) dimension of X (e.g. see [Eng95], [HW41] and [Nag65]). Let X be compact metric space and Y a space with a complete metric d_Y. Let $C(X, Y)$ denote the space consisting of all maps $f : X \to Y$. We equip $C(X, Y)$ with the metric d defined by

$$d(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

Recall that $C(X, Y)$ is a complete metric space and hence Baire's category theorem holds in $C(X, Y)$.

A map $g : X \to Y$ of separable metric spaces is *n-dimensional* ($n = 0, 1, 2, ...$) if $\dim g^{-1}(y) \leq n$ for each $y \in Y$. Note that a closed map $g : X \to Y$ is 0-dimensional if and only if for any 0-dimensional subset D of Y, $\dim g^{-1}(D) \leq 0$ (see [Eng95, Hurewic's theorem (1.12.4)]). A map $T : X \to X$ is *doubly 0-dimensional* if for each closed set $A \subset X$ of dimension 0, one has $\dim T^{-1}(A) \leq 0$ and $\dim T(A) = 0$.

If K is a subset of a space X, then $\mathrm{cl}(K)$, $\mathrm{bd}(K)$ and $\mathrm{int}(K)$ denote the closure, the boundary and the interior of K in X, respectively. A subset A of a space X is an F_σ-set of X if A is a countable union of closed subsets of X. Also, a subset B of X is a G_δ-set of X if B is an intersection of countably many open subsets of X.

An indexed family $(C_s)_{s \in S}$ of subsets of a set X will by abuse of notation also be denoted by $\{C_s\}_{s \in S}$ or $\{C_s : s \in S\}$. Hence if $\mathcal{C} = \{C_s\}_{s \in S}$ is such a family then its members C_s and C_t will be considered as different whenever $s \neq t$. We then put

$$\mathrm{ord}(\mathcal{C}) = \sup\{\mathrm{ord}_x(\mathcal{C}) : x \in X\}, \text{ where } \mathrm{ord}_x(\mathcal{C}) = |\{s \in S \mid x \in C_s\}|.$$

Note that $\mathrm{ord}(\mathcal{C})$ so defined is by 1 larger than it would be according to the usual definition, as e.g. in [Eng95, (1.6.6) Definition].

Modifying the definition of TSP in [Kat20], we define the notion of (k, η) trajectory-separation property for $k \in \mathbb{N}$ and $\eta > 0$ which is very important in this paper.

Definition 2.1. *Let $T : X \to X$ be a map of a compact metric space X with $\dim X = d < \infty$ and let $k \in \mathbb{N}, \eta > 0$. Then T has the (k, η) trajectory-separation property ((k, η)-TSP for short) provided that there is a closed set H of X such that*
(1) $X \setminus H$ is a union of finitely many disjoint open sets of diameter at most η, and
(2) $\mathrm{ord}\{T^{-p}(H)\}_{p=0}^{k} \leq d$.

3. RECONSTRUCTION SPACES OF DYNAMICAL SYSTEMS

For a space K, we consider the (one-sided) shift $\sigma : K^{\mathbb{N}} \to K^{\mathbb{N}}$ which is defined by

$$\sigma(x_0, x_1, x_2, x_3 \dots) = (x_1, x_2, x_3 \dots),\ x_i \in K.$$

Let (X, T) and (X', T') be dynamical systems. If a map $h : X \to X'$ satisfies the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ \downarrow T & & \downarrow T' \\ X & \xrightarrow{h} & X' \end{array}$$

then we say that $h : (X, T) \to (X', T')$ is a *morphism* of dynamical systems.

In this article, we need the following definition from [Kat20].

Definition 3.1. *Let $T : X \to X$ be a map of a compact metric space X.*
(a) Given a set $S \subset \mathbb{N}$ and a map $f : X \to \mathbb{R}$, the map $(fT^j)_{j \in S} : X \to \mathbb{R}^S$ will be denoted by $I_{T,f}^S$. We call this map the delay observation map at times $j \in S$. Note that $I_{T,f} := I_{T,f}^{\mathbb{N}} : (X, T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$ is a morphism of dynamical systems. We call $I_{T,f}$ the infinite delay observation map for (T, f).
(b) We say that I_f^S is a trajectory-embedding if $I_f^S(x) \neq I_f^S(y)$ whenever $T^j(x) \neq T^j(y)$ for all $j \in S$.

Let (X, T) be a dynamical system of a compact metric space X. For $n \geq 1$, let $P_n(T)$ be the set of all periodic points of T with period $\leq n$ and $P(T)$ the set of all periodic points of T, i.e.

$$P_n(T) = \{x \in X |\ \text{there is an } i \text{ such that } 1 \leq i \leq n \text{ and } T^i(x) = x\}$$

$$\text{and } P(T) = \bigcup_{n \geq 1} P_n(T).$$

Two points x and y of X are *trajectory-separated* for T if $T^j(x) \neq T^j(y)$ for $j \in \mathbb{N}$. A morphism $h : (X, T) \to (X', T')$ is a *trajectory-monomorphism* if

$h(x), h(y)$ are trajectory-separated for T', whenever $x, y \in X$ are trajectory-separated for T.

For $x, y \in X$, let $o_T(x) = (T^i(x))_{i\in\mathbb{N}}$ and $o_T(y) = (T^i(y))_{i\in\mathbb{N}}$ be two orbits of T. We say that the orbit $o_T(x)$ is *eventually equivalent* to the orbit $o_T(y)$ if the orbits will be equal in the future, i.e., there exists an $n \in \mathbb{N}$ such that $T^i(x) = T^i(y)$ for each $i \geq n$. In this case, we wright $o_T(x) \sim_e o_T(y)$. We see that this relation is an equivalence relation. So we have the equivalence class

$$[o_T(x)] = \{o_T(y) |\ o_T(x) \sim_e o_T(y)\}$$

containing $o_T(x)$ and we put

$$[O(T)] = \{[o_T(x)] |\ x \in X\}.$$

Note that if $T : X \to X$ is injective, the function $o : X \to [O(T)]$ defined by $x \mapsto [o_T(x)]$ is bijective, i.e., $o : X \cong [O(T)]$. Also, note that if $h : (X,T) \to (X',T')$ is a morphism of dynamical systems, then h induces the function $h : [O(T)] \to [O(T')]$ defined by $h([o_T(x)]) = [o_{T'}(h(x))]$ for $x \in X$. A morphism $h : (X,T) \to (X',T')$ of dynamical systems is a *trajectory-isomorphism* if h induces the bijection $h : [O(T)] \cong [O(T')]$.

Proposition 3.2. *Suppose that a morphism $h : (X,T) \to (X',T')$ is a trajectory-monomorphism and h is surjective, i.e., $h(X) = X'$. Then h is a trajectory-isomorphism:*

$$h : [O(T)] \cong [O(T')]$$

We need the definition of topological entropy and we give the definition by Bowen [Bow78]. Let $T : X \to X$ be any map of a compact metric space X. A subset E of X is (n,ϵ)-separated if for any $x, y \in E$ with $x \neq y$, there is an integer j such that $0 \leq j < n$ and $d(T^j(x), T^j(y)) \geq \epsilon$. If K is any nonempty closed subset of X, $s_n(\epsilon; K)$ denotes the largest cardinality of any set $E \subset K$ which is (n,ϵ)-separated. Also we define

$$s(\epsilon; K) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\epsilon; K),$$

$$h(T; K) = \lim_{\epsilon\to 0} s(\epsilon; K).$$

It is well known that the topological entropy $h(T)$ of T is equal to $h(T; X)$ (see [Bow78]).

Let (X,T) and (Y,S) be one-sided dynamical systems of compact metric spaces. The *inverse limit* of T is the space

$$\varprojlim(X,T) = \{(x_i)_{i=0}^{\infty} \mid T(x_{i+1}) = x_i \text{ for each } i \in \mathbb{N}\} \subset X^{\mathbb{N}}$$

which has the topology inherited as a subspace of the product space $X^{\mathbb{N}}$. If $h : (X,T) \to (Y,S)$ is a morphism of dynamical systems, then the map

$$\varprojlim h : \varprojlim(X,T) \to \varprojlim(Y,S)$$

is defied by $\varprojlim h((x_i)_i) = (h(x_i))_i$ for $(x_i)_i \in \varprojlim(X,T)$. Note that if T is a homeomorphism, then $X \cong \varprojlim(X,T)$.

Now, we will introduce the notion of *reconstruction space* of dynamical systems.

Definition 3.3. *A compact metric space X is a reconstruction space of dynamical systems if there exists a G_δ-dense set E of $C(X,X) \times C(X,\mathbb{R})$ such that for $(T,f) \in E$, the infinite delay observation map*

$$I_{T,f} := I_{T,f}^{\mathbb{N}} : (X,T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$$

satisfies the following conditions (1) and (2):
(1) $I_{T,f} : [O(T)] \cong [O(\sigma_{T,f})]$, *where* $\sigma_{T,f} = \sigma | I_{T,f}(X)$, *and*
(2) $\varprojlim I_{T,f} : \varprojlim (X,T) \to \varprojlim (I_{T,f}(X), \sigma_{T,f})$ *is a homeomorphism.*

$$\begin{array}{ccccc} X & \overset{I_{T,f}}{\longrightarrow} & I_{T,f}(X) & \subset & \mathbb{R}^{\mathbb{N}} \\ \downarrow T & & \downarrow \sigma_{T,f} & & \downarrow \sigma \\ X & \overset{I_{T,f}}{\longrightarrow} & I_{T,f}(X) & \subset & \mathbb{R}^{\mathbb{N}} \end{array}$$

Remark. In the above definition, (1) implies that we can understand the structure of orbits of (X,T) from the analysis of time series $(I_{T,f}(X), \sigma_{T,f})$, and (2) implies that $(I_{T,f}(X), \sigma_{T,f})$ reflects topological and dynamical properties of (X,T). Let $\mathcal{P}$ be any dynamical property such that (X,T) has $\mathcal{P}$ if and only if $(\varprojlim(X,T), \varprojlim T)$ has $\mathcal{P}$; e.g. minimal, topological transitive, topological mixing, sensitive, etc. Then (X,T) has $\mathcal{P}$ if $(I_{T,f}(X), \sigma_{T,f})$ has $\mathcal{P}$, because that we have the following commutative diagram of homeomorphisms:

$$\begin{array}{ccc} \varprojlim(X,T) & \overset{\varprojlim I_{T,f}}{\longrightarrow} & \varprojlim(I_{T,f}(X), \sigma_{T,f}) \\ \downarrow \varprojlim T & & \downarrow \varprojlim \sigma_{T,f} \\ \varprojlim(X,T) & \overset{\varprojlim I_{T,f}}{\longrightarrow} & \varprojlim(I_{T,f}(X), \sigma_{T,f}) \end{array}$$

In this article, we know that many compact metric spaces (e.g. PL-manifolds, branched manifolds, Menger manifolds, Sierpiński carpet, Sierpiński gasket and many fractal sets) are reconstruction spaces of dynamical systems. Our result means that almost all dynamical systems (X,T) on a reconstruction space X can be reconstructed from (observation) maps $f : X \to \mathbb{R}$ in the sense of 'eventually equivalent orbits', and so it forms a bridge between the theory of nonlinear dynamical systems and nonlinear time series analysis.

4. Trajectory-embeddings in $(\mathbb{R}^{\mathbb{N}}, \sigma)$

In this section, we study some fundamental properties of trajectory-embeddings.

Proposition 4.1. *Let (X,T) be a dynamical system and $f : X \to \mathbb{R}$ a map. Let $k \in \mathbb{N}$ and suppose that $I_{T,f}^{(0,1,..,k)} : X \to \mathbb{R}^{k+1}$ is a trajectory-embedding.*

Then the following properties (1)-(4) *hold.*

(1) *There is the unique map* $\sigma_{T,f}^{(0,1,..,k)} : I_{T,f}^{(0,1,..,k)}(X) \to I_{T,f}^{(0,1,..,k)}(X)$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} X & \overset{I_{T,f}^{(0,1,..,k)}}{\longrightarrow} & I_{T,f}^{(0,1,..,k)}(X) \subset \mathbb{R}^{k+1} \\ \downarrow T & & \downarrow \sigma_{T,f}^{(0,1,..,k)} \\ X & \overset{I_{T,f}^{(0,1,..,k)}}{\longrightarrow} & I_{T,f}^{(0,1,..,k)}(X) \subset \mathbb{R}^{k+1}. \end{array}$$

In other words, the map $\sigma_{T,f}^{(0,1,..,k)}$ *defined by* $(fT^i(x))_{i=0}^k) \mapsto (fT^i(x))_{i=1}^{k+1})$ $(x \in X)$ *is well-defined. And* $I_{T,f}^{(0,1,..,k)} : (X,T) \to (I_{T,f}^{(0,1,..,k)}(X), \sigma_{T,f}^{(0,1,..,k)})$ *is a trajectory-isomorphism. In particular,* $I_{T,f} := I_{T,f}^{\mathbb{N}} : (X,T) \to (R^{\mathbb{N}}, \sigma)$ *is a trajectory-monomorphism.*

(2) *Let* $p_{(0,1,..,k)} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{k+1}$ *be the projection defined by* $(x_i)_{i\in\mathbb{N}} \mapsto (x_i)_{i=0}^k$*. Then* $p_{(0,1,..,k)} : (I_{T,f}(X), \sigma_{T,f}) \to (I_{T,f}^{(0,1,..,k)}(X), \sigma_{T,f}^{(0,1,..,k)})$ *is an isomorphism of dynamical systems, i.e.,* $p_{(0,1,..,k)}$ *is a homeomorphism.*

(3) $h(T) = h(\sigma_{T,f}) = h(\sigma_{T,f}^{(0,1,..,k)})$.

(4) $\varprojlim I_{T,f}^{(0,1,..,k)} : \varprojlim(X,T) \to \varprojlim(I_{T,f}^{(0,1,..,k)}(X), \sigma_{T,f}^{(0,1,..,k)})$ *is a homeomorphism.*

By Proposition 4.1 and [Kat20, Theorem 3.1], we have the following result.

Theorem 4.2. *Let* X *be a compact metric space with* $\dim X = d < \infty$ *and let* $T : X \to X$ *be a doubly 0-dimensional map with* $\dim P(T) \leq 0$*. Then there is a dense* G_δ*-set* D *of* $C(X,\mathbb{R})$ *such that for all* $f \in D$,

$$I_{T,f} = T_{T,f}^{\mathbb{N}} : (X,T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$$

satisfies the following conditions:
(a) $I_{T,f} : [O(T)] \cong [O(\sigma_{T,f})]$,
(b) $\varprojlim I_{T,f} : \varprojlim(X,T) \to \varprojlim(I_{T,f}(X), \sigma_{T,f})$ *is a homeomorphism,*
(c) $h(T) = h(\sigma_{T,f})$ *and*
(d) *if* $x, y \in X$ *are trajectory-separated for* T*, then*

$$|\{i \in \mathbb{N}|\ I_{T,f}(x)_i = I_{T,f}(y)_i\}| \leq 2d.$$

5. Reconstruction theorem in the one-sided shift $(\mathbb{R}^{\mathbb{N}}, \sigma)$

Let X, Y be compact metric spaces and let $\varphi : X \to 2^Y \cup \{\emptyset\}$ be a set-valued function, where 2^Y denotes the set of all nonempty closed subsets of Y. Then $\varphi : X \to 2^Y \cup \{\emptyset\}$ is *upper semi-continuous* if for any $x \in X$ and any open neighborhood V of $\varphi(x)$ in Y, there is an open neighborhood U of x in X such that $\varphi(x') \subset V$ for any $x' \in U$.

Let (X,T) be any one-sided dynamical system. A point $x \in X$ is a *chain recurrent point* of T if for any $\epsilon > 0$ there is a finite sequence $x = x_0, x_1, \cdots, x_m = x$ $(m \geq 1)$ of points of X such that $d(T(x_i), x_{i+1}) < \epsilon$ for each $i = 0, 1, \cdots, m-1$. Let $CR(T)$ be the set of all chain recurrent points of T. Note that $P(T) \subset CR(T)$, $CR(T)$ is a nonempty closed subset of X and the set-valued function

$$CR : C(X, X) \to 2^X,\ T \mapsto CR(T)$$

is upper semi-continuous (see [BF85]).

We will define the following class 0-$\mathcal{DCR}$ of compact metric spaces.

Definition 5.1. *Let 0-$\mathcal{DCR}$ be the class of all compact metric spaces X satisfying the following two conditions:*
(0-$\mathcal{D}$) *The set of doubly 0-dimensional maps $T : X \to X$ is dense in $C(X, X)$.*
(0-$\mathcal{CR}$) *The set of maps $T : X \to X$ with $\dim CR(T) = 0$ is dense in $C(X, X)$.*

Remark. Note that for a compact metric space X, both the set of 0-dimensional maps $T : X \to X$ and the set of maps $T : X \to X$ with $\dim CR(T) = 0$ are G_δ-sets of $C(X, X)$ (e.g. see [KOU16]). So note that if X belongs to 0-$\mathcal{DCR}$, then the set of all maps $T : X \to X$ such that T is a 0-dimensional map with $\dim CR(T) = 0$ is a dense G_δ-set of $C(X, X)$.

Let A be a (nonempty) closed subset of a compact metric space X. Here we need the following notion: $D(A) < \eta$ if A can be decomposed into finitely many mutually disjoint closed sets A_i with $\mathrm{diam}(A_i) < \eta$ for each i, i.e. $A = \bigcup_i A_i$, $\mathrm{diam}(A_i) < \eta$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. Note that $\dim A = 0$ if and only if $D(A) < \eta$ for each $\eta > 0$.

Modyfying the proof of [KM20, Lemma 3.11], we have the following.

Lemma 5.2. (c.f. [KM20, Lemma 3.11]) *Let $\eta > 0$ and $k \in \mathbb{N}$. Suppose that $T : X \to X$ is a doubly 0-dimensional map of a compact metric space X such that $\dim X = d < \infty$ and $D(\mathrm{cl}[\cup_{p=0}^{4k} T^{-p}(P(T))]) < \eta$. Then T has (k, η)-TSP.*

Lemma 5.3. (A version of Borsuk's homotopy extension theorem, c.f. [Bor67, (8.1)Theorem] and [Mil01, Theorem 4.1.3]) *Let X be a compact metric space and M a closed subset of X, and let maps $f', g' : M \to \mathbb{R}^k$ satisfy $d(f', g') < \epsilon$. If $g : X \to \mathbb{R}^k$ is an extension of g', then f' has an extension $f : X \to \mathbb{R}^k$ such that $d(f, g) < \epsilon$.*

Let X be any compact metric space. For each $\alpha > 0$ and $S \subset \mathbb{N}$ a set of cardinarity $2d+1$, let $E(\alpha; S)$ be the subset of $C(X, X) \times C(X, \mathbb{R})$ consisting of all pairs (T, f) such that $I_{T,f}^S : X \to \mathbb{R}^S$ is an α *trajectory-embedding* (i.e., $I_{T,f}^S(x) \neq I_{T,f}^S(y)$ whenever $x, y \in X$ with $d(T^j(x), T^j(y)) \geq \alpha$ for all $j \in S$).

The main theorem is the following.

Main Theorem 5.4. (Reconstruction theorem of dynamical systems) *Let X be a compact metric space with $\dim X = d$. Suppose that X belongs to the class 0-$\mathcal{DCR}$.Then the following assertions $(1)-(3)$ hold.*

(1) (α trajectory-embedding) *Let* $\alpha > 0$ *and* $S \subset \mathbb{N}$ *a set of cardinarity* $2d+1$. *Then the set* $E(\alpha; S)$ *is a dense open set of* $C(X,X) \times C(X,\mathbb{R})$.

(2) (Trajectory-embedding) *There exists a* G_δ*-dense set* E *of* $C(X,X) \times C(X,\mathbb{R})$ *such that if* $(T,f) \in E$, *for any* $S \subset \mathbb{N}$ *of cardinality* $2d+1$

$$I^S_{T,f} : X \to \mathbb{R}^S$$

is a trajectory-embedding.

(3) (Infinite delay observation) *If* E *is the set as in the above (2), then for any* $(T,f) \in E$,

$$I_{T,f} = T^{\mathbb{N}}_{T,f} : (X,T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$$

satisfies the following conditions:

(*a*) $I_{T,f} : [O(T)] \cong [O(\sigma_{T,f})]$,

(*b* $\varprojlim I_{T,f} : \varprojlim(X,T) \to \varprojlim(I_{T,f}(X), \sigma_{T,f})$ *is a homeomorphism,*

(*c*) $h(T) = h(\sigma_{T,f})$ *and*

(*d*) *if* $x, y \in X$ *are trajectory-separated for* T, *then*

$$|\{i \in \mathbb{N} |\ I_{T,f}(x)_i = I_{T,f}(y)_i\}| \leq 2d.$$

In particular, X *is a reconstruction space of dynamical systems.*

$$\begin{array}{ccccc} X & \xrightarrow{I_{T,f}} & I_{T,f}(X) & \subset & \mathbb{R}^{\mathbb{N}} \\ \downarrow T & & \downarrow \sigma_{T,f} & & \downarrow \sigma \\ X & \xrightarrow{I_{T,f}} & I_{T,f}(X) & \subset & \mathbb{R}^{\mathbb{N}} \end{array}$$

6. The class 0-$\mathcal{DCR}$

In this section, we consider the following general problem.

Problem 6.1. *What kinds of compact metric spaces belong to the class 0-*$\mathcal{DCR}$ *?*

We will show that PL-manifolds, some branched manifolds and some fractal sets, e.g. Menger manifolds, Sierpiński carpet, Sierpiński gasket and dendrites, belong to the class 0-$\mathcal{DCR}$.

In [KOU16] Krupski, Omiljanowski and Ungeheuer defined the class 0-$\mathcal{CR}$ which is the family of all compact metric spaces X such that the set $CR(T)$ is 0-dimensional for a generic map $T \in C(X,X)$. They proved the following result.

Theorem 6.2. ([KOU16, Theorem 5.1]) *If* X *is a (compact) polyhedron, then* $X \in 0\text{-}\mathcal{CR}$. *Moreover, if* X *is a compact metric space that admits an* ϵ*-retraction* $r_\epsilon : X \to P$ *onto a polyhedron* $P \subset X$ *for each* $\epsilon > 0$ *(i.e.,* $d(r_\epsilon, id_X) < \epsilon$ *and* $r_\epsilon | P = id_P$*), then* $X \in 0\text{-}\mathcal{CR}$.

Now, we will consider the family 0-$\mathcal{D}$ of all compact metric spaces X such that all doubly 0-dimensional maps on X is dense in $C(X,X)$. A map $T : X \to X$ is said to be a *piecewise embedding* if there is a countable family $\{F_i\}_{i\in\mathbb{N}}$ of closed subsets of X such that $X = \bigcup_{i\in\mathbb{N}} F_i$ and $T|F_i : F_i \to X$

is injective for each $i \in \mathbb{N}$. Note that if a map $T : X \to X$ is a piecewise embedding, then T is doubly 0-dimensional because that $\dim T^{-1}(x)$ is a countable set for each $x \in X$ and

$$\dim T(A) = \max\{\dim T(A \cap F_i) \mid i \in \mathbb{N}\} \leq 0$$

for any 0-dimensional closed set A of X (see the countable sum theorem for dimension [Eng95, Theorem 3.1.8]).

A (compact) d-dimensional polyhedron P ($d \geq 1$) is called a *manifold with branch structures* if $P = \bigcup_{j \in J} M_j \cup M$, where
(1) $\{M_j\}_{j \in J}$ is a finite family of mutually disjoint closed sets of P such that for each $j \in J$,

$$M_j = N_j \cup_{\varphi_\alpha} \bigcup\{N_{j,\alpha} | \alpha \in J_j\},$$

where J_j is a finite set, $N_j, N_{j,\alpha}$ ($\alpha \in J_j$) are d-dimensional manifolds with boundaries, and M_j is obtained from N_j by attaching $N_{j,\alpha}$ ($\alpha \in J_j$) via locally embedding maps $\varphi_\alpha : N'_{j,\alpha} \to \partial N_j$ from a $(d-1)$-dimensional (compact) submanifold $N'_{j,\alpha}$ of $\partial N_{j,\alpha}$ into ∂N_j, i.e., M_j is the quotient space of the topological sum $N_j \amalg_{\alpha \in J_j} N_{j,\alpha}$ under the identifications $x \sim \varphi_\alpha(x)$ for $x \in N'_{j,\alpha} \subset \partial N_{j,\alpha}$ and the quotient map is denoted by $q_j : N_j \amalg_{\alpha \in J_j} N_{j,\alpha} \to M_j$ $(= N_j \cup \bigcup\{q_j(N_{j,\alpha}) | \alpha \in J_j\})$,
(2) M is a d-dimensional compact manifold in P with

$$M \cap \bigcup\{\varphi_\alpha(N'_{j,\alpha}) \mid j \in J, \alpha \in J_j\} = \emptyset$$

and
(3) $P \setminus \bigcup\{\varphi_\alpha(N'_{j,\alpha}) \mid j \in J, \alpha \in J_j\}$ is a d-dimensional (non-compact) manifold.

Remark. All PL-manifolds and some branched manifolds are manifolds with branch structures. The associated template of the well-know Lorenz attractor is a manifold with branch structures [GL02].

Proposition 6.3. *Let P be a manifold with branch structures. Then the set of all piecewise embedding maps $T : P \to P$ is dense in $C(P, P)$. In particular, P belongs to 0-$\mathcal{DCR}$. Hence P is a reconstruction space of dynamical systems.*

Many dynamical properties of Cantor sets have been studied by many authors. Now we consider dynamical properties of higher dimensional fractal sets.

For $0 \leq k < n$, we will construct a space L^n_k in the n-simplex $M_0 =< v_0, v_1, ..., v_n >$ by Lefshetz's method (see [Chi96, p.129] and [Lef31]). We define a sequence $\{(M_i, L_i)\}_{i \in \mathbb{N}}$ of compact n-dimensional polyhedra M_i with triangulations L_i inductively as follows. Let M_0 be the n-simplex $< v_0, v_1, ..., v_n >$ with the standard simplicial complex structure L_0. Suppose

(M_i, L_i) has been defined. Let

$$M_{i+1} = \bigcup\{\mathrm{St}(v, \beta^2(L_i)) \mid v \text{ is a vertex of } \beta(L_i^{(k)})\}$$

and

$$L_{i+1} = \beta^2 L_i | M_{i+1}.$$

Note that M_{i+1} may be regarded as a regular neighborhood of the k-skeleton of L_i. Then $\{M_i\}_{i \in \mathbb{N}}$ is a decreasing sequence and we obtained a compact metric space

$$L_k^n = \bigcap_{i \in \mathbb{N}} M_i.$$

Note that L_0^1 is a Cantor set and L_d^{2d+1} $(= \mu^d)$ is called the d-dimensional *Menger compactum*. Also L_1^2 is called the *Sierpiński carpet*. A space X is a d-dimensional *Menger manifold* if X is compact and each point x of X has a neighborhood W of x in X such that W is homeomorphic to the d-dimensional Menger compactum μ^d (for many geometric properties of μ^d, see [Bes88]).

Also the Sierpiński gasket can be constructed from an equilateral triangle by repeated removal of (open) triangular subsets: Start with an equilateral triangle. Subdivide it into four smaller congruent equilateral triangles and remove the central (open) triangle. Repeat this step with each of the remaining smaller triangles infinitely. So we have a sequence $\{X_i\}_{i \in \mathbb{N}}$ of continua in the plane and the intersection $X = \bigcap_{i \in \mathbb{N}} X_i$ is called the *Sierpiński gasket*.

A compact connected metric space (=continuum) X is said to be a *dendrite* if X is a 1-dimensional locally connected continuum which contains no simple closed curve.

Proposition 6.4. *Let M be a d-dimensional Menger manifold. Then M belongs to 0-$\mathcal{DCR}$ and hence M is a reconstruction space. More precisely, there exists a G_δ-dense set E' of $C(M, M) \times C(M, \mathbb{R})$ such that if $(T, f) \in E'$, then for any $S \subset \mathbb{N}$ of cardinality $2d + 1$, $I_{T,f}^S : M \to \mathbb{R}^S$ is an embedding and so*

$$I_{T,f} = T_{T,f}^{\mathbb{N}} : (M, T) \to (\mathbb{R}^{\mathbb{N}}, \sigma)$$

is an embedding.

We show that the Sierpiński carpet belongs to 0-$\mathcal{DCR}$. In [Why58, p.323], Whyburn proved that the Sierpiński carpet is homeomorphic to any S-curve X (=plane locally connected 1-dimensional continuum whose complement in the plane consists of countably many components with frontiers being mutually disjoint simple closed curves $\{S_i\}_{i \in \mathbb{N}}$, and moreover, if K_1, K_2 are S-curves and C_1, C_2 are frontiers of components of complements of K_1, K_2 in the plane $\mathbb{R}^2$, respectively, then each homeomorphism of C_1 onto C_2 can be extended to a homeomorphism of K_1 onto K_2. Such simple closed curves $\{S_i\}_{i \in \mathbb{N}}$ are called the *rational circles* of the S-curve X. The union of all these circles $\{S_i\}_{i \in \mathbb{N}}$ is called the *rational part* of X, and the remainder $X \setminus (\bigcup_{i \geq 0} S_i)$ is called the *irrational part* of X.

Proposition 6.5. *Let $X = L_1^2 \subset \mathbb{R}^2$ be the Sierpiński carpet. Then X belongs to 0-$\mathcal{DCR}$.*

Proposition 6.6. *Let X be the Sierpiński gasket. Then X belongs to 0-$\mathcal{DCR}$.*

Proposition 6.7. *Let X be any dendrite. Then X belongs to 0-$\mathcal{DCR}$.*

Finally, we obtain the following consequence.

Theorem 6.8. *Let X be one of the following spaces: PL-manifold, manifold with branch structures, Menger manifold, Sierpiński carpet, Sierpiński gasket and dendrite. Then X is a reconstruction space of dynamical systems.*

7. Application: Reconstructions of one-sided dynamical systems from nonlinear time series analysis

There have been attempts to reconstruct dynamical models directly from data, and nonlinear methods for the analysis of time series data have been extensively investigated. This research is an inverse problem to the numerical analysis of dynamical systems model, in that it seeks to identify models that fit data.

Time-delay embedding is well-known for nonlinear time series analysis, and it is used in several research fields such as physics, meteorology, informatics, neuroscience and so on. In laboratories, experimentalists are striving to find principles of phenomenons from a lot of data and they use delay embedding for reconstructing the dynamical systems from experimental time series. For smooth dynamical systems on manifolds, the celebrated Takens' reconstruction theorem ensures validity of the delay embedding analysis. Takens' theorem means that many dynamics theoretically can be reconstructed by the delay coordinate system, more precisely almost all (two-sided) dynamical systems can be reconstructed from observation maps (see Takens [Tak81, Tak02] and Sauer,Yorke and Casdagli [SYC91]). So Takens' theorem is the basis for nonlinear time series analysis and form a bridge between the theory of nonlinear differential dynamical systems on smooth manifolds and nonlinear time series analysis.

However, unfortunately the systems may not to be two-sided and moreover, they may not be systems on manifolds. Recently we freqently encounter a situation where we have to study dynamical systems of spaces that cannot have differential structure. In natural sciences and physical engineering, there has been an increase in importance of fractal sets and more complicated spaces, and also in mathematics, the dynamical properties and stochastic analysis of such spaces have been studied by many authors. Our reconstruction theorem theoretically ensures validity of the delay embedding analysis for (topological) dynamical systems on such complicated compact metric spaces, i.e., almost all one-sided dynamical systems (X, T) of spaces X belonging to 0-$\mathcal{DCR}$ can be reconstructed from observation maps $f : X \to \mathbb{R}$ in the sense of "trajectory embedding", i.e., the delay observation map

$$I_{T,f}^{(0,1,2,\cdots,k)} : (X,T) \to (I_{T,f}^{(0,1,2,\cdots,k)}(X), \sigma_{T,f}^{(0,1,\cdots k)})$$

is a trajectory-embedding for a natural number $k \geq 2\dim X$, and so the dynamical system

$$(I_{T,f}^{(0,1,2,\cdots,k)}(X), \sigma_{T,f}^{(0,1,2,\cdots,k)})$$

may reflect many dynamical and topological properties of the original dynamical system (X,T). Especially,

$$I_{T,f} : [O(T)] \cong [O(\sigma_{T,f}^{(0,1,2,\cdots,k)})]$$

and

$$\varprojlim I_{T,f} : \varprojlim(X,T) \cong \varprojlim(I_{T,f}^{(0,1,2,\cdots,k)}(X), \sigma_{T,f}^{(0,1,2,\cdots,k)})$$

$$\begin{array}{ccc} X & \xrightarrow{I_{T,f}^{(0,1,2,\cdots,k)}} & I_{T,f}^{(0,1,2,\cdots,k)}(X) \subset \mathbb{R}^{k+1} \\ \downarrow T & & \downarrow \sigma_{T,f}^{(0,1,2,\cdots,k)} \\ X & \xrightarrow{I_{T,f}^{(0,1,2,\cdots,k)}} & I_{T,f}^{(0,1,2,\cdots,k)}(X) \subset \mathbb{R}^{k+1}. \end{array}$$

In laboratories, experimentalists may understand how the system (X,T) will go in the future in the sense of orbital classification from the analysis of experimental time series and they understand the geometric properties of (X,T) by use of the inverse limit space $\varprojlim(\mathrm{I}_{T,f}^{(0,1,2,\cdots,k)}(X), \sigma_{T,f}^{(0,1,2,\cdots,k)})$. More precisely, for $x, y \in X$, if one can find a time $n \in \mathbb{N}$ such that

$$|\{i \in \mathbb{N}|\ fT^i(x) = fT^i(y), 0 \leq i \leq n\}| = 2\dim X + 1,$$

then $T^j(x) = T^j(y)$ for $j \geq n$ and hence $[o_T(x)] = [o_T(y)]$.

For more general case where a d-dimensional compact metric space X does not belong to 0-$\mathcal{DCR}$ and (X,T) is any one-sided dynamical system, we have an extension (μ^d, T') of (X,T), where μ^d is the d-dimensional Menger compactum containing X and $T' : \mu^d \to \mu^d$ is an extension of T (see [Bes88]). By Proposition 6.4, there is a possibility to be able to investigate the approximate properties of the dynamical system (X,T) by use of time-delay embedding of the dynamical system (μ^d, T').

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