DISCRETE ORTHONORMAL STOCKWELL TRANSFORMS

Abstract. Since its appearing in 1996, the Stockwell transform (S-transform) has been used as a tool in medical imaging, geophysics and signal processing in general. In [2] Riba and Wong gave a definition of a multi-dimensional version of the Stockwell transform highlighting a link among the Stockwell transform, the Gabor transform and the Wavelet transform. In [1] Battisti and Riba studied in details the discrete counterpart of the Stockwell transform proving that the system of functions known in the literature as DOST basis is indeed an orthonormal basis of $L^2([0,1])$. In this paper, we present a summary of the results obtained in [1] in order to provide a $\mathcal{O}(N \log N)$-fast algorithm to compute the DOST-coefficients.

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1. Introduction

Let f be a signal in $L^2([0,1])$ and let φ be a window in $L^2([0,1])$. Then, following M. W. Wong and H. Zhu [6], we define the Stockwell transform (S-transform) $\mathrm{S}_\varphi f$ as

$$(\mathrm{S}_\varphi f)(b,\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi \mathrm{i}\, t\xi} f(t)\, |\xi|\, \overline{\varphi(\xi(t-b))}\, dt, \qquad b,\xi \in \mathbb{R}. \tag{1.1}$$

It is possible to rewrite the S-transform with respect to the Fourier transform of the analyzed signal using

$$(\mathrm{S}_\varphi f)(b,\xi) = \int_{\mathbb{R}} e^{2\pi \mathrm{i}\, b\zeta} \hat{f}(\zeta+\xi)\, \overline{\hat{\varphi}\left(\frac{\zeta}{\xi}\right)}\, d\zeta, \qquad b,\xi \in \mathbb{R}, \quad \xi \neq 0, \tag{1.2}$$

where $\hat{f}$ is the Fourier transform of the signal f, given by

$$\hat{f}(\xi) = (\mathrm{F} f)(\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi \mathrm{i}\, t\xi} f(t) dt, \qquad \xi \in \mathbb{R}.$$

In the following, we denote with $\check{f}$ or $\mathrm{F}^{-1} f$ the inverse Fourier transform of a signal f.

The S-transform was initially defined by R. G. Stockwell, L. Mansinha and R. P. Lowe in [4] using a Gaussian window

$$g(t) = e^{-t^2/2}, \qquad t \in \mathbb{R}.$$

In this case,

$$(\mathrm{S}_g f)(b,\xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-2\pi\mathrm{i}\, t\xi} f(t)\, |\xi|\, e^{-(t-b)^2\xi^2/2}\, dt, \qquad b,\xi \in \mathbb{R},$$

which in the alternative formulation becomes

$$(1.3) \qquad (\mathrm{S}_g f)(b,\xi) = \int_{\mathbb{R}} e^{2\pi\mathrm{i}\,\zeta b}\, \widehat{f}(\zeta+\xi)\, e^{-2\pi^2\zeta^2/\xi^2}\, d\zeta, \qquad b,\xi \in \mathbb{R}, \quad \xi \neq 0.$$

The natural discretization of (1.3), introduced in [4], is given by

$$(1.4) \qquad (\mathrm{S}_g f)(j,n) = \sum_{m=0}^{N-1} e^{2\pi\mathrm{i}\, mj/N}\, \widehat{f}(m+n)\, e^{-2\pi^2 m^2/n^2},$$

where $j = 0,\dots,N-1$ and $n = 1,\dots,N-1$. For $n = 0$, it is set

$$(\mathrm{S}_g f)(j,0) = \frac{1}{N}\sum_{k=0}^{N-1} f(k), \qquad j = 0,\dots,N-1.$$

In the literature, (1.4) is called redundant discrete Stockwell transform.

2. DOST Fucntions

Unfortunately, the redundant discrete Stockwell transform transform has an high computational cost – $\mathcal{O}\left(N^2\log N\right)$). In order to overcome this problem, R. G. Stockwell introduced in [3], without a mathematical proof, a basis of $L^2([0,1])$ given by

$$(2.1) \qquad \bigcup_{p\in\mathbb{Z}} D_p = \bigcup_{p\in\mathbb{Z}} \{D_{p,\tau}\}_{\tau=0}^{\beta(p)-1}.$$

The decomposition of a signal f in this basis is called in the literature the discrete orthonormal Stockwell transform (DOST transform) and can be accomplished with an algorithm of complexity $\mathcal{O}(N\log N)$.

We recall here the definition of the so-called DOST functions, introduced in [3]

Definition. *For* $p = 0$, *we set*

$$\nu(0) = 0, \quad \beta(0) = 1, \quad \tau(0) = 0,$$

for $p = 1$

$$\nu(1) = 1, \quad \beta(1) = 1, \quad \tau(1) = 0,$$

for all $p \geqslant 2$

$$\nu(p) = 2^{p-1} + 2^{p-2}, \quad \beta(p) = 2^{p-1}, \quad \tau(p) = 0, \ldots, \beta(p) - 1.$$

Setting, for each p*, the* p*-frequency band*

$$[\beta(p), 2\beta(p) - 1] = \left[\nu(p) - \frac{\beta(p)}{2}, \nu(p) + \frac{\beta(p)}{2} - 1\right],$$

we obtain a partition of $\mathbb{N}$*; notice that* $\nu(p)$ *is the center of each* p*-frequency band.*

Now introduce the set of functions D_p *as*

$$D_0\,(t) = 1, \qquad t \in \mathbb{R},$$

$$D_1\,(t) = e^{2\pi \mathrm{i}\, t}, \qquad t \in \mathbb{R},$$

and

$$D_p\,(t) = \{D_{p,\tau}(t)\}_{\tau=0,\ldots,\beta(p)-1}\,, \qquad t \in \mathbb{R},$$

where

$$D_{p,\tau}(t) = \frac{1}{\sqrt{\beta(p)}} \sum_{f=\nu(p)-\beta(p)/2}^{\nu(p)+\beta(p)/2-1} e^{2\pi \mathrm{i}\, f t} e^{-2\pi \mathrm{i}\, f\tau/\beta(p)}, \qquad t \in \mathbb{R}.$$

We set for all negative integers p

$$D_{p,\tau}(t) = \overline{D_{-p,\tau}(t)}, \qquad \tau = 0, \ldots, \beta\,(|p|) - 1.$$

For each $p \in \mathbb{N}$*,* $\nu(-p) = -\nu(p)$ *and* $\beta(-p) = -\beta(p)$*.*

Then

$$\bigcup_{p \in \mathbb{Z}} D_p$$

is called Stockwell basis and the coefficients $f_{p,\tau}$ *defined as*

$$f_{p,\tau} = (f, D_{p,\tau})_{L^2([0,1])}\,,$$

for $f \in L^2\,([0,1])$ *are called DOST coefficients.*

It is clear that the DOST function are not dilations nor translations of a single functions. Nevertheless, for each p,

$$D_p\,(t) = \left\{ \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} e^{2\pi \mathrm{i}\,(\beta(p)+j)(t-\tau/\beta(p))} \right\}_{\tau=0,\ldots,\beta(p)-1}$$

is formed by translation of $\tau/\beta(p)$ of the same function. Roughly speaking, we can state that the DOST basis is non self similar globally, but it is self similar in each band, see Figure 2. Hence, the S-transform in this setting appears different from the wavelet transform because the mother wavelet changes as the frequencies increases, in contrast to the usual formulation.

Theorem 1. *$\bigcup_{p\in\mathbb{Z}} D_p$ is an orthornomal basis of $L^2\left([0,1]\right)$.*

The DOST functions are related to the Sotckwell transform due to the following Proposition.

Proposition 2. *Let f be a signal in $L^2\left([0,1]\right)$ and let φ be an admissible window, i.e. let*

$$\varphi \in Z = \left\{ \varphi \in \mathcal{S}'\left(\mathbb{R}\right) \middle| \int_{\mathbb{R}} \left|\widehat{\varphi}\left(\xi\right)\right|^2 \frac{d\xi}{|1+\xi|} < \infty \right\}.$$

Then, we can write

$$\left(\mathrm{S}_{\varphi}\, f\right)\left(b,\xi\right) = \sum f_{p,\tau}\left(\mathrm{S}_{\varphi}\, D_{p,\tau}\right)\left(b,\xi\right), \tag{2.2}$$

where

$$f_{p,\tau} = \left(f, D_{p,\tau}\right)_{L^2([0,1])}$$

and the sum in (2.2) *is over all $D_{p,\tau}$ functions.*

R. G. Stockwell proposed this basis because it is an efficient compromise between frequency localization in low frequencies and spatial localization for high frequencies. Certainly, on one hand, for high frequencies, we do not have a precise frequency localization, but just a localization in a certain band, which is wider as the frequency increases and, on the other hand, in low frequencies, we loose spatial localization. The key of the spatial localization for high frequencies is that the basis $D_{p,\tau}$ are, broadly speaking, local at $t = \tau/\beta(p)$. This sentence must be understood in density terms, because it is not true that the function has compact support in time, but the mass is concentrated near the point $t = \tau/\beta(p)$.

Proposition 3. *For each $D_{p,\tau}(t)$ we have*

$$\|D_{p,\tau}\|_{L^2(I_{p,\tau})} = \left(\int_{\frac{2\tau-1}{2\beta(p)}}^{\frac{2\tau+1}{2\beta(p)}} |D_{\nu(p),\beta(p),\tau}(t)|^2 dt \right)^{1/2} > 0,85,$$

i.e. the L^2-norm is concentrated in the interval

$$I_{p,\tau} = \left[\frac{\tau}{\beta(p)} - \frac{1}{2\beta(p)}, \frac{\tau}{\beta(p)} + \frac{1}{2\beta(p)} \right].$$

Since $\|D_{p,\tau}\| = 1$, we can also state that the L^2-norm of $D_{p,\tau}$ is less that $0,15$ out of $I_{p,\tau}$. For $\tau = 0$, $I_{p,0}$ must be considered as an interval in circle, that is

$$I_{p,0} = \left[0, \frac{1}{2\beta(p)}\right) \cup \left(\frac{2\beta(p)-1}{2\beta(p)}, 1\right].$$

3. DOST Fucntions, a generalization

The DOST basis is not suited to the standard S-transform with Gaussian window (1.1), rather to a S-transform associated with a characteristic function (boxcar window). This fact was already pointed out by R. G. Stockwell himself in [3]. In the following we determine a basis of $L^2([0,1])$ adapted to a window φ satisfying some mild technical conditions which extends 2.1.

Theorem 4. *Let φ be a window satisfying conditions*

$$\widehat{\varphi}(\xi) = 0, \qquad \xi \in \mathbb{R} \backslash \left[-\frac{1}{3}, \frac{1}{3}\right).$$

and

$$c^{\varphi}_{p,j}(\nu(p)) \neq 0, \quad \forall j = 0, \ldots, \beta(p) - 1.$$

where

$$c^{\varphi}_{p,j}(\xi) = \overline{\widehat{\varphi}\left(\frac{\beta(p) + j - \xi}{\xi}\right)}.$$

Then, setting

$$E^{\varphi}_{p,\tau}(t) = \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} \left[c^{\varphi}_{p,j}(\nu(p))\right]^{-1} e^{2\pi \mathrm{i}\,(\beta(p)+j)\left(t - \frac{\tau}{\beta(p)}\right)},$$

we get

$$\left(\mathrm{S}_{\varphi}\, E^{\varphi}_{p,\tau}\right)(b, \nu(p)) = e^{-2\pi \mathrm{i}\, b\nu(p)} D_{p,\tau}(b). \tag{3.1}$$

Moreover,

$$\bigcup_{p \in \mathbb{Z}} E^{\varphi}_{p},$$

where

$$E^{\varphi}_{p} = \left\{E^{\varphi}_{p,\tau}\right\}_{\tau = 0, \ldots, \beta(|p|) - 1}$$

is a basis of $L^2([0,1])$.

Remark 5. *The functions defined in Theorem 4 are a proper generalization of the DOST functions. In fact,*

$$E^{\check{\chi}}_{p,\tau}(t) = D_{p,\tau}(t).$$

4. Algorithm Analysis

The computational complexity of the algorithm suggested by R. G. Stockwell for computing the DOST coefficients is still high: $\mathcal{O}(N^2)$. In 2009, Y. Wang and J. Orchard [5] proposed a fast algorithm which reduces drastically the complexity to $\mathcal{O}(N \log N)$; the same complexity of the FFT. This achievement allowed the application of the S-transform to image analysis. The following property proves that we can compute the $E^{\varphi}_{p,\tau}$ coeffiecients in $\mathcal{O}\,(N \log N)$.

Proposition 6. *Let $E^{\varphi}_{p,\tau}$ as in Theorem 4 and let f be a discrete signal. Then the evaluation of the coefficients*

$$f^{\varphi}_{p,\tau} = \left(f, E^{\varphi}_{p,\tau}\right)_{L^2([0,1])}$$

has computational complexity $\mathcal{O}(N \log N)$, where N is the length of f.

It is possible to check explicitly the computational complexity of the algorithm. To perform this task, we start evaluating the column vector f^{φ}_{p} given by

$$\begin{aligned}
f^{\varphi}_{p} &= \{f^{\varphi}_{p,\tau}\}_{\tau=0}^{\beta(p)-1} \\
&= \left\{\left(f, E^{\varphi}_{p,\tau}\right)_{L^2([0,1])}\right\}_{\tau=0}^{\beta(p)-1} \\
&= \left\{\left(\hat{f}, \widehat{E^{\varphi}_{p,\tau}}\right)_{l^2(\mathbb{Z})}\right\}_{\tau=0}^{\beta(p)-1} \\
&= \left\{\left(\hat{f}, \frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} \left[c^{\varphi}_{p,j}(\nu(p))\right]^{-1} e^{-2\pi \mathrm{i}\,(\beta(p)+j)(\tau/\beta(p))} \delta_{\beta(p)+j}(\cdot)\right)_{l^2(\mathbb{Z})}\right\}_{\tau=0}^{\beta(p)-1} \\
&= \left\{\frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} \hat{f}(\beta(p)+j)\overline{\left[c^{\varphi}_{p,j}(\nu(p))\right]^{-1}} e^{2\pi \mathrm{i}\,(\beta(p)+j)(\tau/\beta(p))}\right\}_{\tau=0}^{\beta(p)-1} \\
&= \left\{\frac{1}{\sqrt{\beta(p)}} \sum_{j=0}^{\beta(p)-1} \hat{f}(\beta(p)+j)\overline{\left[c^{\varphi}_{p,j}(\nu(p))\right]^{-1}} e^{2\pi \mathrm{i}\, j(\tau/\beta(p))}\right\}_{\tau=0}^{\beta(p)-1} \\
&= \left(\mathrm{F}^{-1}_{j\mapsto\tau}\left(\left(R^{\varphi}\,\hat{f}\right)|_{\beta(p),\ldots,2\beta(p)-1}(j)\right)\right)(\tau)
\end{aligned}$$

where R^{φ} is a sequence in $\mathbb{Z}$ such that

$$R^{\varphi}(\beta(p)+j) = \overline{\left[c^{\varphi}_{p,j}(\nu(p))\right]^{-1}}$$

for all p and j.

Therefore, first we have to perform the FFT of the signal f ($\mathcal{O}\,(N \log N)$, and the multiplication by R^{φ} ($\mathcal{O}(N)$), then at each p band we need to use the FFT to perform the anti Fourier transform with computational complexity $\mathcal{O}\left(\beta(p)\log\beta(p)\right)$. Summing up the contribution for each p-band we get the computational complexity of $\mathcal{O}(N\log N)$.

A Matlab implementation of this algorithm can be found on the MathWorks page `https://it.mathworks.com/matlabcentral/fileexchange/53910-fft-fast-s-transforms-dost-dcst-dost2-and-dcst2`.

5. Conclusions

In this article we surveyed preceding work by the author and his collaborators regarding the discrete orthonormal Stockwell transform and its properties. In particulare, we higlighted a simple and $\mathcal{O}\left(N\log N\right)$-fast algorithm to compute Stockwell coefficients.

In summary, the DOST basis associated to the boxcar function satisfy the following properties:

(i) D_j^φ is an orthonormal basis of $L^2\left([0,1]\right)$;
(ii) $\left(\mathrm{S}_\varphi D_j^\varphi\right)(b,f)$ is *local* in time;
(iii) $\left(\mathrm{S}_\varphi D_j^\varphi\right)(b,f)$ is *local* in frequency;
(iv) the coefficients f_j^φ can be computed with a fast $\mathcal{O}\left(N\log N\right)$ algorithm.

Furthermore, it is possible to generalize this basis working with functions associated to an admissible window.

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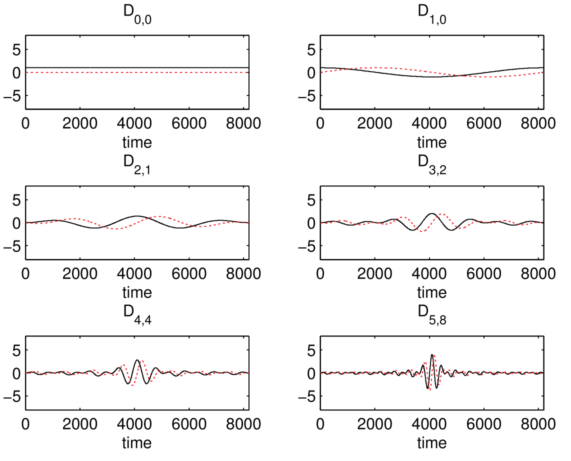

FIGURE 1. DOST basis functions in increasing frequency p-bands. Black line = real, red line = imaginary. See Figure 2 in [3] for a comparison.

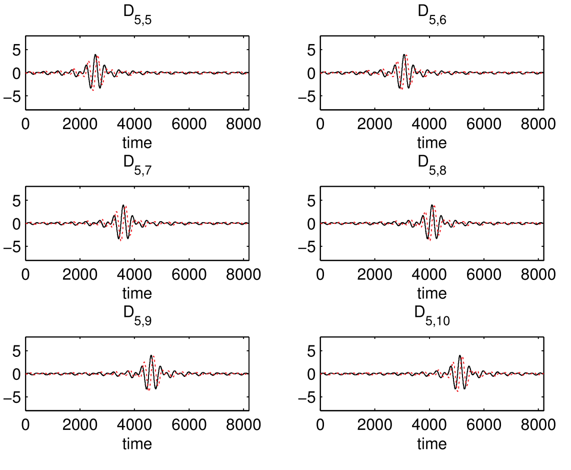

FIGURE 2. DOST basis functions in the same p-band ($p = 5$). Black line = real, red line = imaginary. See Figure 1 in [3] for a comparison.

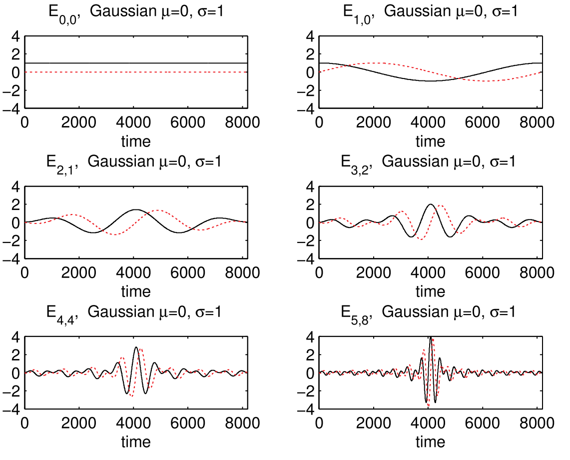

FIGURE 3. $E^{\varphi}_{p,\tau}$ basis functions in increasing frequency p-bands. Black line = real, red line = imaginary. $\widehat{\varphi}$ is a Gaussian window with $\mu = 0$ and $\sigma = 1$. Notice the similarities with Figure 1.

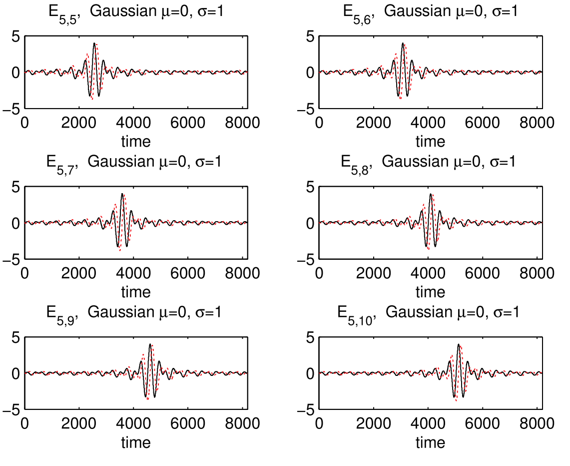

FIGURE 4. $E^{\varphi}_{p,\tau}$ basis functions in the same frequency p-band ($p = 5$). Black line = real, red line = imaginary. $\widehat{\varphi}$ is a truncated Gaussian window with $\mu = 0$ and $\sigma = 0.2$. See Figure 2 for comparison.

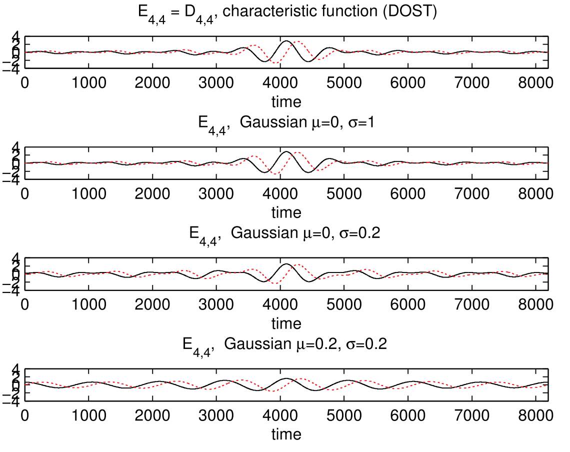

FIGURE 5. $E^{\varphi}_{p,\tau}$ basis functions with $p = 4$ and $\tau = 4$ with different windows. Black line = real, red line = imaginary. $\hat{\varphi}$ are $\chi_{[-1/3,1/3)}$ or truncated Gaussian windows with varying μ and σ.

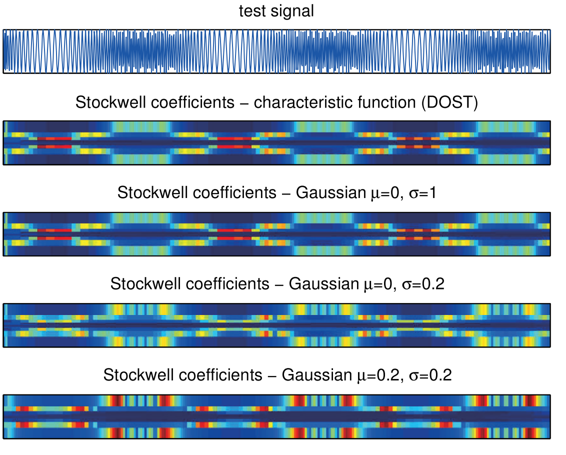

FIGURE 6. Decompositions of a given test signal on different windowed basis.

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