RIMS Kôkyûroku Bessatsu
B85 (2021), 15–34

Area and moment in *De Analysi* by Isaac Newton

By

Naoki Osada*

Abstract

In 1669, Isaac Newton wrote *De Analysi* to claim the priority of the analysis with infinite series. The method of infinite series requires the antiderivative of a simple curve $ax^{\frac{m}{n}}$ and the derivative of an object to be sought. In the October 1666 tract and *De Metodis* (1671), Newton expressed the antiderivative using fluxional equations, and the derivative by the ratio of the fluxions. On the other hand, in *De Analysi*, Newton represented the antiderivative as the pair of the region described by the ordinate $ax^{\frac{m}{n}}$ and its signed area, and he introduced the term *momentum* (moment) to represent the differential. In *De Analysi*, Newton replaced the terms and concepts of the fluxional method with those of geometry.

§ 1. Introduction

In 1665, Isaac Newton derived

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \cdots \tag{1.1}$$

by division, and discovered the infinite series expansion of $\log(1+x)$, i.e.,

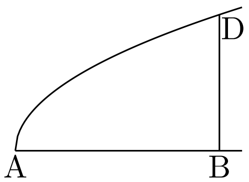
$$\log(1+x) = \int_0^x \frac{1}{1+x} dx = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 \cdots , \tag{1.2}$$

by integrating (1.1) by terms. He used (1.2) to calculate values such as $\log 1.1$, $-\log 0.9$, $\log 1.01$, $-\log 0.99$, and so on, with high accuracy.

Received February 27, 2021; Revised May 15, 2021.
2020 Mathematics Subject Classification(s): 01A45
Key Words: *Isaac Newton, antiderivative, area, moment, infinite series*
This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.
*Tokyo Woman's Christian University 167-8585, Japan.
e-mail: `osada@lab.twcu.ac.jp`

Newton wrote *De Analysi per æquationes numero terminorum infinitas* , abbreviated as *De Analysi*, in 1669 to claim the priority of the above method. He gave three rules for the method of analysis using infinite series in *De Analysi.* The first two rules, i.e., Rule I and II, are as follows:

> To the base AB of some curve AD let the ordinate BD be perpendicular and let AB be called x and BD y. Let again $a, b, c, \ldots$ be given quantities and m, n integers. Then



> **The Quadrature of simple Curves**
> RULE I. If $ax^{\frac{m}{n}} = y$, then will $\frac{na}{m+n}x^{\frac{m+n}{n}}$ equal the area ABD.
> The matter will be evident by example. [...]
>
> **And of those compounded of simple ones**
> RULE II. If the value of y is compounded of several terms of that kind the area also will be compounded of the areas which arise separately from each of those terms. [13, pp.206-209]

In some presentations, e.g., [1, p.12] and [4, p.154], Rule I is discussed as applied only to equation

$$\text{area ABD} = \int_0^x ax^{\frac{m}{n}}\,dx = \frac{na}{m+n}x^{\frac{m+n}{n}},$$

in which $m/n > 0$, which is the case considered by Newton in the figure illustrating the Rule quoted above. However, Newton dealt with the case of $m/n < -1$ as in Example 4, and so on.

Whiteside's annotations[1] suggest that Whiteside interpreted Rule I as

$$\begin{aligned} \text{area ABD} &= \int_0^x ax^{\frac{m}{n}}\,dx = \frac{na}{m+n}x^{\frac{m+n}{n}}, && \text{if } \frac{m}{n} > 0,\\ \text{area } \alpha\text{BD} &= -\int_\infty^x ax^{\frac{m}{n}}\,dx = -\frac{na}{m+n}x^{\frac{m+n}{n}}, && \text{if } \frac{m}{n} < -1. \end{aligned}$$

[1] "Since AB(x) is zero when B is at A, the lower bound of the integral is zero and Newton correctly evaluates $\int_0^x ax^{m/n}.dx$. In examples 4 and 5 following, however, he avoids the difficulty of having an integrand which is infinite when $x = 0$ by assuming a lower (or rather an upper) bound A$\alpha = \infty$"[13, p.207 note (5)] and "What Newton intends, it would seem, is to say that the integration bounds are in effect reversed: $\int_x^\infty x^{-2}.dx = -\int_\infty^x x^{-2}.dx = x^{-1}$." [13, p.209 note (6)]

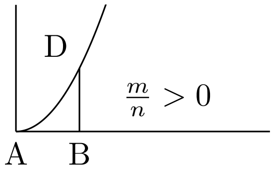


Fig. 1

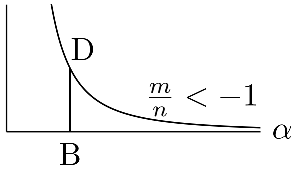


Fig. 2

However, in Example 4 of Rule I, Newton specified the region described by the line segment $\text{BD} = x^{-2}$ as αBD, and its area as

$$-x^{-1} = \int_{\infty}^{x} x^{-2} dx = \alpha\text{BD}.$$

We note that the sign is opposite to the area given by Whiteside.

On the other hand Hara[2] and Nagaoka[3] interpret Rule I as

$$\int a x^{\frac{m}{n}} dx = \frac{na}{m+n} x^{\frac{m+n}{n}}. \tag{1.3}$$

The equation (1.3) is true for $m/n \neq -1$, but it ignores the area that Newton carefully explained and does not follow Newton's intention.

In this paper, we show that Newton gave the antiderivative of the simple curve $ax^{\frac{m}{n}}$ by the area of the specified region in rule I in order to describe the method of infinite series without using the fluxional method. We also show that the moment which Newton introduced before the part where he found the arc length represents differentials, if BK(1) is corrected to BK(o).

We quote Newton's papers and figures, except Fig. 1-9, from [12], [13] and [14]. The formulas used in the English translation by Whiteside are reverted to the original Latin formulas where appropriate. Newton sometimes expressed "the" as y$^{\text{e}}$, "that" as y$^{\text{t}}$, "than" as y$^{\text{n}}$ and "which" as w$^{\text{ch}}$, but we keep the original text. The formulas in parentheses [ ] are supplementary explanations in modern calculus.

§ 2. Examples of Rule I

Newton gave six examples of Rule I. The first three examples of Rule I are the case $\frac{m}{n} > 0$, Examples 4 and 5 are the case $\frac{m}{n} < -1$, and Example 6 is the case $\frac{m}{n} = -1$.

---

[2] "*De Analysi* begins with three rules for integration, followed by Rule 1 and 2 those state $\int \sum a_r x^r dx = \sum a_r x^{r+1}/(r+1)$ for the rational number r, [...]" [5, p.305].

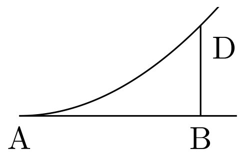
[3] "Newton summarizes the general quadrature calculation in three rules. For the sake of simplicity, the first two are

$$\text{Rule 1} \int a x^{m/n} dx = \frac{na}{m+n} x^{(m+n)/n},$$

$$\text{Rule 2} \int \sum_k f_k(x) dx = \sum_k \int f_k(x) dx,$$

in a somewhat modern way." [6, p.116]

Example 1. If $x^2(= 1 \times x^{\frac{2}{1}}) = y$, that is, if $a = n = 1$ and $m = 2$, then $\frac{1}{3}x^3 = \mathrm{ABD}$.



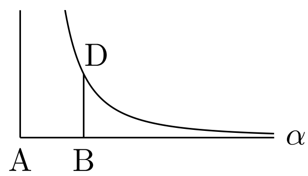
[13, pp.208-209]

Since the curve $y = x^2$ in Example 1 passes through the origin A, just like the curve from the figure given in Rule I, so we can be easily apply Rule I.

Example 1 shows that when $\frac{m}{n} > 0$, Newton defined the region [4] described by the line segment[5] $\mathrm{BD} = ax^{\frac{m}{n}}$ as ABD, and its area as $\dfrac{na}{n+m}x^{\frac{m+n}{n}}$ which is not only the ordinary area of the region ABD but also the antiderivative with the integral constant zero[6] of $ax^{\frac{m}{n}}$. In the rest of this paper, the integral constant of the antiderivative is set to 0.

The problem is Example 4.

Example 4 If $\frac{1}{x^2}(= x^{-2}) = y$, that is, if $a = n = 1$ and $m = -2$, then $\left(\dfrac{1}{-1}x^{\frac{-1}{1}} =\right) - x^{-1} \left(= \dfrac{-1}{x}\right) = \alpha\mathrm{BD}$ infinitely extended in the direction of α: the computation sets its sign negative because it lies on the further side of the line BD.



[13, pp.208-209]

Example 4 is represented as

$$\text{area } \alpha\mathrm{BD} = \int_{\infty}^{x} x^{-2}dx = -x^{-1}. \tag{2.1}$$

Since the signed area of $\alpha\mathrm{BD} < 0$ in (2.1), Newton stated “the computation sets its sign negative because it lies on the further side of the line BD”. Example 4 shows that when $\frac{m}{n} < -1$, Newton defined the region described by the line segment $\mathrm{BD} = ax^{\frac{m}{n}}$ by $\alpha\mathrm{BD}$, and its area is $\dfrac{na}{n+m}x^{\frac{m+n}{n}}$. Newton made the area negative because it matched the sign of the antiderivative.

Example 6 If $\frac{1}{x}(= x^{-1}) = y$, then $\dfrac{1}{0}x^{\frac{0}{1}} = \dfrac{1}{0} \times 1 =$ infinite, just as the area of the hyperbola is on each side of the line BD. [13, pp.208-209]

---

[4] Newton used the term *superficies* (surface) for the region described by the line segment BD in all examples of Rule II.

[5] In examples of Rule II, Newton used the term *linea* (line) for the line segment. Westfall stated “*De Analysi* did not confine itself to the method of calculating areas. It also expounded Newton’s concept of the generation of areas by the motion of lines, whereby infinitesimal moments are continuously added to the finite area already generated.” [11, p.205]

[6] Edward remarked “It may be noted that Newton habitually ignored the ‘constant of integration’, taking all of this curves to pass through the origin.”[2, p.196]

Example 6 is expressed as

$$\int_0^x \frac{1}{x}dx = \lim_{\epsilon\to+0}\int_\epsilon^x \frac{1}{x}dx = \lim_{\epsilon\to+0}(\log x - \log\epsilon) = +\infty,$$
$$\int_x^\infty \frac{1}{x}dx = \lim_{b\to\infty}\int_x^b \frac{1}{x}dx = \lim_{b\to\infty}(\log b - \log x) = +\infty,$$

in modern calculus. In *De Analysi*, Newton expressed the antiderivative of $\frac{1}{x}$ as $\boxed{\frac{1}{x}}$. See section 4.2 and section 9 of this paper.

§ 3. Examples of Rule II

Newton gave three examples of Rule II with $y = a_1x^{\frac{m_1}{n_1}} + a_2x^{\frac{m_2}{n_2}}$. The first examples are the case $\frac{m_1}{n_1} > 0$ and $\frac{m_2}{n_2} > 0$.

> Let its first examples be these. If $x^2 + x^{\frac{3}{2}} = y$, then $\frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} =$ ABD. For if there be always BF $= x^2$ and FD $= x^{\frac{3}{2}}$, then by the preceding rule $\frac{1}{3}x^3 =$ the surface AFB described by the line BF and $\frac{2}{5}x^{\frac{5}{2}} =$ AFD described by DF; and consequently $\frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} =$ the whole ABD. [...] [13, pp.208-209]

For the example $x^2 + x^{\frac{3}{2}} = y$, let BE $= x^{\frac{3}{2}}$ (see Fig. 3). By Cavalieri's principle, the areas of AFD and ABE are each equal to $\frac{2}{5}x^{\frac{5}{2}}$.

Newton gave

$$\text{ABD} = \text{ABF} + \text{AFD} = \frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}}.$$

On the other hand, by Rule II,

$$\text{ABD} = \text{ABF} + \text{ABE} = \frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}}.$$

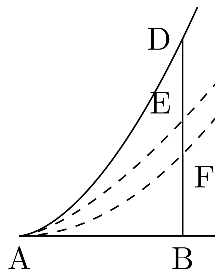
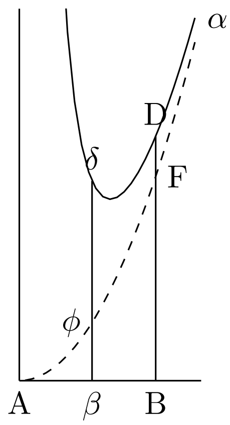


Fig. 3

The third examples are the case $\frac{m_1}{n_1} > 0$ and $\frac{m_2}{n_2} < -1$. The first of these is the curve $y = x^2 + x^{-2}$.

> Third examples. If $x^2 + x^{-2} = y$, then $\frac{1}{3}x^3 - x^{-1}$ =the surface described. But here you should note that the parts of the said surface thus found lie on opposite sides of the line BD: precisely, on setting BF $= x^2$ and FD $= x^{-2}$, then $\frac{1}{3}x^3 =$ the surface ABF described by BF and $-x^{-1} =$ DFα describes by DF.



[13, pp.210-211]

Newton represented the surface described by the line segment $\mathrm{BD} = x^2 + x^{-2}$ as

$$\mathrm{ABF} \cup \mathrm{DF}\alpha.$$

By Cavalieri's principle , the areas of $\mathrm{DF}\alpha$ and $\mathrm{BE}\alpha'$ are equal, and by Example 1 and Example 4 of Rule I, the area of $\mathrm{ABF} \cup \mathrm{DF}\alpha$ is

$$\frac{1}{3}x^3 - x^{-1}.$$

This area coincides with the antiderivative of $x^2 + x^{-2}$. Here $\mathrm{BE} = x^{-2}$, and α' is the point at infinity on the x-axis.

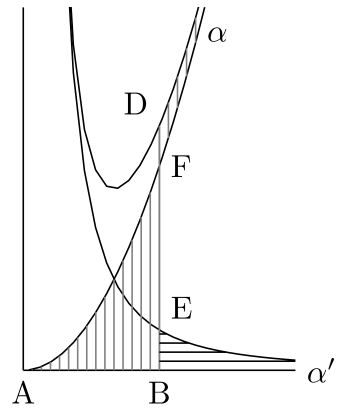


Fig. 4

§ 4. Applications of Rule II to infinite series

§ 4.1. Extraction by division

In *De Analysi*, Newton obtained infinite series expansions by division, and applied Rule II to those series.

> In the same way if $\dfrac{1}{1+xx} = y$, by division there arises $y = 1 - xx + x^4 - x^6 + x^8$&c. Hence by Rule II there will be $\mathrm{ABDC} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$&c [see Fig. 5]. Or, if the term xx be set first in the divisor, in this way $xx+1)1$, there appears $x^{-2} - x^{-4} + x^{-6} - x^{-8}$&c for the value of y; and hence by Rule II there will be $\mathrm{BD}\alpha = -x^{-1} + \frac{x^{-3}}{3} - \frac{x^{-5}}{5} + \frac{x^{-7}}{7}$&c [see Fig. 6]. Proceed by the former way when x is small enough, by the latter when it is taken large enough.
>
> [13, pp.212-215]

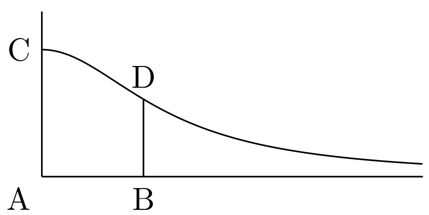


Fig. 5

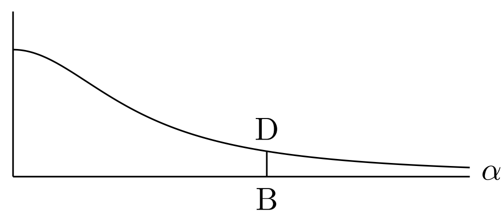


Fig. 6

The series $-x^{-1} + \frac{1}{3}x^{-3} - \frac{1}{5}x^{-5} + \frac{1}{7}x^{-7} - \cdots$ is the antiderivative of $x^{-2} - x^{-4} + x^{-6} - x^{-8} + \cdots$, not the ordinary area of $\mathrm{BD}\alpha$. In modern calculus, when $x > 1$ the ordinary area of $\mathrm{BD}\alpha$ is

$$\int_x^\infty \frac{1}{1+x^2} = \frac{\pi}{2} - \tan^{-1} x = \tan^{-1}\frac{1}{x} = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{2j-1}x^{-(2j-1)}.$$

§ 4.2. The literal resolution of affected equation

In *De Analysi*, Newton gave the literal resolution of affected equations[7] and applied Rule II to the quotient.

> Suppose now that the literal equation $y^3 + aay - 2a^3 + axy - x^3 = 0$ has to be resolved. [...]
> Finally, that quotient $\left(a - \frac{x}{4} + \frac{xx}{64a} \&c\right)$ will, by Rule II, yield $ax - \frac{xx}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2} + \frac{509x^5}{81920a^3} \&c$ for the area sought, an expansion which approaches more rapidly to the truth the smaller x is. [13, pp.222-223, 226-227]

By using modern calculus, let $f(x, y) = y^3 + a^2y - 2a^3 + axy - x^3$. The implicit function $\phi(x)$ of $f(x, y) = 0$ with $\phi(0) = a$ can be expanded to

$$\phi(x) = a - \frac{1}{4}x + \frac{1}{64a}x^2 + \frac{131}{512a^2}x^3 + \frac{509}{16384a^3}x^4 + \cdots . \tag{4.1}$$

See [9] for Newton's algorithm and its modern proof. Let C be the y-intercept of the graph of $y = \phi(x)$ as in Fig. 7. Then the area ABDC (see Fig. 7) is

$$\int_0^x \phi(x)dx = ax - \frac{1}{8}x^2 + \frac{1}{192a}x^3 + \frac{131}{2048a^2}x^4 + \frac{509}{81920a^3}x^5 + \cdots , \tag{4.2}$$

which is the antiderivative of (4.1).

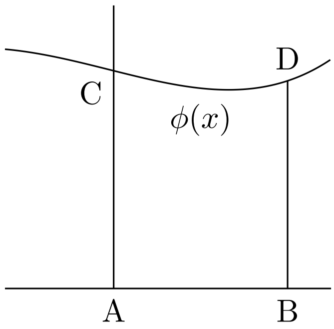


Fig. 7 ($a = 1$)

Moreover Newton applied Rule II to the quotient of the literal resolution of an affected equation for x sufficient large.

> But if you wish that the value of the area should approach nearer the truth the greater x is, take this an example:$y^3 + axy + x^2y - a^3 - 2x^3 = 0$. [...]

[7] The affected equation is an algebraic equation $f(x, y) = 0$, and its quotient is the implicit function defined by $f(x, y) = 0$.

Then I suppose $x+p=y$ and so proceed as in the former example until I have the quotient $x-\dfrac{a}{4}+\dfrac{aa}{64x}+\dfrac{131a^3}{512xx}+\dfrac{509a^4}{16384x^3}$&c, so that the area is

$$\frac{x^2}{2}-\frac{ax}{4}+\boxed{\frac{aa}{64x}}-\frac{131a^3}{512x}-\frac{509a^4}{32768x^2}.$$

Relating to this see the third examples of Rule II. [...] [13, pp.226-227]

As Newton wrote in Example 6 of Rule I, the area of the region described by the line segment $\frac{aa}{64x}$ is infinite. Thus $\boxed{\frac{aa}{64x}}$ is not the area but the antiderivative of $\frac{aa}{64x}$, that is $\frac{a^2}{64}\log x$, in modern form. Whiteside corrected [13, p.227] the area[8] of the quotient as

$$\frac{1}{2}x^2-\frac{1}{4}ax+\left[\int\frac{a^2}{64x}.dx\right][+]\frac{131a^3}{512x}[+]\frac{509a^4}{32768x^2}\cdots,$$

but two corrections [+] do not conform to Newton's intention. This is because Newton represents the antiderivative as the area.

§ 5. Newton's true intentions of Rule I and Rule II

In modern calculus Newton's true intentions of Rule I is as follows: The region described by the line segment $\mathrm{BD}=ax^{\frac{m}{n}}$ and its signed area are

$$\frac{na}{m+n}x^{\frac{m+n}{n}}=\begin{cases}\displaystyle\int_0^x ax^{\frac{m}{n}}\,dx=\mathrm{ABD}, & \text{if } \frac{m}{n}>0,\\ \displaystyle\int_\infty^x ax^{\frac{m}{n}}\,dx=\alpha\mathrm{BD}, & \text{if } \frac{m}{n}<-1,\end{cases} \tag{5.1}$$

where A is the origin and α is the point at infinity on the x-axis (see Fig. 1 and 2). In Rule I, Newton gave the antiderivative of a simple curve $ax^{\frac{m}{n}}$ as the pair of the region ABD or αBD and its signed area.

[8] In modern mathematics, let the quotient

$$\phi(x)\sim x-\frac{a}{4}+\frac{a^2}{64x}+\frac{131a^3}{512x^2}+\frac{509a^4}{16384x^3}\cdots.$$

Then

$$\int_x^\infty\left(\phi(x)-x+\frac{a}{4}-\frac{a^2}{64x}\right)dx\sim\frac{131a^3}{512x}+\frac{509a^4}{32768x^2}\cdots,$$

where $\sim$ is represented as an asymptotic expansion. See [3, p.21].

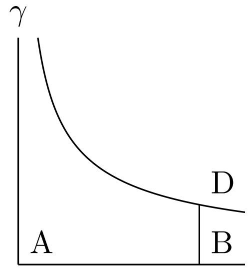


Fig. 8

There is a term $x^{-\frac{3}{5}}$ in the last curve $2x^3 - 3x^5 + x^{-\frac{3}{5}} - \frac{2}{3}x^{-4} = y$ in the third examples of Rule II. Newton gave the area of the region described by the line segment $\mathrm{BD} = x^{\frac{-3}{5}}$ as $\frac{5}{2}x^{\frac{2}{5}}$ without explanation. To explain using modern calculus,

$$\gamma\mathrm{ABD} = \int_0^x x^{-\frac{3}{5}}\,dx = \lim_{\epsilon\to+0}\int_\epsilon^x x^{-\frac{3}{5}}\,dx = \tfrac{5}{2}x^{\frac{2}{5}},$$

where γ is the point at infinity on the y-axis (see Fig. 8).

This example can be generalized that if $-1 < \frac{m}{n} < 0$,

$$\frac{na}{m+n}x^{\frac{m+n}{n}} = \int_0^x ax^{\frac{m}{n}}\,dx = \gamma\mathrm{ABD}. \tag{5.2}$$

However, in *De Analysi*, Newton neither did explicitly state (5.2), nor did he show a figure[9] corresponding to Fig. 8.

Newton gave Rule II to the sum of a finite number of terms, but, as we saw in the previous section, he applied it to infinite series. Newton's true intention of Rule II is that the antiderivative of

$$y = a_0x^{-1} + \sum_i a_ix^{\frac{m_i}{n_i}}, \quad (\tfrac{m_i}{n_i} \neq -1)$$

is

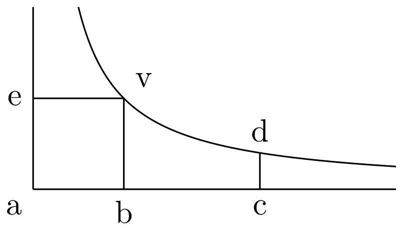
$$\int_1^x a_0x^{-1}dx + \sum_{\frac{m_i}{n_i}>-1}\int_0^x a_ix^{\frac{m_i}{n_i}}\,dx + \sum_{\frac{m_i}{n_i}<-1}\int_\infty^x a_ix^{\frac{m_i}{n_i}}\,dx$$
$$= \boxed{\frac{a_0}{x}} + \sum_i \frac{n_ia_i}{m_i+n_i}x^{\frac{m_i+n_i}{n_i}}.$$

§ 6. History of the antiderivative by Newton

§ 6.1. The antiderivative in Newton's early works

Newton gave the antiderivative of $y = a^2/(a+x)$ in the manuscript [7, 78v] which is estimated to have been written autumn 1665 by Whiteside [12, p.122].

[9]In the first examples of Problem 9 of *De Methodis*, Newton gave a figure similar to Fig. 8. See section 6.3 of this paper.



If $\bar{e}$a||vb||dc⊥ac||$\bar{e}$v $= a$. & bc $= x$. & dc $= y = \dfrac{aa}{a+x}$. [...] The product will bee y[e] area vbcd. viz

$$\text{vbcd} = ax - \frac{xx}{2} + \frac{x^3}{3a} - \frac{x^4}{4a^2} + \frac{x^5}{5a^3} - \frac{x^6}{6a^4} + \frac{x^7}{7a^5} - \frac{x^8}{8a^6} + \frac{x^9}{9a^7} - \frac{x^{10}}{10a^8} + \frac{x^{11}}{11a^9} - \frac{x^{12}}{12a^{10}} \&c.$$

[12, pp.134-138]

The area vbcd is the antiderivative of the hyperbola. In modern notation,

$$\text{vbcd} = \int_0^x \frac{a^2}{a+x}dx = a^2 \log(1 + \frac{x}{a}). \tag{6.1}$$

In the same manuscript [7, 81v] Newton wrote

If	Its area[10] is
$\dfrac{x^6}{a+bx} = y.$	$\dfrac{x^6}{6b} - \dfrac{ax^5}{5bb} + \dfrac{aax^4}{4b^3} - \dfrac{a^3x^3}{3b^4} + \dfrac{a^4xx}{2b^5} - \dfrac{a^5x}{b^6}$ $+\text{area of} \left[\dfrac{a^6}{b^7x + ab^6} = z.\right]^{11}$
$\dfrac{x^5}{a+bx} = y.$	$\dfrac{x^5}{5b} - \dfrac{ax^4}{4bb} + \dfrac{aax^3}{3b^3} - \dfrac{a^3xx}{2b^4} + \dfrac{a^4x}{b^5} - \square \text{ of } \left[\dfrac{a^5}{b^6x + ab^5} = z.\right]$

[...] [12, p.342]

The area or $\square$ means the antiderivative of y or z.

Newton provided the prototype of Rule I of *De Analysi* in the manuscript [8, 152v] which was written around the same time as the manuscript above.

If $apx^{\frac{m}{n}} = q.$ then $\frac{na}{m+n}x^{\frac{m+n}{n}} = y.$ [12, p.344]

Here $p = \frac{dx}{dt}$ and $q = \frac{dy}{dt}$, in modern notation. Thus, this statement is equivalent to "If $\frac{dy}{dx} = ax^{\frac{m}{n}}$, then $y = \frac{na}{m+n}x^{\frac{m+n}{n}}$".

§ 6.2. The antiderivative in the October 1666 tract

In 1666, Newton put together a study on the fluxional theory that had been obtained and wrote it as the October 1666 tract.

[10] Similar to (6.1), "Its area" is $\displaystyle\int_0^x \frac{x^6}{bx+a}dx$, and so on.

[11] This is obtained by division

$$\frac{x^6}{bx+a} = \frac{x^5}{b} - \frac{ax^4}{b^2} + \frac{a^2x^3}{b^3} - \frac{a^3x^2}{b^4} + \frac{a^4x}{b^5} - \frac{a^5}{b^6} + \frac{a^6}{b^6}\frac{1}{bx+a},$$

and then termwise integration.

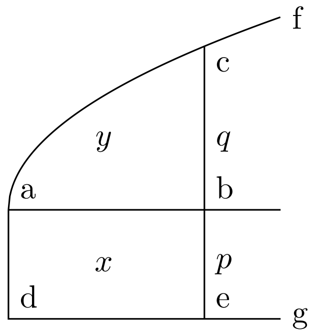
8. If two Bodys A&B, by their velocitys $p\&q$ describe y$^{\rm e}$ lines $x\&y$. & an Equation bee given expressing y$^{\rm e}$ relation twixt one of y$^{\rm e}$ lines x, & y$^{\rm e}$ ratio $\frac{q}{p}$ of their motions $q\&p$; To find y$^{\rm e}$ other line y. [...]
As if $ax^{\frac{m}{n}} = \dfrac{q}{p}$. Then is $\dfrac{na}{n+m}x^{\frac{n+m}{n}} = y$. [12, p.403]

The last statement is also equivalent to "If $\frac{dy}{dx} = ax^{\frac{m}{n}}$, then $y = \frac{na}{m+n}x^{\frac{m+n}{n}}$".

In Problem 5 Newton gave the fundamental theorem of calculus.

Prob. 5$^{\rm t}$. *To find y^{e} nature of y^{e} crooked line whose area is expressed by any given equation.* That is, y$^{\rm e}$ nature of y$^{\rm e}$ area being given to find y$^{\rm e}$ nature of y$^{\rm e}$ crooked line whose area it is.

Resol. If y$^{\rm e}$ relation of ab $= x$, & $\triangle$abc $= y$ bee given y$^{\rm e}$ relation of ab $= x$, &bc $= q$ bee required (bc being ordinately applied at right angles to ab). Make de$||$ab$\perp$ad$||$be $= 1$. & y$^{\rm n}$ is $\square$abed $= x$. Now supposing y$^{\rm e}$ line cbe be parallel motion from ad to describe y$^{\rm e}$ two superficies ae $= x$, & abc $= y$; The velocity w$^{\rm th}$ w$^{\rm ch}$ they increase will bee, as be to bc;



y$^{\rm t}$ is, y$^{\rm e}$ motion by w$^{\rm ch}$ x increaseth will bee bc $= q$. which therefore may bee found by prop: 7$^{\rm th}$. viz: $\dfrac{-\,{\cdot}\mathfrak{X}y}{\mathfrak{X}{\cdot}x} = q = $ bc. [12, p.427]

Here Newton's notation ${\cdot}\mathfrak{X}$ and $\mathfrak{X}{\cdot}$ are homogeneous partial derivatives: i.e.,

$$ {\cdot}\mathfrak{X} = xf_x(x,y), \qquad \mathfrak{X}{\cdot} = yf_y(x,y), $$

for an algebraic curve $\mathfrak{X} \equiv f(x,y) = 0$, in modern calculus. Since

$$ f_x(x,y) + \frac{dy}{dx}f_y(x,y) = 0, $$

Newton proved

$$ q = \frac{-\,{\cdot}\mathfrak{X}y}{\mathfrak{X}{\cdot}x} = \frac{dy}{dx}. \tag{6.2} $$

Therefore, the area y is the antiderivative of the ordinate bc $= q$.

Based on Problem 5 (the fundamental theorem of calculus[12]), in Rule I in *De Analysi*, Newton expressed the antiderivative of a curve as the area of the region described by the motion of the ordinate of the curve.

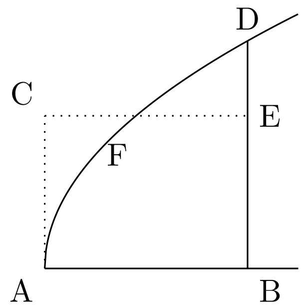
[12]Guicciardini states "An important consequence of this [the fundamental theorem of calculus] is the possibility of computing integrals by using an antiderivative of the function to be integrated, [...]"[4, p.182].

§ 6.3. The antiderivative in *De Methodis*

In 1671 Newton wrote *De Methodis* which is a development of the content of both the October 1666 tract and *De Analysi*. He also used the method of representing the antiderivative by the signed area in *De Methodis*.

PROBLEM 9.
TO DETERMINE THE AREA OF ANY PROPOSED CURVE

The resolution of the problem is based on that of establishing the relationship between fluent quantities from one between their fluxions (by Problem 2)[13].
[...]
Call AB $= x$, therefore, and there will be also ABEC$(= 1 \times x) = x$ and BE $= m[= \frac{dx}{dt}]$: further, call the area AFDB $= z$ and then BD $= r[= \frac{dz}{dt}]$ or, equivalently, $\frac{r}{m}$, since $m = 1$. Consequently, by the equation defining BD is at once defined the fluxional ratio $\frac{r}{m}[= \frac{dz}{dx}]$, and from this (by Problem 2, Case 1) will be elicited the relationship of the fluent quantities x and z.



[14, pp.210-211]

The resolution to Problem 9 for finding the area z of the curve with ordinate BD is to solve the fluxional equation

$$\frac{r}{m}\left[= \frac{dz}{dx}\right] = \mathrm{BD}.$$

Thus the area z is the antiderivative of BD. Since A is the origin,

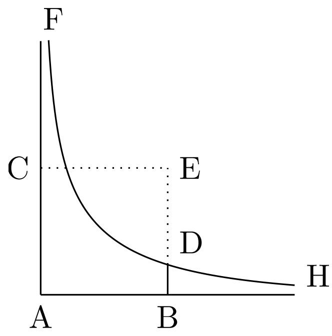
$$z = \int_0^x \mathrm{BD}dx,$$

in modern notation.

FIRST EXAMPLES: when BD (that is $r[= \frac{dz}{dt}]$) is, in value, some simple quantity. Let there be given $\frac{xx}{a} = r$ or $\frac{r}{m}[= \frac{dz}{dx}]$, namely equation to a parabola, and there will (by Problem 2) emerge $\dfrac{x^3}{3a} = z$. Therefore $\dfrac{x^3}{3a}$ or $\frac{1}{3}$AB $\times$ BD is equal to the parabolic area AFDB. [...]

[13]Problem 2 in *De Methodis* is to solve fluxional equations: "When an equation involving the fluxions of quantities is exhibited, to determine the relation of the quantities one to another." [14, pp.82-83]

Let there be given $\frac{a^3}{xx} = r$ or $a^3x^{-2} = r$, the equation to a second-order hyperbola, and there will emerge $-a^3x^{-1} = z$ or $-\frac{a^3}{x} = z$: that is, AB $\times$ BD equals the infinitely extended area HDBH lying on the further side of the ordinate BD, as its negative value conveys. [...]
Further, let $ax = rr$, the equation again to a parabola, and there will come out $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}}$, that is, $\frac{2}{3}$AB $\times$ BD = area AFDB.



[14, pp.210-211]

The last equation $ax = rr$ is

$$ax = \left(\frac{dz}{dx}\right)^2,$$

in modern calculus. Newton solved $a^{\frac{1}{2}}x^{\frac{1}{2}} = r[= \frac{dz}{dx}]$ as $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} = z$.

The first examples are generalized as follows. Let $ax^{\frac{m}{n}} = r\left[= \frac{dz}{dx}\right]$. Since $\frac{na}{n+m}x^{\frac{n+m}{n}} = \frac{n}{n+m}xax^{\frac{m}{n}}$,

$$\frac{n}{n+m}\mathrm{AB} \times \mathrm{BD} = \begin{cases} \text{the area ADB}, & \text{if } \frac{m}{n} > 0, \\ \text{the area AFDB}, & \text{if } -1 < \frac{m}{n} < 0, \\ \text{the area HDBH}, & \text{if } \frac{m}{n} < -1. \end{cases}$$

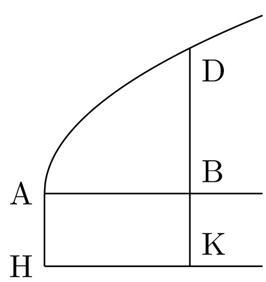
This generalization corresponds to (5.1) for Rule I in *De Analysi*.

§ 7. Moments in *De Analysi*

§ 7.1. Introduction of moments

Newton introduced the term *momentum* (moment) to apply the three rules to the problem of finding the length of a curve or the area under a curve.

Let ABD be any curve and AHKB a rectangle whose side AH or BK is unity. And consider that the straight line DBK describes the areas ABD and AK as it moves uniformly away from AH; that BK(1) is the moment by which AK(x) gradually increases and BD(y) that by which ABD does so; and that, when given continuously the moment of BD, you can by the foregoing rules investigate



the area ABD described by it or compare it with AK(x) described with a unit moment. Now, by the same means as the surface ABD is elicited by the foregoing rules from its continuously given moment, any other quantity will be elicited from its moment thus given. The matter will be clarified by example.

[13, pp.232-233]

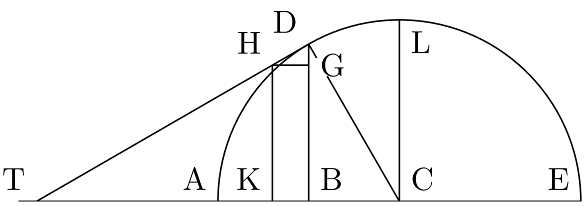
As in Rule I, Newton introduced the moment by examples, but did not give a rigorous definition. The diagram used for the introduction is the same as Problem 5 in the October 1666 tract (see section 6.2), and BD in Problem 5 is the derivative or the velocity of the area ABD. Whereas Newton called BD the moment of the area ABD which is gradually increases.

The meaning of the moment here will be discussed in section 7.2 and 8.4.

§ 7.2. The length of the arc

Newton used the moment to find the length of the arc.

Let ADLE be a circle whose arc length AD is to be discovered. On drawing the tangent DHT, completing the indefinitely small rectangle HGBK and setting AE = 2AC = 1,



there will then be BK or GH (the moment of the base AB) to DH (the moment of the arc AD)[= BK : DH]

$$= \mathrm{BT} : \mathrm{DT} = \mathrm{BD}(\sqrt{x-x^2}) : \mathrm{DC}(\tfrac{1}{2}) = 1(\mathrm{BK}) : \frac{1}{2\sqrt{x-x^2}}(\mathrm{DH})$$

so that $\dfrac{1}{2\sqrt{x-x^2}}$ or $\dfrac{\sqrt{x-x^2}}{2(x-x^2)}$ is the moment of the arc AD. When reduced this becomes

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \frac{35}{256}x^{\frac{7}{2}} + \frac{63}{512}x^{\frac{9}{2}}\&\text{c.} \tag{7.1}$$

Therefore by Rule II the length of the arc AD is

$$x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} + \frac{35}{1152}x^{\frac{9}{2}} + \frac{63}{2816}x^{\frac{11}{2}}\&\text{c}, \tag{7.2}$$

[...]

[13, pp.232-233]

Newton obtained the length (7.2) of $\overset{\frown}{\mathrm{AD}}$ by integrating the series expansion (7.1) of the moment of $\overset{\frown}{\mathrm{AD}}$ by terms. From this fact the moment is not the velocity[14]; if BK = 1

[14] Hara wrote "Isn't it now clear that the moment does not mean speed? It is not the author's intention that the speed of x becomes infinitesimally small in BK. The argument of the next work [*De Methodis*], which calls differential not velocity the moment, has already started here." [5, pp.306-307]

then the moment is the derivative, if BK $= o$ then the moment is the differential[15] in modern calculus.

§ 8. History of the moment in the sense of *De Methodis*

§ 8.1. The definition of the moment in *De Methodis*

In 1671 Newton gave the following definition of the moment in *De Methodis*.

> The moments of the fluent quantities (that is, their indefinitely small parts, by addition of which they increase during each infinitely small period of time) are as their speeds of flow. Wherefore if the moment of any particular one, say x, be expressed by the product of its speed $m[=\frac{dx}{dt}]$ and an infinitely small quantity o (that is, by $mo[=\frac{dx}{dt}o]$), then the moments of the others, v, y, z, will be expressed by $lo[=\frac{dv}{dt}o], no[=\frac{dy}{dt}o], ro[=\frac{dz}{dt}o]$ seeing that lo, mo, no, and ro are to one another as l, m, n, and r. [...] [14, pp.78-81]

Since o is "infinitely small period of time", o is the differential dt in modern calculus. Thus the moment of y is

$$\frac{dy}{dt}o = \frac{dy}{dt}dt = dy,$$

which is the differential of y.

§ 8.2. The moment in the October 1666 tract

Newton used the moment in the October 1666 tract. In the proof of Prop 7[16] of the tract, Newton wrote:

> *Prop 7 Demonstrated.* [...]
>
> As if y$^{\mathrm{e}}$ body A w$^{\mathrm{th}}$ y$^{\mathrm{e}}$ velocitys p describe y$^{\mathrm{e}}$ infinitely little line (cd $=$)$p \times o$ in one moment, in y$^{\mathrm{t}}$ moment y$^{\mathrm{e}}$ body B w$^{\mathrm{th}}$ y$^{\mathrm{e}}$ velocitys q will describe y$^{\mathrm{e}}$ line (gh $=$)$q \times o$. [12, p.414]

The above $p \times o$ and $q \times o$ are the moments of x and y, respectively, in the sense of *De Methodis*.

§ 8.3. The proof of Rule I in *De Analysi*

At the end of *De Analysi*, Newton proved Rule I with $\frac{m}{n} > 0$ using the moment in the sense of *De Methodis*.

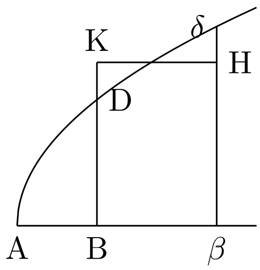
---

[15] Let $y = f(x)$ be a function. The differential dy is defined by $dy = f'(x)dx$, where $f'(x)$ is the differential coefficient of f, and dx is the infinitesimal increment of the variable x.

[16] Prop 7 is "Haveing an Equation expressing y$^{\mathrm{e}}$ relation twixt two or more lines x, y, z,&c: described in y$^{\mathrm{e}}$ same time by two or more moving bodys A, B, C,&c: the relation of their velocities p, q, r,&c may bee thus found, [...]". [12, p.402]

Preparation for demonstrating the first rule.

1. The quadrature of simple curves in Rule I. Let then any curve ADδ have base AB $= x$, perpendicular ordinate BD $= y$ and area ABD $= z$, as before. Likewise take Bβ $= o$, BK $= v$ and the rectangle BβHK(ov) equal to the space B$\beta\delta$D. It is, therefore, A$\beta = x + o$ and A$\delta\beta = z + ov$.



[13, pp.242-245]

In modern calculus, let $o = dx$. Since

$$\frac{\mathrm{A}\beta\delta - \mathrm{ABD}}{o} = \frac{\mathrm{B}\beta\delta\mathrm{D}}{o} = \frac{\mathrm{B}\beta\mathrm{HK}}{o} = \frac{ov}{o} = v,$$

v is the derivative of ABD. Therefore

$$ov = \frac{dz}{dx}dx = dz,$$

which is the differential of z.

Furthermore if we assume that the velocity of x is 1, then $\frac{dz}{dt} = \frac{dz}{dx} = v$, and $o = dx = dt$. In this case vo becomes the moment of z in the sense of *De Methodis*.

§ 8.4. The meaning of the moment in *De Analysi*

In the example of the arc length, Newton described BK(1),i.e., BK $= 1$, and wrote the moment BK of the base AB to be indefinitely small, this is contradictory. Thus we correct BK(1) to BK(o) and "AK(x) described with a unit moment" to "AK(x) described with an indefinitely small moment". Then Newton's example becomes as follows:

To make $o > 0$, K is taken to the right of B (See Fig. 9). As Newton wrote BK is the moment of AB and DH is the moment of the arc AD. Since

$$\frac{\mathrm{BK}}{\mathrm{DH}} = \frac{\mathrm{BD}}{\mathrm{CD}} = \frac{\sqrt{x - x^2}}{\frac{1}{2}},$$

we have

$$\mathrm{DH} = \frac{o}{2\sqrt{x - x^2}}.$$

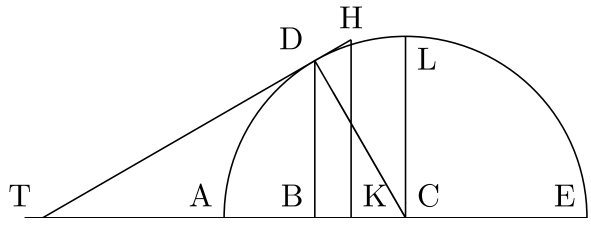


Fig. 9

From here on, we will use the notation of modern calculus[17]. Let $o = dx$, then by expanding to an infinite series

$$\text{the moment of } \widehat{\mathrm{AD}} = \mathrm{DH} = \frac{dx}{2\sqrt{x - x^2}}$$

$$= \left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \frac{35}{256}x^{\frac{7}{2}} + \frac{63}{512}x^{\frac{9}{2}} + \cdots\right)dx, \quad 0 < x < 1,$$

[17] Dunham[1, pp.16-17] did similar rewriting.

and by Rule II

$$\begin{aligned}\widehat{\mathrm{AD}} &= \int_0^x \left(\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{4}x^{\frac{1}{2}} + \frac{3}{16}x^{\frac{3}{2}} + \frac{5}{32}x^{\frac{5}{2}} + \frac{35}{256}x^{\frac{7}{2}} + \frac{63}{512}x^{\frac{9}{2}} + \cdots\right) dx \\ &= x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}} + \frac{35}{1152}x^{\frac{9}{2}} + \frac{63}{2816}x^{\frac{11}{2}} + \cdots \\ &= \sin^{-1}\sqrt{x}, \quad 0 < x < 1.\end{aligned}$$

Since

$$\frac{d}{dx}\sin^{-1}\sqrt{x} = \frac{1}{\sqrt{x - x^2}}, \quad 0 < x < 1,$$

we have

$$\text{the moment of } \widehat{\mathrm{AD}} = \left(\frac{d}{dx}\widehat{\mathrm{AD}}\right) dx = d(\widehat{\mathrm{AD}}).$$

Moreover by "it moves uniformly" and the error "BK(1)", Newton assumed that the velocity of x is unity, i.e., $\frac{dx}{dt} = 1$, and then

$$\frac{d}{dt}\sin^{-1}\sqrt{x} = \frac{d}{dx}\sin^{-1}\sqrt{x} = \frac{1}{\sqrt{x - x^2}}.$$

Thus the moment of $\widehat{\mathrm{AD}}$ can be expressed as $\frac{d}{dt}\widehat{\mathrm{AD}} \times o$, which is the moment of $\widehat{\mathrm{AD}}$ in the sense of *De Methodis*.

Therefore, if BK $= 1$ is corrected to BK $= o$, the moment in *De Analysi* becomes the differential and coincides with that in *De Methodis*.

§ 9. A counter observation on Leibniz' review by Newton in 1713

In 1713, Newton wrote a counter observation on Leibniz' review on *De Analysi*, more than 40 years after writing,

> M$^{\mathrm{r}}$ Newton in his *Analysis* sometimes represents fluents by the areas of curves & their fluxions by y$^{\mathrm{e}}$ Ordinates, & moments by the Ordinates drawn into y$^{\mathrm{e}}$ letter o. So where the Ordinate is $\dfrac{aa}{64x}$ he puts $\boxed{\dfrac{aa}{64x}}$ for the area. And so if the Ordinate be v or y the Area will be $\boxed{v}$ or $\boxed{y}$. And in this way of notation the moments will be $\dfrac{aao}{64x}, vo, yo$. M$^{\mathrm{r}}$ Leibniz instead of the Notes $\boxed{\dfrac{aa}{64x}}$, $\boxed{v}$, $\boxed{y}$ uses the notes $\displaystyle\int \frac{aa}{64x}, \int v, \int y$. [13, p.273]

What Newton said about his earlier work during the priority dispute is not necessarily true, but the statement "represents fluents by the areas of curves & their fluxions by y$^{\mathrm{e}}$ Ordinate" is true as stated in sections 2-4. The statement "moments by the

Ordinates drawn into y$^{\mathrm{e}}$ letter o" referred to the proof of Rule I, not to the moment introduced in *De Analysi*. If BK $=1$ is corrected to BK $=o$, the moment in *De Analysi* can be represented as "the Ordinates drawn into y$^{\mathrm{e}}$ letter o".

Table 1. Comparison of representations

De Analysi	*De Methodis*	modern calculus
ordinate	fluxion $(m[=\frac{dx}{dt}], n[=\frac{dy}{dt}], \dots)$	velocity $(\frac{dx}{dt}, \frac{dy}{dt}, \dots)$
area of the curve	fluent $(x, y, \dots)$	antiderivative
moment †	moment $(mo[=\frac{dx}{dt}o], no,)$	differential $(dx, dy,)$
ordinate drawn into o ‡		

†The case BK $=1$ is corrected to BK $=o$.

‡We assume that the velocity of x is 1.

Following Newton's above statement, we summarize in Table 1 how he represented in *De Analysi* the concepts he used in *De Methodis*. According to Table 1, Newton replaced the concepts and terminology of fluxional theory with those of geometry. From this, it is considered that Newton consciously avoided describing fluxional theory in *De Analysi*.

§ 10. Why did Newton not explicitly use fluxions in *De Analysi*?

Newton wrote *De Analysi* (1669) to claim the priority of the method of analysis using infinite series. The procedure of this method is to expand the derivative or the differential of a sought quantity into an infinite series of the form

$$\sum_i a_i x^{\frac{m_i}{n_i}}, \tag{10.1}$$

and then to obtain the quantity by termwise integration. In order to perform this method it is sufficient that

1. To find the antiderivative of $ax^{\frac{m}{n}}$.

2. The possibility of termwise integration.

3. To expand the derivative or the differential of the quantity into an infinite series of the form (10.1).

Newton gave Rule I, II, and III in *De Analysi* which correspond to the above 1,2, and 3, respectively.

In the October 1666 tract, Newton represented the antiderivative of $ax^{\frac{m}{n}}$ by

$$\text{As if } ax^{\frac{m}{n}} = \frac{q}{p}\left[= \frac{dy}{dx}\right]. \text{ Then is } \frac{na}{m+n}x^{\frac{m+n}{n}} = y,$$

using fluxional equation, but in Rule I of *De Analysi*, he represented it as

$$\text{If } ax^{\frac{m}{n}} = y. \text{ then will } \frac{na}{m+n}x^{\frac{m+n}{n}} \text{ equal the area ABD,}$$

without using the fluxional equations. Newton specified the region ABD (or αBD in Example 4) so that the area of the region described by the ordinate of the curve $y = ax^{\frac{m}{n}}$ would match the antiderivative of $ax^{\frac{m}{n}}$, and he gave the antiderivative of $ax^{\frac{m}{n}}$ as its signed area. He gave not only the area of the region but also the antiderivative of y.

In *De Methodis* (1671), Newton gave some examples corresponding to Rule I of *De Analysi* using the fluxional equations: e.g.,

> If there be given $\frac{n}{m}\left[= \frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right] = \frac{x}{a}$, I multiply $\frac{x}{a}$ by x and there comes $\frac{xx}{a}$. Here since x is of two dimensions I divide by 2 and there comes $\frac{xx}{2a}$, which I set equal to y. [14, pp.82-83]

Only in *De Analysi*, Newton did not use fluxional equations. According to Newton's counter observation in 1713, Newton replaced all the terms and concepts of fluxional method with those of geometry in *De Analysi.*

Why did Newton represent the antiderivative by the region and its area without using the fluxional equation in *De Analysi*? The first reason is that he did not desire to show the fluxional theory[18] at the time of 1669. The second reason is that he was able to describe the method of infinite series without explicitly using fluxional theory. Area and moment in *De Analysi* are alternatives to fluent and ratio of fluxions in *De Methodis*, respectively.

§ 11. Conclusion

Newton intended to claim priority of the method of infinite series, but he did not desire to disclose the fluxional method. To do so, he expressed the antiderivative as a pair of the region described by the ordinate and its signed area and he introduced the moment as an alternative to the ratio of fluxions.

[18] Newton addressed to Oldenburg on October 24, 1676 "But when there appeared that ingenious work, the *Logarithmotechinia* of Nicolas Mercator (whom I suppose to have made his discoveries first), I began to pay less attention to these things, suspecting that either he knew the extraction of roots as well as division of fractions, or at least that others upon the discovery of division would find out the rest before I could reach a ripe age for writing,"[10, pp.114,133].

Due to clever tricks and the mistaking of BK $= 1$ for BK $= o$ in the introduction of moments, it seems that most later historians of mathematics have failed to accurately grasp Newton's true intentions in *De Analysi*.

Acknowledgement

I would like to thank to Professor Niccolò Guicciardini for many helpful comments and suggestions which improve this paper.

References

[1] William Dunham, The Calculus Gallery: Masterpieces from Newton to Lebesgue, Princeton University Press, 2009.
`http://assets.press.princeton.edu/chapters/s7905.pdf`
[2] C.H. Edwards, Jr., The Historical Development of the Calculus, Springer-Verlag, 1979.
[3] A. Erdélyi, Asymptotic Expansions, Dover, 1956.
[4] Niccolò Guicciardini, Isaac Newton on mathematical certainty and method, MIT Press, 2009.
[5] Kokiti Hara, Early-modern Mathematics (in Japanese), in A History of Mathematics, Chikuma Shobō, 1975.
[6] Ryosuke Nagaoka, Mathematics of Newton (in Japanese), in Genealogy of natural philosophy of Newton, Heibonsha, 1987.
[7] Isaac Newton, Early Papers
`https://cudl.lib.cam.ac.uk/view/MS-ADD-03958/`
[8] Isaac Newton, Fluxions
`https://cudl.lib.cam.ac.uk/view/MS-ADD-03960/`
[9] Naoki Osada, Literal resolution of affected equations by Isaac Newton, RIMS Kôkyûroku Bessatsu, B73, 1-20 (2019)
[10] H.W. Turnbull, The Correspondence of Isaac Newton, Vol. II, Cambridge University Press, 1960.
[11] Richard Westfall, Never at Rest, Cambridge University Press, 1980.
[12] D.T. Whiteside, The Mathematical Papers of Isaac Newton, Vol.I, Cambridge at the University Press, 1967.
[13] D.T. Whiteside, The Mathematical Papers of Isaac Newton, Vol.II, Cambridge at the University Press, 1968.
[14] D.T. Whiteside, The Mathematical Papers of Isaac Newton, Vol.III, Cambridge at the University Press, 1969.