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Some explicit calculations of cohomology groups of (φ, Γ)-modules

By

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Contents

§ 1. Introduction
§ 2. Robba rings, (φ, Γ)-modules and analytic sheaves
§ 3. Cutting out representations from (φ, Γ)-modules
§ 4. Principal series
§ 5. Koszul complexes
§ 6. Some examples of cohomology calculations
References

Abstract

This is a survey article that advertises the idea of a p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ in the setting of not necessarily étale (φ, Γ)-modules of rank 2. In particular the cohomology of various (semi)-groups is considered.

§ 1. Introduction

Let $A^+ = \left(\begin{smallmatrix} \mathbf{Z}_p - \{0\} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\overline{P}^+ = \left(\begin{smallmatrix} \mathbf{Z}_p - \{0\} & 0 \\ p\mathbf{Z}_p & 1 \end{smallmatrix}\right)$. This article is devoted to a survey of the cohomology of these (semi)-groups and their appearance in the p-adic Langlands correspondence. Let L be a finite extension of $\mathbf{Q}_p$ and let $\mathscr{R}_L$ denote the Robba ring with coefficients in L, cf. Definition 2.2. Recall the ultimate goal is starting from Δ

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(a (φ,Γ)-module over $\mathscr{R}_L$, cf. Definition 2.4), one would look to produce a suitable L-representation $\Pi(\Delta)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ and vice-versa. To put things into context, let us recall the general history of this correspondence. Fix a linear continuous semi-stable 2-dimensional L-representation V of the absolute Galois group $\mathcal{G}_{\mathbf{Q}_p}$ of $\mathbf{Q}_p$ with distinct Hodge-Tate weights $w_1 < w_2$. To V one can associate a linear algebra object (cf. [6]), a so-called admissible filtered (φ, N)-module. Forgetting the filtration for a moment, this produces a Weil-Deligne representation to which one can in turn attach a smooth admissible representation π of $\mathrm{GL}_2(\mathbf{Q}_p)$ over L by the classical local Langlands correspondence (cf. [9]). The question posed by C. Breuil is starting from w_1, w_2 and π, can one recover V? That is, can one recover the filtration on the associated (φ, N)-module, after passing to π? This filtration should correspond to constructing an appropriate norm on the locally algebraic representation $\det^{w_1}\otimes_L \mathrm{Sym}^{w_2-w_1-1}(L^2)\otimes_L \pi$. In [1], this was achieved for small values of w_2-w_1. Upon completing $\det^{w_1}\otimes_L \mathrm{Sym}^{w_2-w_1-1}(L^2)\otimes_L \pi$ with respect to this norm, one obtains a Banach space representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ and the birth of the p-adic Langlands program.

In [3], [10] and [5], a bijection $V \mapsto \Pi(V)$ between absolutely irreducible 2-dimensional continuous L-representations of $\mathcal{G}_{\mathbf{Q}_p}$ and admissible unitary non-ordinary Banach L-representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ which are topologically absolutely irreducible is established. The point here is that the category of local Galois representations in L-vector spaces is equivalent to that of étale (φ,Γ)-modules, a certain subcategory of (φ,Γ)-modules. These turn out to be very useful intermediary objects for the construction of the correspondence in question. It is natural to ask the question whether such a correspondence can be realized for all (φ,Γ)-modules, not just étale objects. This question is addressed in [4] and [8]. One of the main results in [4] is the following.

Theorem 1.1 ([4, Théorème 0.1])**.** *Let Δ be a (φ,Γ)-module over $\mathscr{R}_L$ of rank 2 which is an extension of $\mathscr{R}_L(\delta_2)$ by $\mathscr{R}_L(\delta_1)$, where $\delta_1,\delta_2 : \mathbf{Q}_p^\times \to L^\times$ are locally analytic characters. There exists a unique extension of Δ to a $\mathrm{GL}_2(\mathbf{Q}_p)$-sheaf $\Delta\boxtimes_\omega \mathbf{P}^1$ of analytic type over[1] $\mathbf{P}^1$ with central character[2] ω. Moreover, there exists a unique admissible locally analytic L-representation $\Pi(\Delta)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$, with central character ω, such that*

$$0 \to \Pi(\Delta)^* \otimes \omega \to \Delta\boxtimes_\omega \mathbf{P}^1 \to \Pi(\Delta) \to 0.$$

Theorem 1.1 generalizes the bijection $V \mapsto \Pi(V)$ and also provides a new construction by working directly on the locally analytic level. Moreover the methods developed in [4], allow the author to work in the setting of $\mathrm{GL}_2(F)$ for any finite extension F of $\mathbf{Q}_p$. A variant of Theorem 1.1 is proved in [8] where the field L is replaced by a $\mathbf{Q}_p$-affinoid

[1] From now on, $\mathbf{P}^1$ will always mean $\mathbf{P}^1(\mathbf{Q}_p)$.

[2] Here $\omega = \delta_1\delta_2\chi^{-1}$ where $\chi(x) = x|x|$ corresponds to the cyclotomic character via local class field theory.

algebra. This is expected to have applications to the study of eigenvarieties. This means in particular that the representations considered in [8] do not live on topological vector spaces over a p-adic field but rather on topological modules over an an affinoid algebra in the sense of Tate. The overall strategy in [8] is similar to that of [4]. Arguing pointwise, the authors in [8] actually use Colmez's results in many places. Nonetheless, there is an additional layer of technical problems they need to overcome. For instance, they need to introduce locally analytic representations on A-modules (where A is a $\mathbf{Q}_p$-affinoid algebra) from scratch and study their homological properties. In this framework the following questions are therefore unavoidable:

Q1 What is a locally convex A-module?

Q2 What is a locally analytic $\mathrm{GL}_2(\mathbf{Q}_p)$-representation in A-modules?

Q3 What is the relation between locally analytic $\mathrm{GL}_2(\mathbf{Q}_p)$-representations in A-modules and modules equipped with a (separately) continuous action of the relative distribution algebra $\mathscr{D}(G, A)$?

These questions are answered in a compatible manner with existing theory (the case when A is finite field extension of $\mathbf{Q}_p$), cf. [8, Theorem 1.6].

We summarize the structure of this paper. In §2, we review basic definitions, so one can make sense of Theorem 1.1. This includes definitions of the Robba ring, (φ, Γ)-modules and analytic sheaves. In §3, we outline the main ideas that are present in both [4] and [8]. The remaining sections expand on these main ideas. In §4, we explictly state the action of $\overline{P}^+$ on $\mathscr{R}_L$ via principal series. In §5, we write down the complexes which are used for cohomology calculations. Since the cohomology theory considered uses locally analytic cochains, this is a slightly non-trivial matter. Finally in §6, we give some explicit examples of cohomology calculations using the complexes in §5 and the actions in §4. These calculations underpin the heart of [4] and [8] and are interesting in their own right.

§ 2. Robba rings, (φ, Γ)-modules and analytic sheaves

In this section we briefly recall the definitions of the Robba ring and (φ, Γ)-modules. For a more complete reference, we refer the reader to [4]. For $0 < r < s \leq \infty$ (with r and s rational, except possibly $s = \infty$), and v_p a p-adic valuation on L with $v_p(p) = 1$, one begins with the following definition.

Definition 2.1. Let $\mathscr{R}_L^{[r,s]}$ be the ring of rigid analytic functions on the annulus $\{z \in L \mid r \leq v_p(z) \leq s\}$.

Remark 1. In Definition 2.1, one views the ring $\mathscr{R}_L^{[r,s]}$ as a subset of the Laurent power series $\sum_{k\in\mathbf{Z}} a_k T^k$ with $a_k \in L$. Equipping $\mathscr{R}_L^{[r,s]}$ with the valuation

$$v^{[r,s]} = \min\left(\inf_{k\in\mathbf{Z}}(v_p(a_k)+rk), \inf_{k\in\mathbf{Z}}(v_p(a_k)+sk)\right)$$

endows it with a Banach ring structure.

One then defines the Robba ring as an inductive limit of rigid analytic functions on the open annulus.

Definition 2.2. Let $\mathscr{R}_L^{]r,s]} := \cap_{t\in]r,s]}\mathscr{R}_L^{[t,s]}$ and $\mathscr{R}_L := \cup_{s>0}\mathscr{R}_L^{]0,s]}$.

Remark 2. In Definition 2.2, one sets $\mathscr{R}_L^+ := \mathscr{R}_L^{]0,\infty]}$. Then $\mathscr{R}_L^+$ is the set of rigid analytic functions on the open unit rigid disk. This no longer has a Banach structure, but has Fréchet topology. The quotient $\mathscr{R}_L^- := \mathscr{R}_L/\mathscr{R}_L^+$ is an inductive limit of Banach spaces (an LB-space).

In order to introduce (φ,Γ)-modules over $\mathscr{R}_L$, we need to equip $\mathscr{R}_L$ with actions of φ and Γ. From now on we set $\Gamma = \mathbf{Z}_p^\times$.

Definition 2.3. For s small enough, one defines an L-linear ring endomorphism $\varphi\colon \mathscr{R}_L^{[r,s]} \to \mathscr{R}_L^{[r/p,s/p]}$ via $\varphi(T) = (1+T)^p - 1$. This induces an operator φ on $\mathscr{R}_L$ Similarly for $a \in \mathbf{Z}_p^\times$, one sets $\sigma_a(T) = (1+T)^a - 1$.

Remark 3. The actions of φ and $\sigma_a \in \Gamma$ commute and $\mathscr{R}_L^+$ is stable under these actions.

We can now define the notion of a (φ,Γ)-modules over $\mathscr{R}_L$.

Definition 2.4. A (φ,Γ)-modules over $\mathscr{R}_L$ is a free $\mathscr{R}_L$-module (of finite rank) equipped with semi-linear and continuous actions of φ and Γ. Furthermore these actions are required to commute.

Example 2.5. Suppose $\delta\colon \mathbf{Q}_p^\times \to L^\times$ is a locally analytic character. Then the *twisted* (φ,Γ)-module by δ of rank 1, $\mathscr{R}_L(\delta)$ has actions of φ and $\sigma_a \in \Gamma$ twisted by $\delta(p)$ and $\delta(a)$, respectively.

Remark 4. Let Δ be a (φ,Γ)-module. Then Δ is naturally equipped with an action of A^+ via

$$\begin{pmatrix} p^k a & 0 \\ 0 & 1 \end{pmatrix} \cdot z = \varphi^k \circ \sigma_a(z)$$

for $k \in \mathbf{Z}_{\geq 0}$, $a \in \mathbf{Z}_p^\times$.

Remark 4 gives some geometric flavour to a (φ, Γ)-module. This leads one to the following definition.

Definition 2.6. Let H be a semi-group and X an H-space (totally disconnected, compact space on which H acts continuously). An H-sheaf $\mathscr{M}$ over X is the datum:

1. For every compact open $U \subset X$, an A-module $\mathscr{M} \boxtimes U$, with $\mathscr{M} \boxtimes \emptyset = 0$
2. For each $U \subset V$ of compact opens, there are restriction maps:

$$\mathrm{Res}_U^V : \mathscr{M} \boxtimes V \to \mathscr{M} \boxtimes U,$$

   such that if $U = \cup_{i=1}^n U_i$ and $s_i \in \mathscr{M} \boxtimes U_i$ for $1 \leq i \leq n$, such that

$$\mathrm{Res}_{U_i \cap U_j}^{U_i} s_i = \mathrm{Res}_{U_i \cap U_j}^{U_j} s_j,$$

   then there exists a unique $s \in \mathscr{M} \boxtimes U$, such that $\mathrm{Res}_{U_i}^U s = s_i$ for all i.
3. There are continuous isomorphisms:

$$g_U : \mathscr{M} \boxtimes U \cong \mathscr{M} \boxtimes gU$$

   for every $g \in H$ and U compact open, satisfying the cocycle condition, $g_{hU} \circ h_U = (gh)_U$ for every $g, h \in H$ and U compact open.

Moreover, we say that an H-sheaf $\mathscr{M}$ over X is analytic if for all open compact $U \subset X$, $\mathscr{M} \boxtimes U$ is an LF-space (an inductive limit of Fréchet spaces) and a $\mathscr{D}(K, \mathbf{Q}_p)$-module[3] for all open compact subgroups $K \subset H$, stabilizing U.

Indeed starting from a (φ, Γ)-module Δ, one can naturally produce an A^+-sheaf M_Δ over $\mathbf{Z}_p$ in the sense of Definition 2.6. The global sections of M_Δ are Δ while sections over a small neighbourhood of the origin $p^n\mathbf{Z}_p$ are given by the image of $\varphi^n \circ \psi^n$ acting on Δ (here ψ is a left-inverse of φ, cf. [8, §2.1.3]).

In the next section we will use the language of analytic sheaves to see how one can adjoin elements to Δ, to obtain a (larger) object $\Delta \boxtimes_\omega \mathbf{P}^1$, a $\mathrm{GL}_2(\mathbf{Q}_p)$-sheaf over $\mathbf{P}^1$. The underlying point is starting from an A^+-module we produce something equipped with an action of $\mathrm{GL}_2(\mathbf{Q}_p)$.

§ 3. Cutting out representations from (φ, Γ)-modules

Let us now describe some of the main ingredients common to [4] and [8]. For simplicity we will work in the setting of a field as in [4] and address the additional work

[3] Here $\mathscr{D}(K, \mathbf{Q}_p)$ is the dual of the space of locally analytic functions from K to $\mathbf{Q}_p$.

required in the relative setting along the way. Let $\delta_1, \delta_2 \colon \mathbf{Q}_p^\times \to L^\times$ be two locally analytic characters. Suppose that we have an exact sequence of (φ, Γ)-modules over $\mathscr{R}_L$

$$0 \to \mathscr{R}_L(\delta_1) \to \Delta \to \mathscr{R}_L(\delta_2) \to 0.$$

Starting from this datum, the aim is construct a suitable L-representation $\Pi(\Delta)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$. This is done in two steps.

Step 1 Enlarge Δ and construct an object which is suggestively denoted by $\Delta \boxtimes_\omega \mathbf{P}^1$. This will be a $\mathrm{GL}_2(\mathbf{Q}_p)$-sheaf over $\mathbf{P}^1$.

Step 2 Cut out the representation $\Pi(\Delta)$ from $\Delta \boxtimes_\omega \mathbf{P}^1$.

Let us begin by explaining **Step 1**. How to construct an object $\Delta \boxtimes_\omega \mathbf{P}^1$ equipped with a *nice*[4] action of $\mathrm{GL}_2(\mathbf{Q}_p)$ is a difficult problem. On the other hand, geometrically this should be two copies of Δ glued along its $\mathbf{Z}_p^\times$-sections. In other words, forgetting the action for the moment and if one thinks of $\Delta \boxtimes_\omega \mathbf{P}^1$ as a sheaf over $\mathbf{P}^1$, then the $\mathbf{Z}_p$-sections are precisely Δ. Now $\mathscr{R}_L(\delta_1)$ and $\mathscr{R}_L(\delta_2)$ are both 1-dimensional and it so happens that there is not much choice for how to construct $\mathscr{R}_L(\delta_1) \boxtimes_\omega \mathbf{P}^1$ and $\mathscr{R}_L(\delta_2) \boxtimes_\omega \mathbf{P}^1$ (we will review this in the next section). This provides some guidance for the construction of $\Delta \boxtimes_\omega \mathbf{P}^1 (=?)$ as it should sit in an exact sequence

$$0 \to \mathscr{R}_L(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to ? \to \mathscr{R}_L(\delta_2) \boxtimes_\omega \mathbf{P}^1 \to 0.$$

As just explained, taking $\mathbf{Z}_p$-sections of $\mathscr{R}_L(\delta_1) \boxtimes_\omega \mathbf{P}^1$ recovers $\mathscr{R}_L(\delta_1)$, but the crucial observation (coming from the equivariant nature of $\mathscr{R}_L(\delta_1) \boxtimes_\omega \mathbf{P}^1$), is that $\mathscr{R}_L(\delta_1)$ is now equipped with an action of $\overline{P}^+$! This is because $\overline{P}^+$ stabilizes $\mathbf{Z}_p$, when considering the usual action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on $\mathbf{P}^1$. Indeed if $x \in \mathbf{Z}_p$, then so is $\frac{ax}{pbx+1} \in \mathbf{Z}_p$ for any $a, b \in \mathbf{Z}_p$.

The following definition is now appropriate.

Definition 3.1. We note $\mathscr{R}_L(\delta_1, \delta_2)$ to be the $\overline{P}^+$-module

$$\mathscr{R}_L(\delta_1, \delta_2) := (\mathscr{R}_L(\delta_1) \boxtimes_\omega \mathbf{Z}_p) \otimes \delta_2^{-1}.$$

Remark 5. As A^+-modules, $\mathscr{R}_L(\delta_1, \delta_2)$ is nothing but $\mathscr{R}_L(\delta_1 \delta_2^{-1})$. The notation $\mathscr{R}_L(\delta_1, \delta_2)$ is introduced to emphasize the action of $\overline{P}^+$.

Of course the crux of **Step 1** is to construct an appropriate[5] action of the involution $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on $\Delta \boxtimes_\omega \mathbf{P}^1$, but this seems difficult to do in one step. As an intermediate step the first question to ask now is as follows: can we compare $\mathrm{Ext}^1_{A^+}(\mathscr{R}_L, \mathscr{R}_L(\delta_1 \delta_2^{-1}))$ and $\mathrm{Ext}^1_{\overline{P}^+}(\mathscr{R}_L, \mathscr{R}_L(\delta_1, \delta_2))$? The answer is yes.

[4]The action should at least be locally analytic in some sense. More precisely it should be analytic in the sense of Definition 2.6.

[5]Here appropriate can be summarized as something that is compatible with the already present action of A^+ on Δ.

Theorem 3.2. *The restriction morphism from $\overline{P}^{+}$ to A^{+} induces a surjection*

$$H^1(\overline{P}^{+}, \mathscr{R}_L(\delta_1, \delta_2)) \to H^1(A^{+}, \mathscr{R}_L(\delta_1\delta_2^{-1})).$$

Theorem 3.2 is the heart of completing **Step 1**. This is contained in [4, Proposition 5.18], however in loc.cit. the mentioned morphism is claimed to be an isomorphism. It turns out that this is not quite true, cf. [8, Proposition 5.2], however it is sufficient for constructing $\Delta \boxtimes_\omega \mathbf{P}^1$. To prove Theorem 3.2, Colmez essentially calculates the dimensions of each of the cohomologies. This turns out to be sufficient for the case of a field, but when L is replaced by some $\mathbf{Q}_p$-affinoid algebra say (in the context of families), understanding the higher Ext^i becomes important. Thus one is required to calculate the higher cohomology groups $H^i(\overline{P}^{+}, -)$ and $H^i(A^+, -)$ for $i = 2, 3$. The author believes that understanding these cohomology groups are crucial for achieving a p-adic Langlands correspondence for GL_n for $n \geq 3$. For this reason, we have decided to illustrate some explicit calculations of these cohomology groups. This is the goal of §4, 5 and 6.

We end this section by briefly explaining **Step 2**. At this stage we have enlarged Δ to $\Delta \boxtimes_\omega \mathbf{P}^1$ to obtain something equipped with an action of $\mathrm{GL}_2(\mathbf{Q}_p)$. One could stop there, but upon further inspection this action is not quite locally analytic. In fact it turns out to be a hybrid of something locally analytic and dual to something locally analytic. This is one reason why we decide to cut out the relavant part $\Pi(\Delta)$ from $\Delta \boxtimes_\omega \mathbf{P}^1$. This $\Pi(\Delta)$ will be a locally analytic L-representation of $\mathrm{GL}_2(\mathbf{Q}_p)$. In the relative setting, one needs to make sense of locally analytic topological modules (as mentioned in the introduction) to properly state $\Pi(\Delta)$, cf. [8, §A]. Now **Step 1** gives us[6]

$$\Delta \boxtimes_\omega \mathbf{P}^1 = [\mathscr{R}_L(\delta_1) \boxtimes_\omega \mathbf{P}^1 - \mathscr{R}_L(\delta_2) \boxtimes_\omega \mathbf{P}^1].$$

As mentioned in §2, the Robba ring $\mathscr{R}_L$ is made of parts $\mathscr{R}_L^+$ (this is a distribution algebra, that is dual to something locally analytic) and $\mathscr{R}_L^-$ (this is locally analytic). Therefore in the search for $\Pi(\Delta)$, it seems reasonable to first cut up the Robba ring parts $\mathscr{R}_L(\delta_1) \boxtimes_\omega \mathbf{P}^1$ into their $+/-$ constituents. Fortunately, from the way $\mathscr{R}_L(\delta_i) \boxtimes_\omega \mathbf{P}^1$ are constructed, we also have

$$\mathscr{R}_L(\delta_i) \boxtimes_\omega \mathbf{P}^1 = [\mathscr{R}_L^+(\delta_i) \boxtimes_\omega \mathbf{P}^1 - \mathscr{R}_L^-(\delta_i) \boxtimes_\omega \mathbf{P}^1]$$

for $i = 1, 2$. Therefore

$$\Delta \boxtimes_\omega \mathbf{P}^1 = [\mathscr{R}_L^+(\delta_1) \boxtimes_\omega \mathbf{P}^1 - \mathscr{R}_L^-(\delta_1) \boxtimes_\omega \mathbf{P}^1 - \mathscr{R}_L^+(\delta_2) \boxtimes_\omega \mathbf{P}^1 - \mathscr{R}_L^-(\delta_2) \boxtimes_\omega \mathbf{P}^1].$$

[6] The notation $M = [M_1 - M_2 - \ldots - M_n]$ means that M admits an increasing filtration $0 \subseteq F_1 \subseteq \ldots \subseteq F_n = M$ by subobjects such that $M_i = F_i/F_{i-1}$ for $i = 1, \ldots, n$.

The key to completing **Step 2** is now to show that the middle extension

$$[\mathscr{R}_L^-(\delta_1) \boxtimes_\omega \mathbf{P}^1 - \mathscr{R}_L^+(\delta_2) \boxtimes_\omega \mathbf{P}^1]$$

is split. This allows one to separate the locally analytic parts and the distribution algebras. Moreover this explains the appearance of both $\Pi(\Delta)$ and its dual $\Pi(\Delta)^*$ in Theorem 1.1.

§ 4. Principal series

In analogy with Definition 3.1 we have

Definition 4.1. We set $\mathscr{R}_L^+(\delta_1, \delta_2)$ the $\overline{P}^+$-submodule of $\mathscr{R}_L(\delta_1, \delta_2)$ corresponding to $\mathscr{R}_L^+$, and $\mathscr{R}_L^-(\delta_1, \delta_2)$ to be the quotient of $\mathscr{R}_L(\delta_1, \delta_2)$ by $\mathscr{R}_L^+(\delta_1, \delta_2)$.

Remark 6. Similarly to Remark 5, as A^+-modules, $\mathscr{R}_L^+(\delta_1, \delta_2)$ and $\mathscr{R}_L^-(\delta_1, \delta_2)$ are just $\mathscr{R}_L^+(\delta_1\delta_2^{-1})$ and $\mathscr{R}_L^-(\delta_1\delta_2^{-1}\chi^{-1})$, respectively. The character χ appears when identifying the quotient $\mathscr{R}_L^-$ with locally analytic functions (cf. [4, Théorème 2.3(ii)]).

We now have an exact sequence of $\overline{P}^+$-modules

$$0 \to \mathscr{R}_L^+(\delta_1, \delta_2) \to \mathscr{R}_L(\delta_1, \delta_2) \to \mathscr{R}_L^-(\delta_1, \delta_2) \to 0, \tag{4.1}$$

and one can describe the actions of $\overline{P}^+$ on $\mathscr{R}_L^+(\delta_1, \delta_2)$ and that of the Lie algebra on $\mathscr{R}_L^-(\delta_1, \delta_2)$. In order to achieve this, one relates the given objects to principal series, where the actions are understood.

We define $B_L(\delta_1, \delta_2)$ to be the space of locally analytic functions $\phi\colon \mathbf{Q}_p \to L$, such that $\delta(x)\phi\left(\frac{1}{x}\right)$ extends to an analytic function on a neighbourhood of 0, where $\delta = \delta_1\delta_2^{-1}\chi^{-1}$. One can equip $B_L(\delta_1, \delta_2)$ with an action of $\mathrm{GL}_2(\mathbf{Q}_p)$ defined by

$$\left(\begin{pmatrix} a\; b \\ c\; d \end{pmatrix} \cdot \phi\right)(x) = \delta_2(ad - bc)\delta(a - cx)\phi\left(\frac{dx - b}{a - cx}\right). \tag{4.2}$$

Further one can show that $B_L(\delta_1, \delta_2)$ is in fact a principal series representation, that is $B_L(\delta_1, \delta_2) = \mathrm{Ind}_{\overline{B}}^{G}(\delta_1\chi^{-1} \otimes \delta_2)$ (where $\overline{B}$ is the lower-half Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$).

Lemma 4.2. *We have*

- $\mathscr{R}_L^+(\delta_1) \boxtimes_\omega \mathbf{P}^1 \cong B_L(\delta_2, \delta_1)^* \otimes \omega$
- $\mathscr{R}_L^-(\delta_1) \boxtimes_\omega \mathbf{P}^1 \cong B_L(\delta_1, \delta_2)$.

Lemma 4.2 is the content of [4, Corollaire 4.11]. Coupled with equation (4.2), it is now a formality to obtain the action of $\overline{P}^+$ on $\mathscr{R}_L^+(\delta_1,\delta_2)$:

$$\int_{\mathbf{Z}_p}\phi\cdot\left(\begin{pmatrix}a&0\\c&1\end{pmatrix}\star\mu\right)=\delta_1\delta_2^{-1}(a)\int_{\mathbf{Z}_p}\delta(cx+1)\phi\left(\frac{ax}{cx+1}\right)\cdot\mu(x),\quad \left(\begin{smallmatrix}a&0\\c&1\end{smallmatrix}\right)\in\overline{P}^+,\tag{4.3}$$

where $\mu\in\mathscr{R}_L^+(\delta_1,\delta_2)$ and $\phi\in\mathscr{R}_L^-(\delta_1,\delta_2)$. Indeed by restricting the (dual) action of (4.2) to $\overline{P}^+\subset\mathrm{GL}_2(\mathbf{Q}_p)$, one obtains formula (4.3). At first glance this formula may be hard on the eyes but is explictly given by[7]

(4.4)

$$\sigma_a(t^j)=\delta_1\delta_2^{-1}(a)a^jt^j,\quad \varphi(t^j)=\delta_1\delta_2^{-1}(p)p^jt^j,\quad \tau(t^j)=\sum_{h=0}^{j}\binom{-\kappa(\delta_1\delta_2^{-1})-1-h}{j-h}p^{j-h}t^h,$$

for the generators $\sigma_a=\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)$, $\varphi=\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)$ and $\tau=\left(\begin{smallmatrix}1&0\\p&1\end{smallmatrix}\right)$.

Let $a^+:=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $u^-:=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$. The action of the Lie algebra on $\mathscr{R}_L^-(\delta_1,\delta_2)$ can be summarized as follows.

Lemma 4.3. *Let $f\in\mathscr{R}_L^-(\delta_1,\delta_2)$. Under the identification (as modules) $\mathscr{R}_L^-(\delta_1,\delta_2)=$ $\mathrm{LA}(\mathbf{Z}_p,L)$ (the space of locally analytic functions from $\mathbf{Z}_p$ to L), the infinitesimal actions of a^+ and u^- and the action of φ are given by* [8]

$$(a^+f)(x)=\kappa(\delta)f(x)-xf'(x),$$

$$(u^-f)(x)=\kappa(\delta)xf(x)-x^2f'(x),$$

$$(\varphi f)(x)=\delta(p)f\left(\frac{x}{p}\right).$$

Proof. First note for $\begin{pmatrix}a&0\\b&1\end{pmatrix}\in\overline{P}^+$ and $f\in\mathscr{R}_L^-(\delta_1,\delta_2)$, the action of $\overline{P}^+$ on $\mathscr{R}_L^-(\delta_1,\delta_2)$ is given by

$$\begin{pmatrix}a&0\\b&1\end{pmatrix}\cdot f=\delta(a-bx)f\left(\frac{x}{a-bx}\right).$$

The above action can be deduced from identifying $\mathscr{R}_L^-(\delta_1)\boxtimes_\omega\mathbf{P}^1$ with the space of locally analytic functions from $\mathbf{Q}_p$ to L with some finiteness condition at ∞ (as explained in

[7] As usual we have set $t:=\log(1+T)$ and $\kappa(\epsilon)=\epsilon'(1)$ is the weight for a character ϵ.

[8] Observe that, in the formula for φ below, $f\left(\frac{x}{p}\right)$ is taken to be zero whenever $x\in\mathbf{Z}_p^\times$, so the precise formula should be $(\varphi f)(x)=\mathbf{1}_{p\mathbf{Z}_p}(x)\delta(p)f\left(\frac{x}{p}\right)$.

the previous paragraph). The action of φ is now evident. By definition

$$\begin{aligned}(a^+f)(x) &= \lim_{a\to 1}\frac{\begin{pmatrix} a\,0 \\ 0\,1\end{pmatrix} f(x) - f(x)}{a-1} \\ &= \lim_{a\to 1}\frac{\delta(a)f\left(\frac{x}{a}\right) - f(x)}{a-1} \\ &= \delta'(a)f\left(\frac{x}{a}\right) - xa^{-2}\delta(a)f'\left(\frac{x}{a}\right)|_{a=1} \\ &= \kappa(\delta)f(x) - xf'(x).\end{aligned}$$

Viewing $\mathscr{R}_L^-(\delta_1,\delta_2)$ as the module $\mathscr{R}_L^-$ equipped with action of $\overline{P}^+$, we have by [7, Théorème 1.1],

$$u^- = -t^{-1}\nabla(\nabla + \kappa(\delta_2\delta_1^{-1})),$$

where here $\nabla = t\frac{d}{dt}$. Thus

$$\begin{aligned}(u^-f)(x) &= -\frac{d}{dt}(\nabla + \kappa(\delta_2\delta_1^{-1}))(f)(x) \\ &= -\frac{d}{dt}((1+\kappa(\delta_2\delta_1^{-1}))f(x) + xf'(x)) \\ &= -\frac{d}{dt}(-\kappa(\delta)f(x) + xf'(x)) \\ &= \kappa(\delta)xf(x) - x^2f'(x).\end{aligned}$$

$\square$

We will use the actions developed in this section to calculate cohomology.

§ 5. Koszul complexes

In this section we review complexes which calculate A^+-cohomology and $\overline{P}^+$-cohomology. Assume for simplicity $p \geq 3$, let a be a topological generator of $\mathbf{Z}_p^\times$ and note

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}, \quad \varphi = \begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ p & 1\end{pmatrix},$$

for the topological generators of the (semi-)group $\overline{P}^+$. The relations

$$\varphi\gamma = \gamma\varphi,$$

$$\gamma\tau = \tau^{a^{-1}}\gamma,$$

$$\varphi\tau^p = \tau\varphi,$$

give a finite presentation of $\overline{P}^+$. Let M be a $\overline{P}^+$-module such that the action of $\overline{P}^+$ extends to an action of the Iwasawa algebra $\mathbf{Z}_p[[\overline{P}^+]]$. Recall that the Herr complex

$$\mathscr{C}_{\varphi,\gamma} : 0 \to M \xrightarrow{E'} M \oplus M \xrightarrow{F'} M \to 0,$$

where

$$E'(x) = ((1-\varphi)x, (\gamma - 1)x)$$
$$F'(x,y) = (\gamma - 1)x + (\varphi - 1)y,$$

calculates the A^+-cohomology of M. In addition the following complex calculates the $\overline{P}^+$-cohomology of M (cf. [8, Lemma 4.8])

$$\mathscr{C}_{\tau,\varphi,\gamma} : 0 \to M \xrightarrow{X} M \oplus M \oplus M \xrightarrow{Y} M \oplus M \oplus M \xrightarrow{Z} M \to 0$$

where the arrows are defined as

$$X(x) = ((1-\tau)x, (1-\varphi)x, (\gamma - 1)x)$$
$$Y(x,y,z) = ((1-\varphi\delta_p)x + (\tau - 1)y, (\gamma\delta_a - 1)x + (\tau - 1)z, (\gamma - 1)y + (\varphi - 1)z)$$
$$Z(x,y,z) = (\gamma\delta_a - 1)x + (\varphi\delta_p - 1)y + (1-\tau)z$$

and where we have noted,

$$\delta_p = \frac{1-\tau^p}{1-\tau} = 1 + \tau + \ldots + \tau^{p-1}$$

and

$$\delta_a = \frac{\tau^a - 1}{\tau - 1}.$$

One can check that these are well defined elements as M is a sufficiently nice $\overline{P}^+$-module.

The key difference between the complexes $\mathscr{C}_{\varphi,\gamma}$ and $\mathscr{C}_{\tau,\varphi,\gamma}$ is given by some *twisting* by τ. More precisely there is a distinguished triangle

$$\mathscr{C}_{\tau,\varphi,\gamma} \to \mathscr{C}_{\varphi,\gamma} \xrightarrow{1-\tau} \mathscr{C}^{\text{twist}}_{\varphi,\gamma}$$

where

$$\mathscr{C}^{\text{twist}}_{\varphi,\gamma} : 0 \to M \xrightarrow{E''} M \oplus M \xrightarrow{F''} M \to 0.$$

Here the maps are *twisted* versions of E' and F'

$$E''(x) = ((1-\varphi\delta_p)x, (\gamma\delta_a - 1)x)$$
$$F''(x,y) = ((\gamma\delta_a - 1)x + (\varphi\delta_p - 1)y.$$

Finally it is often the case that M is equipped with additional structure (for instance it is a locally analytic representation of $\overline{P}^+$) which allows one to calculate $H^i(A^+, M)$ and $H^i(\overline{P}^+, M)$ via a *linearized* Lie algebra complex. We note

$$a^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ a^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ u^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ u^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the usual generators of the Lie algebra $\mathfrak{gl}_2$ of GL_2. Let $H^i_{\mathrm{Lie}}(A^+, M)$ denote the cohomology groups of the complex

$$\mathscr{C}_{\varphi,a^+} : 0 \to M \xrightarrow{X} M \oplus M \xrightarrow{Y} M \to 0$$

where

$$X(x) = ((\varphi - 1)x, a^+x)$$
$$Y(x, y) = a^+x - (\varphi - 1)y.$$

Let $A^0 := \begin{pmatrix} \mathbf{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$. Then the A^0-invariants of $H^i_{\mathrm{Lie}}(A^+, M)$ are precisely the groups $H^i(A^+, M)$, cf. [4, Lemme 5.6].

In a similar fashion denote by $H^i_{\mathrm{Lie}}(\overline{P}^+, M)$ the cohomology groups of the complex

$$\mathscr{C}_{u^-,\varphi,a^+} : 0 \to M \xrightarrow{X'} M \oplus M \oplus M \xrightarrow{Y'} M \oplus M \oplus M \xrightarrow{Z'} M \to 0$$

where

$$X'(x) = ((\varphi - 1)x, a^+x, u^-x)$$
$$Y'(x, y, z) = (a^+x - (\varphi - 1)y, u^-y - (a^+ + 1)z, (p\varphi - 1)z - u^-x)$$
$$Z'(x, y, z) = u^-x + (p\varphi - 1)y + (a^+ + 1)z.$$

Let $\tilde{P} := \begin{pmatrix} \mathbf{Z}_p^\times & 0 \\ p\mathbf{Z}_p & 1 \end{pmatrix}$. Then the $\tilde{P}$-invariants of $H^i_{\mathrm{Lie}}(\overline{P}^+, M)$ are precisely the groups $H^i(\overline{P}^+, M)$, cf. [8, Lemma 5.3]. To simplify matters we have two distinguished triangles

$$\mathscr{C}_{u^-,\varphi} \to \mathscr{C}_{u^-} \xrightarrow{[\varphi-1]} \mathscr{C}_{u^-},$$

$$\mathscr{C}_{u^-,\varphi,a^+} \to \mathscr{C}_{u^-,\varphi} \xrightarrow{[a^+]} \mathscr{C}_{u^-,\varphi}.$$

The morphisms and complexes are

$$\begin{tikzcd} \mathscr{C}_{u^-} : 0 \arrow[r] & M \arrow[r] \arrow[d, "\varphi-1"] & M \arrow[r] \arrow[d, "p\varphi-1"] & 0 \\ \mathscr{C}_{u^-} : 0 \arrow[r] & M \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{array}{ccccccccc} \mathscr{C}_{u^-,\varphi}: 0 & \longrightarrow & M & \longrightarrow & M \oplus M & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a^+} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle a^+ +1} & & \\ \mathscr{C}_{u^-,\varphi}: 0 & \longrightarrow & M & \longrightarrow & M \oplus M & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where $s(x, y) = ((a^+ + 1)x, a^+ y)$. Unpacking these complexes gives rise to the following:

1. $H^0(\mathscr{C}_{u^-,\varphi,a^+}) = H^0([a^+] \colon H^0(\mathscr{C}_{u^-,\varphi}))$.
2. We have the following exact sequences in cohomology:

$$0 \to H^1([a^+] \colon H^0(\mathscr{C}_{u^-,\varphi})) \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to H^0([a^+] \colon H^1(\mathscr{C}_{u^-,\varphi})) \to 0, \tag{5.1}$$

$$0 \to H^1([\varphi - 1] \colon H^0(\mathscr{C}_{u^-})) \to H^1(\mathscr{C}_{u^-,\varphi}) \to H^0([\varphi - 1] \colon H^1(\mathscr{C}_{u^-})) \to 0. \tag{5.2}$$

3. We have the following exact sequences in cohomology:

$$0 \to H^1([a^+] : H^1(\mathscr{C}_{u^-,\varphi})) \to H^2(\mathscr{C}_{u^-,\varphi,a^+}) \to H^0([a^+] : H^2(\mathscr{C}_{u^-,\varphi})) \to 0, \tag{5.3}$$

$$H^2(\mathscr{C}_{u^-,\varphi}) \cong H^1([\varphi - 1] : H^1(\mathscr{C}_{u^-})). \tag{5.4}$$

4. $H^3(\mathscr{C}_{u^-,\varphi,a^+}) = H^1([a^+] \colon H^2(\mathscr{C}_{u^-,\varphi}))$.

§ 6. Some examples of cohomology calculations

We want to understand the $\overline{P}^+$-cohomology of $\mathscr{R}_L(\delta_1, \delta_2)$. This is a key step in proving Theorem 3.2. Exact sequence (4.1) comes to our aid and we can reduce the problem by understanding the cohomology of $\mathscr{R}_L^+(\delta_1, \delta_2)$ and $\mathscr{R}_L^-(\delta_1, \delta_2)$ separately. Now $\mathscr{R}_L^+(\delta_1, \delta_2)$ and $\mathscr{R}_L^-(\delta_1, \delta_2)$ are of different topological nature (they are dual to each other). On the one hand $\mathscr{R}_L^+(\delta_1, \delta_2)$ is some kind of distribution algebra (a Fréchet space), while $\mathscr{R}_L^-(\delta_1, \delta_2)$ is a space of locally analytic functions (an LB-space). Thus we can expect the method of calculating cohomology to differ. This is also part of the reason why we can use Lie-algebra cohomology on the $\mathscr{R}_L^-(\delta_1, \delta_2)$ part.

We begin with the $\overline{P}^+$-cohomology of $\mathscr{R}_L^+(\delta_1, \delta_2)$. For this part [2, Lemme 2.9(ii)] provides a significant reduction. The point is that depending on the valuation of $\delta(p)$, the $\overline{P}^+$-cohomology on $T^{N+1}\mathscr{R}_L^+(\delta_1, \delta_2)$ vanishes for large enough N. Rewriting (cf. Lemme 2.9(i) in loc.cit.)

$$\mathscr{R}_L^+(\delta_1, \delta_2) = \bigoplus_{i=0}^{N} L \cdot t^i \oplus T^{N+1}\mathscr{R}_L^+$$

for N large enough, it suffices to calculate the $\overline{P}^+$-cohomology of $\oplus_{i=0}^N L\cdot t^i$ (recall that $t := \log(1+T)$). This is convenient as the action of $\overline{P}^+$ on powers of t are easily described by (4.4).

Example 6.1 ($\delta_1\delta_2^{-1} = x^{-1}$)**.** In this case one can take $N = 2$, that is

$$H^i(\overline{P}^+, \mathscr{R}_L^+(\delta_1,\delta_2)) = H^i(\overline{P}^+, L\oplus L\cdot t\oplus L\cdot t^2).$$

Let us denote by $M_2 = L\oplus L\cdot t\oplus L\cdot t^2$. We can see from (4.4) that $\tau(t^0) = t^0$, $\tau(t^1) = t^1$ and $\tau(t^2) = -pt + t^2$. Let $\overline{U} = \left(\begin{smallmatrix} 1 & 0 \\ p\mathbf{Z}_p & 1\end{smallmatrix}\right)$. We have a spectral sequence

$$H^i(A^+, H^j(\overline{U}, M_2)) \implies H^{i+j}(\overline{P}^+, M_2) \tag{6.1}$$

The existence of spectral sequence (6.1) requires proof as $\overline{P}^+$ is a semigroup and one needs to check that the relavant actions are continuous, cf. the proof of [8, Proposition 3.3]. From the description of the action of τ above, one easily obtains $H^0(\overline{U}, M_2) = L\oplus L\cdot t$ and $H^1(\overline{U}, M_2) = L\oplus L\cdot t^2$ as the only non-zero $\overline{U}$-cohomology groups. We obtain from the description of the action of σ_a and φ on t^i (cf. (4.4))

$$\dim_L H^i(A^+, H^0(\overline{U}, M_2)) = 1, 2, 1 \text{ if } i = 0, 1, 2 \text{ resp.} \tag{6.2}$$

and keeping in mind that the action of A^+ on $H^1(\overline{U}, M_2)$ is twisted one obtains

$$\dim_L H^i(A^+, H^1(\overline{U}, M_2)) = 1, 2, 1 \text{ if } i = 0, 1, 2 \text{ resp.} \tag{6.3}$$

To combine (6.2) and (6.3) and obtain a final result for $H^i(\overline{P}^+, M_2)$, one needs to understand the transgression map

$$tr\colon H^0(A^+, H^1(\overline{U}, M_2))) \to H^2(A^+, H^0(\overline{U}, M_2))).$$

on the second page. According to (6.2) and (6.3), this is a morphism of one-dimensional L-vector spaces, so it is either the zero morphism, or it is an isomorphism. We claim that it is the zero morphism. To prove this we use the following trick, cf. [8, Proposition 5.2]:

Lemma 6.2. *let G be a semi-group, H a normal subgroup of G and M a G-module. Suppose that there exists a submodule M' of M which is stable by the action of G and such that the natural map $H^0(G/H, H^1(H, M')) \to H^0(G/H, H^1(H, M))$ induced by the inclusion $M' \subseteq M$ is an isomorphism. Then we claim that the transgression map $tr_M : H^0(G/H, H^1(H, M)) \to H^2(G/H, H^0(H, M))$ sits in a commutative diagram:*

$$\begin{tikzcd} H^0(G/H, H^1(H, M)) \arrow[r, "tr_M"] \arrow[d, "\cong"] & H^2(G/H, H^0(H, M)) \\ H^0(G/H, H^1(H, M')) \arrow[r, "tr_{M'}"] & H^2(G/H, H^0(H, M')). \arrow[u] \end{tikzcd}$$

With $M = M_2$, $M' = L$, $G = \overline{P}^+$ and $H = \overline{U}$, it is easy to check that Lemma 6.2 applies in this case. However in this case the target of the transgression map $tr_{M'}$ (i.e. $H^2(A^+, H^0(\overline{U}, M')))$ is trivial. This gives the result:

$$\dim_L H^i(\overline{P}^+, \mathscr{R}_L^+(\delta_1, \delta_2)) = 1, 3, 3, 1 \text{ if } i = 0, 1, 2, 3 \text{ resp.}$$

We now turn our attention to the $\overline{P}^+$-cohomology of $\mathscr{R}_L^-(\delta_1, \delta_2)$. Recall in this situation, we can use the Lie-algebra cohomology, as an intermediate step.

Example 6.3 ($\delta = \delta_1\delta_2^{-1}\chi^{-1} = x$)**.** The idea is to use the list of cohomologies developed at the end of §2, together with Lemma 4.3.

It is easy to check that $1 - p\varphi$ is injective on $\mathrm{LA}(\mathbf{Z}_p, L)$. This immediately gives $H^0(\mathscr{C}_{u^-,\varphi,a^+}) = 0$ and so $H^0(\overline{P}^+, \mathscr{R}_L^-(\delta_1, \delta_2)) = 0$. Next we begin calculating $H^2(\mathscr{C}_{u^-,\varphi})$. For this one writes (cf. [4, Lemme 5.9])

$$\mathscr{R}_L^-(\delta_1, \delta_2) = \big(\mathrm{LA}(\mathbf{Z}_p^\times, L) + (p\varphi - 1)\mathscr{R}_L^-(\delta_1, \delta_2)\big) \oplus L \cdot x^2.$$

According to (5.4), we need to understand the operator $u^- \colon \mathrm{LA}(\mathbf{Z}_p^\times, L) \to \mathrm{LA}(\mathbf{Z}_p^\times, L)$. The proof of the following Lemma is omitted in [8], so we include it here for completeness.

Lemma 6.4. *The operator u^- restricted to $\mathrm{LA}(\mathbf{Z}_p^\times, L)$ is surjective on $\mathrm{LA}(\mathbf{Z}_p^\times, L)$.*

Proof. Take $f \in \mathrm{LA}(\mathbf{Z}_p^\times, L)$, $i \in \mathbf{Z}_p^\times$ and choose an $N > 0$ such that f is analytic in the ball $i + p^N\mathbf{Z}_p$. Writing

$$f(x) = \sum_{n \geq 0} a_n (x - i)^n \text{ for } x \in i + p^N\mathbf{Z}_p$$

and developing (in the ball $i + p^N\mathbf{Z}_p$) $u^- f = \kappa(\delta)xf(x) - x^2 f'(x) =: h(x) = \sum_{n\geq 0} b_n(x - i)^n$ gives

$$\begin{aligned} b_0 &= i\kappa(\delta)a_0 - i^2 a_1 \\ b_n &= (\kappa(\delta) - n + 1)a_{n-1} + i(\kappa(\delta) - 2n)a_n - i^2(n+1)a_{n+1} \ \forall\, n \geq 1. \end{aligned}$$

To show u^- is surjective on $\mathrm{LA}(\mathbf{Z}_p^\times, L)$, it suffices to show that given $(b_j)_{j\geq 0}$ such that $v_p(b_n) + nr \xrightarrow{n\to\infty} \infty$ (for some fixed $r > 0$), there exists $(a_j)_{j\geq 0}$ satisfying the above system of equations and for some fixed $r' > 0$,

$$v_p(a_n) + nr' \xrightarrow{n\to\infty} \infty.$$

Setting $a_0 = 0$, we obtain uniquely the (a_j) in terms of the (b_j) and it is easy to prove by induction that

$$v_p(a_n) \geq \min_{m<n}(v_p(b_m)) + c - v_p(n),\ \forall\, n \geq 1,$$

where $c = \min(v_p(\kappa(\delta)), 0)$ is a constant.

Given $C > 0$, $\exists N_0 > 0$ such that $v_p(b_n) + nr > C$ for all $n > N_0$. Let $N_1 > 0$ be such that $\min_{m<N_0}(v_p(b_m)) + nr > C$ for all $n > N_1$. Then, for any $n > \max(N_0, N_1)$, we have

$$v_p(a_n) + nr + v_p(n) - c \geq \min_{m<n}(v_p(b_m)) + nr > C.$$

Finally let $N_2 > 0$ be such that $nr > v_p(n) - c$, $\forall n > N_2$. Then for all $n > \max(N_0, N_1, N_2)$, we get

$$v_p(a_n) + n(2r) > C.$$

Hence for $r' = 2r$, $v_p(a_n) + nr' \xrightarrow{n\to\infty} \infty$, which completes the proof. $\square$

Thus by Lemma 6.4, we get $H^2(\mathscr{C}_{u^-,\varphi})$ is of dimension 1 generated by $[x^2]$. Recalling that $H^3(\mathscr{C}_{u^-,\varphi,a^+}) = H^1([a^+]\colon H^2(\mathscr{C}_{u^-,\varphi}))$, and seeing that the action of $[a^+]$ on $H^2(\mathscr{C}_{u^-,\varphi})$ is given by $a^+ + 1$, we get that $H^3(\mathscr{C}_{u^-,\varphi,a^+})$ is of dimension 1 generated by $[x^2]$. Finally by taking the $\tilde{P}$-invariants, one arrives at $H^3(\overline{P}^+, \mathscr{R}_L^-(\delta_1, \delta_2))$ is of dimension 1 generated by $[x^2]$. Similar calculations yield results for the other $H^i(\overline{P}^+, \mathscr{R}_L^-(\delta_1, \delta_2))$, $i \leq 2$.

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