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Survey on the theory of G-zips

By

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§ 1. Introduction

The theory of F-zips was first introduced by Moonen-Wedhorn in [2]. Roughly speaking, this theory aims at classifying geometric objects in positive characteristic. For example, let E be an elliptic curve over an algebraically closed field k of characteristic p and consider the p-torsion part $E(k)[p]$ of the group $E(k)$. There are two cases:

- If $E(k)[p] \simeq \mathbf{Z}/p\mathbf{Z}$, we say that E is ordinary.
- If $E(k)[p] = \{0\}$, we say that E is supersingular.

Hence, the group $E(k)[p]$ is a discrete invariant for elliptic curves over k, in the sense that the number of possible cases is finite. The theory of F-zips is a similar attempt to attach invariants to geometric objects.

If we consider a family of elliptic curves $\mathscr{E} \to S$ over a base scheme S of characteristic p, then the fibers $\mathscr{E}_s$ for $s \in S$ are usual elliptic curves over fields. Hence S is naturally the (set-theoretic) disjoint union

$$S = S^{\mathrm{ord}} \sqcup S^{\mathrm{ss}} \tag{1.1}$$

where S^{ord} (resp. S^{ss}) is the set of $s \in S$ such that $\mathscr{E}_s$ is ordinary (resp. supersingular). It turns out that the ordinary locus S^{ord} is always open. This is related to the fact that an ordinary elliptic curve has a good deformation theory.

A concrete way of seeing that S^{ord} is open, is to express it as the non-vanishing locus of a section of a line bundle over S. Denote by ω the line bundle over S obtained

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by pulling back the sheaf $\Omega^1_{\mathscr{E}/S}$ along the unit section $S \to \mathscr{E}$ of the S-group scheme $\mathscr{E}$. There exists a section $\mathrm{Ha} \in H^0(S, \omega^{p-1})$ called the Hasse invariant such that

$$S^{\mathrm{ord}} = \{s \in S \mid \mathrm{Ha}(s) \neq 0\}. \tag{1.2}$$

The theory of F-zips provides a geometric object (an algebraic stack), which carries naturally the line bundle ω and the Hasse invariant Ha. Concretely, let B denote the group of upper-triangular matrices in GL_2, let B_- be the lower-triangular ones, and let T be the diagonal torus. Consider the set of pairs $(x, y) \in B_- \times B$ such that the diagonal coefficients of y are the p-th powers of the diagonal coefficients of x. In other words, x, y have the form

$$x = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \quad , \quad y = \begin{pmatrix} a^p & b \\ 0 & d^p \end{pmatrix} \tag{1.3}$$

for some $a, d \in k^\times$ and $b, c \in k$. The set of such pairs forms a group $E \subset B_- \times B$. Let E act on GL_2 by the rule $(x, y) \cdot g = xgy^{-1}$. Then one sees that there are exactly two orbits for this action, an open orbit and a closed one. The open orbit is

$$GL_2^{\mathrm{ord}} := \left\{ \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in GL_2 \mid a \neq 0 \right\}. \tag{1.4}$$

The closed one is GL_2^{ss}, defined by the condition $a = 0$.

The stack of F-zips in this context is the quotient stack $\mathcal{X} = [E \backslash GL_2]$, its underlying topological space consists of two points. We will see in this survey that the datum of an elliptic curve $\mathscr{E}$ over S induces a map $S \to \mathcal{X}$. The ordinary and supersingular loci are the fibers of this map. This geometrization is very useful. The stack $\mathcal{X}$ has a rich structure, and S inherits it by way of pulling back. For example, the section Ha previously mentioned is actually pulled back from a section $\mathrm{Ha} \in H^0(\mathcal{X}, \omega^{p-1})$ for a certain line bundle ω on $\mathcal{X}$.

In the papers [3] and [4], Pink-Wedhorn-Ziegler define the notion of G-zips. The formalism of G-zips makes it possible to work with arbitrary reductive groups G, in place of GL_2 in the previous example. We will see that the Hasse invariant Ha possesses a vast generalization as well.

§ 2. The category of F-zips

We start be recalling the definition of F-zips, as introduced by Moonen-Wedhorn in [2]. Basically, an F-zip over a scheme S of characteristic p is a locally free module endowed with two filtrations and Frobenius-linear isomorphisms between the graded pieces. Specifically, let S be a scheme and let $\mathcal{M}$ be a locally free $\mathcal{O}_S$-module. By a descending filtration on $\mathcal{M}$, we mean a sequence of locally free $\mathcal{O}_S$-submodules $(\mathcal{C}^i)_{i\in\mathbf{Z}}$ such that

(i) For all $i \in \mathbf{Z}$, $\mathcal{C}^{i+1} \subset \mathcal{C}^i$ is Zariski locally a direct factor of $\mathcal{C}^i$.

(ii) One has $\mathcal{C}^i = 0$ for $i \gg 0$ and $\mathcal{C}^i = \mathcal{M}$ for $i \ll 0$.

We define $\mathrm{gr}^i(\mathcal{C}^\bullet) := \mathcal{C}^i/\mathcal{C}^{i+1}$, by assumption (i) it is a locally free $\mathcal{O}_S$-module. We say that $(\mathcal{D}_i)_{i\in\mathbf{Z}}$ is an ascending filtration if $(\mathcal{D}_{-i})_{i\in\mathbf{Z}}$ is a descending filtration. In this case, we write $\mathrm{gr}_i(\mathcal{D}_\bullet) := \mathcal{D}_i/\mathcal{D}_{i-1}$. For an $\mathcal{O}_S$-module $\mathcal{F}$, we denote by $\mathcal{F}^{(p)}$ the pullback of $\mathcal{F}$ under the absolute Frobenius map $\mathrm{Fr}_S : S \to S$.

Definition 2.1. An F-zip over S is a tuple $\underline{\mathcal{M}} = (\mathcal{M}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \iota_\bullet)$, where

(1) $\mathcal{M}$ is a locally free $\mathcal{O}_S$-module of finite rank.

(2) $\mathcal{C}^\bullet = (\mathcal{C}^i)_{i\in\mathbf{Z}}$ is a descending filtration on $\mathcal{M}$.

(3) $\mathcal{D}_\bullet = (\mathcal{D}_i)_{i\in\mathbf{Z}}$ is an ascending filtration on $\mathcal{M}$.

(4) For each $i \in \mathbf{Z}$, ι_i is an isomorphism $\mathrm{gr}^i(\mathcal{C}^\bullet)^{(p)} \to \mathrm{gr}_i(\mathcal{D}_\bullet)$ of $\mathcal{O}_S$-modules.

Next we describe homomorphisms of F-zips. If $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ are F-zips, a homomorphism $f : \underline{\mathcal{M}} \to \underline{\mathcal{N}}$ is a morphism of $\mathcal{O}_S$-modules $f : \mathcal{M} \to \mathcal{N}$ such that $f(\mathcal{C}^i\mathcal{M}) \subset \mathcal{C}^i\mathcal{N}$ and $f(\mathcal{D}^i\mathcal{M}) \subset \mathcal{D}^i\mathcal{N}$ for all $i \in \mathbf{Z}$, and such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{gr}^i(\mathcal{C}^\bullet\mathcal{M})^{(p)} & \xrightarrow{\iota_i} & \mathrm{gr}_i(\mathcal{D}_\bullet\mathcal{M}) \\ \downarrow{\scriptstyle \mathrm{gr}^i(\mathcal{C}^\bullet f)^{(p)}} & & \downarrow{\scriptstyle \mathrm{gr}_i(\mathcal{D}_\bullet f)} \\ \mathrm{gr}^i(\mathcal{C}^\bullet\mathcal{N})^{(p)} & \xrightarrow{\iota_i} & \mathrm{gr}_i(\mathcal{D}_\bullet\mathcal{N}). \end{array} \tag{2.1}$$

The category of F-zips is denoted by `F-Zip`(S), it is $\mathbf{F}_p$-linear (but not $\mathcal{O}_S$-linear, due to the presence of the Frobenius isogeny). We will see now that it is a tensor category. First, if $\mathcal{M}, \mathcal{N}$ are locally free $\mathcal{O}_S$-modules endowed with descending filtrations $\mathcal{C}^\bullet\mathcal{M}$ and $\mathcal{C}^\bullet\mathcal{N}$ respectively, then $\mathcal{M} \otimes \mathcal{N}$ has a descending filtration defined by:

$$\mathcal{C}^i(\mathcal{M} \otimes \mathcal{N}) := \sum_{j\in\mathbf{Z}} \mathcal{C}^j\mathcal{M} \otimes \mathcal{C}^{i-j}\mathcal{N}. \tag{2.2}$$

There is of course a similar statement with ascending filtrations. Hence if $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ are F-zips over S, the locally free $\mathcal{O}_S$-module $\mathcal{M} \otimes \mathcal{N}$ is endowed with a descending filtration $\mathcal{C}^\bullet(\mathcal{M} \otimes \mathcal{N})$ and an ascending filtration $\mathcal{D}_\bullet(\mathcal{M} \otimes \mathcal{N})$. Furthermore, it is easy to see that the isomorphisms $\iota_\bullet$ for $\mathcal{M}$ and $\mathcal{N}$ induces similar isomorphisms between the graded pieces of these filtrations. We obtain an F-zip structure on $\mathcal{M} \otimes \mathcal{N}$, which we call the tensor product of $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ and denote it by $\underline{\mathcal{M}} \otimes \underline{\mathcal{N}}$. This shows that $\texttt{F-Zip}(S)$ is an $\mathbf{F}_p$-linear tensor category.

There is a notion of type for an F-zip $\underline{\mathcal{M}} = (\mathcal{M}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \iota_\bullet)$ over a base scheme S. Let $\eta : \mathbf{Z} \to \mathbf{Z}_{\geq 0}$ be a function with finite support (i.e $\eta(i) \neq 0$ for finitely many $i \in \mathbf{Z}$), then we say that $\underline{\mathcal{M}}$ has type η if the locally free sheaf $\mathrm{gr}^i(\mathcal{C}^\bullet)$ has rank $\eta(i)$ for all $i \in \mathbf{Z}$. Denote by $\texttt{F-Zip}^\eta(S)$ the full subcategory of F-zips of type η over S.

For an F-zip $\underline{\mathcal{M}}$, it is possible to shift the indexation of the filtrations by an index $r \in \mathbf{Z}$. The F-zip $\mathcal{M}[r]$ is defined by $\mathcal{C}^\bullet(\underline{\mathcal{M}}[r]) := \mathcal{C}^{\bullet+r}(\underline{\mathcal{M}})$ and $\mathcal{D}_\bullet(\underline{\mathcal{M}}[r]) := \mathcal{D}_{\bullet+r}(\underline{\mathcal{M}})$. Denote by $\mathcal{O}_S[r]$ the unique F-zip whose underlying sheaf is $\mathcal{O}_S$ and whose type has support $\{r\}$. Then $\underline{\mathcal{M}}[r]$ is simply $\underline{\mathcal{M}} \otimes \mathcal{O}_S[r]$. Hence if we denote by $\eta[r]$ the function $i \mapsto \eta(r+i)$, then the categories $\texttt{F-Zip}^\eta(S)$ and $\texttt{F-Zip}^{\eta[r]}(S)$ are equivalent.

§ 3. F-zips and Dieudonne spaces

We say that a commutative group scheme G over k is n-torsion (for an integer $n \in \mathbf{Z}_{\geq 1}$) if multiplication by n is the zero map of $G(R)$ for any k-algebra R. Recall the classification of finite, commutative, p-torsion group schemes over k by Dieudonne theory.

Theorem 3.1 ([14, page 69]). *There is a contravariant functor $\mathbf{D} : G \mapsto \mathbf{D}(G)$ between the category of finite, commutative, p-torsion group schemes over k and the category of triples (M, F, V) where M is a finite-dimensional k-vector space, $F : M \to M$ is a σ-linear map, $V : M \to M$ is a σ^{-1}-linear map, satisfying the conditions $FV = 0$ and $VF = 0$. Furthermore, this functor is an equivalence of categories.*

Furthermore, if a triple (M, F, V) satisfies the extra conditions $\mathrm{Im}(F) = \mathrm{Ker}(V)$ and $\mathrm{Im}(V) = \mathrm{Ker}(F)$, then we call it a Dieudonne space over k. For example, if A is an abelian variety over k, the p-torsion $A[p]$ is a finite, commutative, p-torsion group scheme over k and its associated object $\mathbf{D}(A[p])$ is a Dieudonne space over k ([15, §3.3.8]). A Dieudonne space (M, F, V) over k gives rise to an F-zip over k as follows.

(i) The filtration $C^\bullet$ of M is defined by $C^0 = M$, $C^1 = \mathrm{Ker}(F)$, $C^2 = 0$.

(ii) The filtration $D_\bullet$ of M is defined by $D_{-1} = 0$, $D_0 = \mathrm{Ker}(V)$, $D_1 = M$.

(iii) The isomorphism $\iota_0 : (C^1)^{(p)} \to M/D_0$ is the inverse of the map induced by V. The isomorphism $\iota_1 : (M/C^1)^{(p)} \to D_0$ is the one induced by F.

Proposition 3.2. *This construction gives an equivalence of categories between the category of Dieudonne spaces over k to the full subcategory of* $\texttt{F-Zip}(k)$ *of F-zips whose type $\eta : \mathbf{Z} \to \mathbf{Z}_{\geq 0}$ has support in $\{0, 1\}$.*

In particular, if A is an abelian scheme over k, we can attach an F-zip to A by applying Proposition 3.2 to the Dieudonne space $\mathbf{D}(A[p])$.

§ 4. F-zips arising in geometry

It is not obvious at first glance why Definition 2.1 is relevant. We will see that F-zips arise naturally in geometry via de Rham cohomology. To define it, first recall the construction of hypercohomology. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and suppose that $\mathcal{A}$ has enough injective objects (i.e., any object has a monomorphism to an injective object). Let $T : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Let $K^\bullet$ be a bounded below complex of objects in $\mathcal{A}$. Choose a quasi-isomorphism $K^\bullet \to I^\bullet$ to a complex of injective objects (this is always possible). Then one defines the hypercohomology of $K^\bullet$ as

$$\mathbf{R}^i T(K^\bullet) := H^i(T(I^\bullet)). \tag{4.1}$$

This gives a well-defined object in $\mathcal{B}$, that we call the hypercohomology of $K^\bullet$. It is possible to construct several spectral sequences that converge to the hypercohomology, by considering different filtrations on a complex $K^\bullet$. In this survey, the two spectral sequences of importance are the following two:

$$'E_1^{a,b} = R^b T(K^a) \Longrightarrow \mathbf{R}^{a+b} T(K^\bullet). \tag{4.2}$$

$$''E_2^{a,b} = R^a T(H^b(K^\bullet)) \Longrightarrow \mathbf{R}^{a+b} T(K^\bullet). \tag{4.3}$$

For example, let X be a scheme of finite-type over an arbitrary field k. The sheaves $\Omega^i_{X/k}$ are coherent $\mathcal{O}_X$-modules. However, since differentials are not $\mathcal{O}_X$-linear, the de Rham complex $\Omega^\bullet_{X/k}$ is a complex in the category of sheaves of k-vector spaces on X. Denote this abelian category by $\mathcal{A}$. Let $\mathcal{B}$ be the category of k-vector spaces, and let $T : \mathcal{A} \to \mathcal{B}$ be the functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$. The formalism of hypercohomology yields a k-vector space $H^i_{\mathrm{dR}}(X/k) := \mathbf{R}^i T(\Omega^\bullet_{X/k})$ and spectral sequences converging to $H^*_{\mathrm{dR}}(X/k)$:

$${}^{\mathrm{H}}E_1^{a,b} := H^b(X, \Omega^a_{X/k}) \Longrightarrow H^{a+b}_{\mathrm{dR}}(X/k). \tag{4.4}$$

$${}^{\mathrm{conj}}E_2^{a,b} := H^a(X, \mathscr{H}^b(\Omega^\bullet_{X/k})) \Longrightarrow H^{a+b}_{\mathrm{dR}}(X/k). \tag{4.5}$$

We call them respectively the Hodge and the conjugate spectral sequences. When k has characteristic zero, things are particularly simple as both sequences degenerate immediately. Furthermore, if $k = \mathbf{C}$, a standard fact in Hodge theory states that the conjugate filtration is obtained from the Hodge filtration by applying complex conjugation.

When k has positive characteristic p, the degeneracy is no longer true, not even when X is proper smooth over k. Hence, we make the following assumption:

Assumption 4.1. The Hodge spectral sequence of X degenerates at E_1.

Let $F_X : X \to X^{(p)}$ be the relative p-power Frobenius map and consider the complex $F_{X,*}(\Omega^\bullet_{X/k})$. This is a complex of $\mathcal{O}_{X^{(p)}}$-modules whose maps are $\mathcal{O}_{X^{(p)}}$-linear (easy computation). Hence the cohomology sheaves $\mathscr{H}^a(F_{X,*}(\Omega^\bullet_{X/k}))$ are $\mathcal{O}_{X^{(p)}}$-modules. The Cartier isomorphisms are natural isomorphisms

$$\Omega^a_{X^{(p)}} \xrightarrow{\sim} \mathscr{H}^a(F_{X,*}(\Omega^\bullet_{X/k})). \tag{4.6}$$

for all $a \geq 0$ (see [16, Theorem 7.2]). Taking the b-th cohomology group over $X^{(p)}$ on each side, we obtain σ-linear isomorphisms for all $a, b \geq 0$:

$$^{\mathrm{H}}E_1^{a,b} \simeq {}^{\mathrm{conj}}E_2^{b,a} \tag{4.7}$$

It also follows from this that the conjugate spectral sequence automatically degenerates at the second page. We now give the construction of the F-zip $\underline{M} := (M, C^\bullet, D_\bullet, \iota_\bullet)$ over k attached to X.

(i) Take $M = H^n_{\mathrm{dR}}(X/k)$.

(ii) Denote by $C^\bullet = (C^i)_{i\in\mathbf{Z}}$ the filtration obtained by the Hodge spectral sequence, indexed such that $C^\bullet$ is descending and $\mathrm{gr}^i(C^\bullet) = H^{n-i}(X, \Omega^i_{X/k})$.

(iii) Denote by $D_\bullet = (D_i)_{i\in\mathbf{Z}}$ the filtration obtained by the conjugate spectral sequence, indexed such that $D_\bullet$ is ascending and $\mathrm{gr}_i(D_\bullet) = H^{n-i}(X, \mathscr{H}^i(\Omega^\bullet_{X/k}))$.

(iv) Let $\iota_i : \mathrm{gr}_i(C^\bullet)^{(p)} \to \mathrm{gr}^i(D_\bullet)$ be the linearized Cartier isomorphism.

We have just seen that if X is a proper smooth scheme over k satisfying Assumption 4.1, then $H^n_{\mathrm{dR}}(X/k)$ is naturally endowed with an F-zip structure. There are many such schemes, for example abelian varieties, K3 surfaces, complete intersections in projective bundles... Also a theorem of Deligne-Illusie states that a proper smooth scheme X over k which lifts to $W_2(k)$ and of dimension $\dim(X) < p$ satisfies Assumption 4.1.

We now give the generalization of this construction to an arbitrary base scheme. Let $f : X \to S$ be a proper, smooth morphism of $\mathbf{F}_p$-schemes. Following the terminology of [2], we say that $f : X \to S$ satisfies condition (D) if the following properties hold.

(1) The $\mathcal{O}_S$-modules $R^b f_*(\Omega^a_{X/S})$ are locally free of finite rank for all $a, b \geq 0$.

(2) The Hodge spectral sequence degenerates at E_1.

Assume that f satisfies condition (D), then just as in the previous case, for any integer n such that $0 \leq n \leq 2\dim(X/S)$ the locally free $\mathcal{O}_S$-module $H^n_{\mathrm{dR}}(X/S)$ is naturally endowed with an F-zip structure over S. We write $\underline{H}^n_{\mathrm{dR}}(X/S)$ for this F-zip. This construction can be promoted to a contravariant functor

$$\underline{H}^n_{\mathrm{dR}} : \begin{pmatrix}\text{Proper smooth } X \to S \\ \text{satisfying condition (D)}\end{pmatrix} \longrightarrow \mathtt{F\text{-}Zip}(S). \tag{4.8}$$

An important example is the case of abelian schemes $A \to S$. If g denotes the relative dimension of A/S, the F-zip $\underline{H}^1_{\mathrm{dR}}(A/S)$ has type $\eta : \mathbf{Z} \to \mathbf{Z}_{\geq 0}$ where η is defined by

$$\eta(i) = \begin{cases} g & i = 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

In the case $S = \mathrm{Spec}(k)$, recall that we attached an F-zip to a Dieudonne space over k (Proposition 3.2). One can check that the F-zip $\underline{H}^1_{\mathrm{dR}}(A/k)$ coincides with the F-zip attached to $\mathbf{D}(A[p])$.

§ 5. Additional structure

From now on, k will denote an algebraic closure of $\mathbf{F}_p$. It is natural to consider F-zips endowed with additional structure. For example, let (A, λ) be a principally polarized abelian variety over k of dimension g. Let $\underline{H}^1_{\mathrm{dR}}(A/k) = (M, C^\bullet, D_\bullet, \iota_\bullet)$ be the attached F-zip. In particular, we have σ-linear isomorphisms $\iota_0 : M/C^1 \to D_0$ and $\iota_1 : C^1 \to M/D_0$. The polarization λ induces on M a perfect pairing $\langle -, - \rangle : M \times M \to k$ which satisfies the following conditions.

(i) C_1 and D_0 are totally isotropic.

(ii) One has $\langle \iota_0 x, \iota_1 y \rangle = \sigma \langle x, y \rangle$ for all $x \in M$ and all $y \in C^1$ (note that the expression $\langle \iota_0 x, \iota_1 y \rangle$ is well-defined because D_0 is totally isotropic).

More generally, we can define F-zips with G-structure for an arbitrary algebraic group G over $\mathbf{F}_p$. Denote by $\mathrm{Rep}(G)$ the category of algebraic representations of G over $\mathbf{F}_p$. Since we saw that the category of F-zips over a scheme S is a tensor category, we may define the following notion.

Definition 5.1. A G-zip functor over an $\mathbf{F}_p$-scheme S is an exact $\mathbf{F}_p$-linear tensor functor $\mathrm{Rep}(G) \to \texttt{F-Zip}(S)$.

We denote by G-$\texttt{ZipFun}(S)$ the category of G-zip functors over S. Our goal in this section is to explain a more down-to-earth definition of F-zips with G-structure over S. For this, we need to understand how to generalize the notion of type.

First of all, if $G = GL_n$, the category G-$\texttt{ZipFun}(S)$ is equivalent to the category $\texttt{F-Zip}^n(S)$ of F-zips $\underline{\mathcal{M}} = (\mathcal{M}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \iota_\bullet)$ where $\mathcal{M}$ is a locally free $\mathcal{O}_S$-module of rank n. If such an F-zip $\underline{\mathcal{M}}$ has type η, then it must satisfy

$$\sum_{i \in \mathbf{Z}} \eta(i) = n. \tag{5.1}$$

Giving such a function is the same as giving a conjugacy class of cocharacters of GL_n. The bijection is given as follows. If η is a function as above, the corresponding conjugacy class of cocharacters is given by considering a decomposition

$$\mathbf{F}_p^n = \bigoplus_{i\in\mathbf{Z}} V_i \tag{5.2}$$

where V_i has dimension $\eta(i)$ and letting $z \in \mathbf{G}_m$ act on V_i by z^i. Hence, it seems natural that the generalization of the notion of type for G-zip functors is a conjugacy class of cocharacters of G.

Definition 5.2. Let $\mu : \mathbf{G}_{m,k} \to G_k$ be a cocharacter. We say that a G-zip functor $z : \mathrm{Rep}(G) \to \texttt{F-Zip}(S)$ has type μ if for all representations $(V,\rho) \in \mathrm{Rep}(G)$, the F-zip $z(V,\rho)$ has type $\rho \circ \mu$. Denote by $G\text{-}\texttt{ZipFun}^{\mu}(S)$ the full subcategory of G-zip functors of type μ.

Now, we explain an equivalent definition of G-zips. Fix a cocharacter $\mu : \mathbf{G}_{m,k} \to G_k$. One obtains naturally a pair of opposite parabolic subgroups (P_-, P_+) in G_k and a common Levi subgroup $L := P_- \cap P_+ = \mathrm{Cent}(\mu)$. The group $P_+(k)$ consists of those elements $g \in G(k)$ such that the limit

$$\lim_{t\to 0} \mu(t) g \mu(t)^{-1} \tag{5.3}$$

exists, i.e. such that the map $\mathbf{G}_{m,k} \to G_k$, $t \mapsto \mu(t)g\mu(t)^{-1}$ extends to a morphism of varieties $\mathbf{A}_k^1 \to G_k$. The Lie algebra of the parabolic P_+ (resp. P_-) is given by

$$\mathrm{Lie}(P_+) = \bigoplus_{n\geq 0} \mathrm{Lie}(G)_n \quad (\text{resp. } \mathrm{Lie}(P_-) = \bigoplus_{n\leq 0} \mathrm{Lie}(G)_n) \tag{5.4}$$

where $\mathrm{Lie}(G)_n$ is the subspace where $z \in \mathbf{G}_m$ acts by z^n via the cocharacter μ. We set $P := P^-$, $Q := (P^+)^{(p)}$ and $M := L^{(p)}$, so that M is a Levi subgroup of Q. We denote by U and V the unipotent radicals of P and Q, respectively. For a k-scheme S, one defines:

Definition 5.3. A G-zip of type μ over S is a tuple $\underline{I} = (I, I_P, I_Q, \iota)$ where

(i) I is a G-torsor over S,

(ii) $I_P \subset I$ is a P-torsor over S,

(iii) $I_Q \subset I$ is a Q-torsor over S,

(iv) $\iota : (I_P/U)^{(p)} \to I_Q/V$ is an isomorphism of M-torsors.

In the case $G = GL_n$, one recovers the usual notion of F-zip. Denote by $G\text{-}\mathtt{Zip}^{\mu}(S)$ the category of G-zips of type μ. By a result of Pink-Wedhorn-Ziegler ([4, §1.4]), there is an equivalence of categories

$$G\text{-}\mathtt{ZipFun}^{\mu}(S) \simeq G\text{-}\mathtt{Zip}^{\mu}(S). \tag{5.5}$$

§ 6. The stack of G-zips

It is convenient to use the language of stacks to study F-zips and G-zips. Roughly speaking, a stack is an object that generalizes the notion of scheme by allowing automorphisms of points. First, recall that a groupoid is a category in which every map is an isomorphism. A category fibred in groupoids over the category of k-schemes is a family of groupoids $\mathcal{X}(S)$ for each k-scheme S, such that if $\varphi : S \to T$ is a map of k-schemes, there is a functor $\varphi^* : \mathcal{X}(T) \to \mathcal{X}(S)$. This is called a base change functor and is denoted by $(-)_S$. Furthermore, if $\varphi : S \to T$ and $\psi : T \to U$ are maps of k-schemes, there is an isomorphism of functors $(\psi \circ \varphi)^* \simeq \varphi^* \circ \psi^*$ (and these isomorphisms satisfy a cocycle relation). A stack over k is a particular kind of category fibred in groupoids over the category of k-schemes.

Specifically, one requires two conditions to hold:

(1) For all k-schemes S and all $x, y \in \mathcal{X}(S)$, the functor from S-schemes to sets which takes T to $\mathrm{Hom}_{\mathcal{X}(T)}(x_T, y_T)$ is a sheaf for the etale topology.

(2) All descent data are effective.

Roughly speaking, the second condition means that if $(T_i \to S)_{i\in I}$ is an etale covering, we may glue objects $x_i \in \mathcal{X}(T_i)$ to obtain an object $x \in \mathcal{X}(S)$. Specifically, write $V_{ij} := V_i \times_S V_j$. Then if $f_{ij} : (x_i)_{V_{ij}} \to (x_j)_{V_{ij}}$ are isomorphisms satisfying the usual cocycle relation, then there exists an object $x \in \mathcal{X}(S)$ such that $x_i = x_{V_i}$.

For example, a k-scheme X may be viewed as a stack over k. The groupoid $X(S)$ is simply the set $\mathrm{Hom}_k(S, X)$, viewed as a category where the only maps are the identities of objects.

For each k-scheme S, consider the category $\mathbf{F\text{-}Zip}(S)$ whose objects are F-zips over S and whose morphisms are isomorphisms of F-zips. Clearly, this is a groupoid. If $f : T \to S$ is a map of k-schemes, then there is a base change functor $f^* : \mathbf{F\text{-}Zip}(S) \to \mathbf{F\text{-}Zip}(T)$. Indeed, let $\underline{\mathcal{M}} = (\mathcal{M}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \iota_\bullet)$ be an F-zip over S. Then $f^*\underline{\mathcal{M}} = (f^*\mathcal{M}, f^*\mathcal{C}^\bullet, f^*\mathcal{D}_\bullet, f^*\iota_\bullet)$ is its pull-back to T.

Definition 6.1. The above construction gives rise to a stack over k. We denote it by $\mathtt{F\text{-}Zip}$ and call it the stack of F-zips. Similarly, if $\eta : \mathbf{Z} \to \mathbf{Z}_{\geq 0}$ is a function with finite support, then the categories $\mathtt{F\text{-}Zip}^{\eta}(S)$ give rise to a stack over k, that we denote by $\texttt{F-Zip}^{\eta}$. More generally, if G is a connected $\mathbf{F}_p$-reductive group and $\mu : \mathbf{G}_{m,k} \to G_k$ is a cocharacter, the categories $G\text{-}\texttt{Zip}^{\mu}(S)$ give rise to a stack $G\text{-}\texttt{Zip}^{\mu}$ over k.

This turns out to be an algebraic stack. In our case, this means that there is a smooth surjective morphism from a scheme to this stack. For an algebraic stack $\mathcal{X}$, it is possible to define an underlying topological space by taking the equivalence classes of pairs (K, x) where $k \subset K$ is a field extension and $x \in \mathcal{X}(K)$. Two pairs (K, x) and (K', x') are equivalent if there exists a common field extension L of K and K' such that $x_L \simeq x'_L$. The set of equivalence classes is denoted by $|\mathcal{X}|$. This set is endowed with a topology, as follows. Say that a map of stacks $\mathcal{Y} \to \mathcal{X}$ is an open immersion if the map $\mathcal{Y} \times_{\mathcal{X}} X \to X$ is an open immersion of schemes for any scheme X mapping to $\mathcal{X}$. In this case, $|\mathcal{Y}|$ is naturally a subset of $|\mathcal{X}|$. Subsets of this kind form a topology, called the Zariski topology of $|\mathcal{X}|$. Similarly, one can define a closed substack $\mathcal{Y} \to \mathcal{X}$ as a map of stacks that becomes a closed immersion (of schemes) after base change to a scheme $X \to \mathcal{X}$.

For a function $\eta : \mathbf{Z} \to \mathbf{Z}_{\geq 0}$ with finite support, the substack $\texttt{F-Zip}^{\eta} \subset \texttt{F-Zip}$ is both open and closed. The stack $\texttt{F-Zip}$ decomposes as a disjoint union

$$\texttt{F-Zip} = \bigsqcup_{\eta} \texttt{F-Zip}^{\eta} \tag{6.1}$$

and the substacks $\texttt{F-Zip}^{\eta}$ are the connected components of $\texttt{F-Zip}$. In particular, this implies that an F-zip over a connected scheme S has a type, because the corresponding map $S \to \texttt{F-Zip}$ must factor through a certain component $\texttt{F-Zip}^{\eta}$.

§ 7. Representation as a quotient stack

A nice property of stacks is the existence of quotients. If H is a smooth k-algebraic group acting on the left on a k-scheme X, then the quotient stack $\mathcal{X} := [H\backslash X]$ is defined as follows. For any k-scheme S, the groupoid $\mathcal{X}(S)$ is the category of pairs (T, α) where T is an H-torsor on S and $\alpha : T \to X \times_k S$ is an $H \times_k S$-equivariant map. It is clear that $\mathcal{X}(S)$ is a groupoid, and one can check that $\mathcal{X}$ is a stack over k. For example, when $X = \operatorname{Spec}(k)$ is endowed with the trivial action of H, the quotient stack $B(H) = [H\backslash \operatorname{Spec}(k)]$ is the classifying stack of H. For a k-scheme S, a morphism of stacks $S \to B(H)$ is essentially the same as an H-torsor over S.

Fix a connected reductive $\mathbf{F}_p$-group G and a cocharacter $\mu : \mathbf{G}_{m,k} \to G_k$. We will see that the k-stack $G\text{-}\texttt{Zip}^{\mu}$ can be written as a quotient stack. Let P, Q, L, M, U, V be the attached groups, as defined in §5. The Frobenius restricts to a map $\varphi : L \to M$. The isomorphisms $L \simeq P/U$ and $M \simeq Q/V$ yield natural maps $P \to L$ and $Q \to M$ which we both denote by $x \mapsto \overline{x}$. Define the zip group E as:

$$E := \{(a, b) \in P \times Q \mid \varphi(\overline{a}) = \overline{b}\}. \tag{7.1}$$

The group E acts on G by the rule $(a,b)\cdot g := agb^{-1}$.

Theorem 7.1 ([4, Th. 1.5]). *There is an isomorphism* $G\text{-}\mathtt{Zip}^{\mu} \simeq [E\backslash G]$.

In particular, the underlying topological space $|\,G\text{-}\mathtt{Zip}^{\mu}\,|$ coincides with the set of E-orbits in G. Each such orbit is locally closed for the Zariski topology of G. We now give a parametrization of these E-orbits. Fix a Borel pair (B,T) satisfying $B\subset P$ and suppose for simplicity that (B,T) are defined over $\mathbf{F}_p$. After possibly changing μ to a conjugate cocharacter, it is always possible to find such a Borel pair. Denote by Φ the set of T-roots and Δ the set of simple roots. Recall that there is a bijection between subsets of Δ and conjugacy classes of parabolic subgroups of G_k. We normalize this bijection such that Borel subgroups correspond to the empty set. Let $I,J\subset\Delta$ be the types of P,Q respectively. Since $B\subset P$, the set I consists of the simple roots of L. Write $W=N(T)/T$ for the Weyl group of T, it is a Coxeter group. There is a length function $\ell : W\to \mathbf{Z}_{\geq 0}$. Write w_0 for the longest element in W. For a subset $K\subset\Delta$, let $w_{0,K}$ be the longest element of the subgroup $W_K\subset W$ generated by $\{s_\alpha \mid \alpha\in K\}$. Also define W^K as the set of elements $w\in W$ which are of minimal length in the coset wW_K.

For $w\in W$, choose a representative $\dot{w}\in N_G(T)$, such that $(w_1w_2)^{\cdot}=\dot{w}_1\dot{w}_2$ whenever $\ell(w_1w_2)=\ell(w_1)+\ell(w_2)$ (this is possible by choosing a Chevalley system, see [13], Exp. XXIII, §6). Define $z:=w_0w_{0,J}$.

For $w\in W$, define G_w as the E-orbit of $\dot{w}\dot{z}^{-1}$. The E-orbits in G form a stratification of G by locally closed subsets.

Theorem 7.2 ([3, Th. 11.3]). *The map $w\mapsto G_w$ induces a bijection from W^J onto the set of E-orbits in G. Furthermore, for $w\in W^J$, one has*

$$\dim(G_w)=\ell(w)+\dim(P). \tag{7.2}$$

Endow G_w with the reduced subscheme structure. Then the quotient stack $\mathcal{X}_w = [E\backslash G_w]$ is a locally closed substack of $\mathcal{X}=G\text{-}\mathtt{Zip}^{\mu}$. We call $\mathcal{X}_w$ a zip stratum. This gives a stratification of $\mathcal{X}$. Note that the underlying topological space of $\mathcal{X}_w$ is a single point.

§ 8. Vector bundles on G-$\mathtt{Zip}^{\mu}$

It is possible to define a notion of vector bundles for algebraic stacks. If $\mathcal{X}$ is an algebraic stack, one could define a vector bundle over $\mathcal{X}$ as a family of vector bundles $\mathscr{V}=(\mathscr{V}_S)_S$ for each scheme S and each morphism of stacks $S\to\mathcal{X}$. Furthermore, this family should be compatible in an obvious sense. The space of global sections of $\mathscr{V}$ over $\mathcal{X}$ is then defined as an inverse limit of the spaces $H^0(S,\mathscr{V}_S)$.

Let G be a smooth algebraic group over k acting on a k-variety X. Let $\mathcal{X}$ be the quotient stack $[G\backslash X]$. Then there is a natural way to attach a vector bundle on $\mathcal{X}$ to an algebraic representation $\rho : G \to GL(V)$. Specifically, if $S \to \mathcal{X}$ is a map from a scheme S, then by definition of the quotient stack, we have a natural G-torsor on S. Applying the representation ρ, we obtain a $GL(V)$-torsor on S, hence a vector bundle of rank $\dim(V)$. This construction is functorial in S, so we obtain a vector bundle $\mathscr{V}(\rho)$ on the stack $\mathcal{X}$. Explicitly, the space of global sections $H^0(\mathcal{X}, \mathscr{V}(\rho))$ is identified with

$$H^0(\mathcal{X}, \mathscr{V}(\rho)) = \{f : X \to V,\ f(g \cdot x) = \rho(g)f(x),\ \forall g \in G,\ \forall x \in X\}\,. \tag{8.1}$$

Recall that the stack of G-zips of type μ is isomorphic to a quotient stack $[E\backslash G]$, as explained earlier. Hence, the previous construction attaches to each algebraic representation $\rho : E \to GL(V)$ a vector bundle $\mathscr{V}(\rho)$ on $G\text{-}\mathtt{Zip}^\mu$. Furthermore, $\mathscr{V}(\rho)$ is a line bundle if and only if ρ is a character of E.

For the time being, we consider only line bundles. There are natural identifications between characters of E, P and L via the first projection $E \to P$ and the Levi projection $P \to L$. Indeed, all these groups coincide up to a unipotent group, which has no nontrivial characters. Hence, we parametrize line bundles on $G\text{-}\mathtt{Zip}^\mu$ by characters of L : If $\lambda \in X^*(L)$, we denote by $\mathscr{V}(\lambda)$ the line bundle attached to the character $E \to \mathbf{G}_m$, $(a, b) \mapsto \lambda(\overline{a})$.

§ 9. Hasse invariants

One interesting feature of the stack of G-zips is the existence (in many cases, but not always) of Hasse invariants for zip strata. Let us start with a definition of what we mean by a Hasse invariant. Let $\mathcal{X}$ be an algebraic stack. We may thus consider its underlying topological space $|\mathcal{X}|$. Let $\mathcal{Y} \subset \mathcal{X}$ be a locally closed subset, and denote by $\overline{\mathcal{Y}}$ its Zariski closure. Endow both $\mathcal{Y}$ and $\overline{\mathcal{Y}}$ with the reduced substack structure. Finally, let $\mathscr{L}$ be a line bundle over $\mathcal{X}$.

Definition 9.1. A Hasse invariant for $\mathcal{Y}$ with respect to $\mathscr{L}$ is a section $h \in H^0(\overline{\mathcal{Y}}, \mathscr{L}^n)$ (some $n \geq 1$) such that the non-vanishing locus of h is exactly $\mathcal{Y}$.

Recall that any character $\lambda \in X^*(L)$ gives rise to a line bundle $\mathscr{V}(\lambda)$ on the stack $\mathcal{X} = G\text{-}\mathtt{Zip}^\mu$. Taking $\mathcal{Y}$ to be a single zip stratum $\mathcal{X}_w \subset \mathcal{X}$ (for some $w \in W^J$) in Definition 9.1, we have the notion of Hasse invariants for $\mathcal{X}_w$ with respect to $\mathscr{L}(\lambda)$. It is possible to give a combinatorial criterion for the existence of such Hasse invariants. For an element $w \in W$, we write E_w for the set of positive roots α satisfying $ws_\alpha < w$ and $\ell(ws_\alpha) = \ell(w) - 1$. Write σ for the action of Frobenius on W and $X^*(T)$. For $w \in W$ and an integer $n \geq 1$, let $w^{(n)}$ be the product $\sigma^n(w)\sigma^{n-1}(w) \ldots \sigma(w)$ and set by convention $w^{(0)} = 1$. It is easy to see that there exists $r \geq 1$ such that $w^{(r)} = 1$. Furthermore, the set of integers $r \geq 1$ such that $w^{(r)} = 1$ is stable under addition. Hence we can find $r \geq 1$ such that $w^{(r)} = 1$ for all $w \in W$. We fix such an integer $r \geq 1$. We also fix an integer $m \geq 1$ such that T is split over $\mathbf{F}_{p^m}$.

Proposition 9.2 ([6, Prop. 3.2.1]). *Let $w \in W^J$ and $\lambda \in X^*(L)$. The following assertions are equivalent:*

(i) There is a Hasse invariant for $\mathcal{X}_w$ with respect to $\mathscr{L}(\lambda)$.

(ii) For all $\alpha \in E_w$, one has:

$$\sum_{i=0}^{rm-1} \langle (zw^{-1})^{(i)} \sigma^i(\lambda), w\alpha^\vee \rangle p^i > 0. \tag{9.1}$$

First, we want to mention negative results. The above proposition can provide a counter-example for the principal purity of the stratification $(\mathcal{X}_w)_w$. Principal purity means that every stratum admits a Hasse invariant (for some $\lambda \in X^*(L)$). The easiest counter-example that we could find is in the case of $G = Sp(6)$ for a cocharacter μ that corresponds to the middle point of the Dynkin diagram. For the prime number $p = 2$, there exists a stratum $\mathcal{X}_w$ which does not admit Hasse invariants (for any $\lambda \in X^*(L)$).

To obtain a positive results for the existence of Hasse invariants, it is of course very cumbersome to check that condition (ii) is satisfied in general. Hence we want to mention a result which has a much easier statement.

Theorem 9.3. *Assume that $\lambda \in X^*(L)$ satisfies the following conditions:*

(i) One has $\langle \lambda, \alpha^\vee \rangle < 0$ for all $\alpha \in \Delta \setminus I$.

(ii) For all $\alpha \in \Phi$ such that $\langle \lambda, \alpha^\vee \rangle \neq 0$, for all $w \in W$ and all $j \in \mathbf{Z}$ we have

$$\left| \frac{\langle \lambda, w\sigma^j(\alpha)^\vee \rangle}{\langle \lambda, \alpha^\vee \rangle} \right| \leq p - 1. \tag{9.2}$$

Then $\mathscr{L}(\lambda)$ admits Hasse invariants on all zip strata.

Theorem 9.3 is an elementary consequence of Proposition 9.2. One checks that the expression (9.1) is positive as follows: View this expression as a polynomial in p. The leading coefficient is

$$\langle (zw^{-1})^{(rm-1)} \sigma^{rm-1}(\lambda), w\alpha^\vee \rangle = \langle wz^{-1}\sigma^{-1}(\lambda), w\alpha^\vee \rangle = \langle \lambda, \sigma(z\alpha)^\vee \rangle.$$

We claim that this leading coefficient is positive. First, since $w \in W^J$ and $\alpha \in E_w$, we have $\alpha \notin J$. Since $z = w_0 w_{0,J}$, we deduce easily that $z\alpha$ is a negative root not contained in M. It follows that $\sigma(z\alpha)$ is a negative root, not contained in L. Hence one can write $-\sigma(z\alpha)^\vee = \sum_j \alpha_j^\vee$ for $\alpha_j \in \Delta$, with at least one α_j in $\Delta \setminus I$. It follows from Condition (i) that $\langle \lambda, \sigma(z\alpha)^\vee \rangle > 0$, hence the claim.

Divide the expression (9.1) by this leading coefficient. Then Condition (ii) implies that the coefficients of this monic polynomial are $\leq p-1$. The result then follows from the inequality

$$(p-1)\sum_{i=0}^{r-2} p^i = p^{r-1} - 1 < p^{r-1}. \tag{9.3}$$

§ 10. Ekedahl-Oort strata

Consider an abelian variety A of dimension $g \geq 1$ over k. The p-torsion $A[p]$ is a finite commutative p-torsion group scheme over k. Not all finite commutative p-torsion group scheme over k appear in this way. Those which do are exactly those whose Dieudonne module $\mathbf{D}(A[p])$ satisfies $\mathrm{Im}(F) = \mathrm{Ker}(V)$ and $\mathrm{Im}(V) = \mathrm{Ker}(F)$ (we say that $A[p]$ is a BT_1).

If $\mathscr{A} \to S$ is an abelian scheme over a base scheme of characteristic p, then S is naturally decomposed as a (set-theoretic) disjoint union

$$S = \bigsqcup_\gamma S_\gamma \tag{10.1}$$

where γ varies in the set of isomorphism classes of BT_1's. The subset S_γ is the set of points $s \in S$ such that $\mathscr{A}_s[p]$ is in γ. By a theorem of Oort, this decomposition is locally closed. However, in general it may not be a stratification of S in the sense that the closure of S_γ may not be a union of $S_{\gamma'}$ for certain γ'.

As we explained, we may attach to $\mathscr{A} \to S$ an F-zip $\underline{\mathcal{M}} = (\mathcal{M}, \mathcal{C}^\bullet, \mathcal{D}_\bullet, \iota_\bullet)$ over S whose type η has support in $\{0,1\}$ and satisfies $\eta(0) = \eta(1) = g$. This gives rise to a morphism of stacks

$$\zeta : S \longrightarrow \texttt{F-Zip}^\eta\,. \tag{10.2}$$

The strata S_γ of S coincide with the fibers of this morphism. The stack $\texttt{F-Zip}^\eta$ coincides with the stack $G\text{-}\texttt{Zip}^\mu$ for the group $G = GL_{2g}$ and the cocharacter

$$\mu : z \mapsto \begin{pmatrix} z\,\mathrm{Id}_g & \\ & \mathrm{Id}_g \end{pmatrix}. \tag{10.3}$$

In particular, the Levi subgroup $L \subset G$ attached to μ is the set of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad \text{where } A, D \in GL_g. \tag{10.4}$$

Thus $X^*(L)$ is a free $\mathbf{Z}$-module of rank 2. Consider the character $\lambda_\omega \in X^*(L)$ given by mapping the matrix (10.4) to $\det(A)^{-1}$. We have an associated line bundle $\mathscr{V}(\lambda_\omega)$ over $\mathcal{X} = \texttt{F-Zip}^\eta$.

The Hodge vector bundle Ω of the abelian scheme $\mathscr{A} \to S$ is defined as the pullback along the unit section $S \to \mathscr{A}$ of the sheaf of relative differentials $\Omega^1_{\mathscr{A}/S}$. It is a rank g vector bundle on S. Denote by $\omega = \wedge^g \Omega$ its determinant. Then one has the following equation.

$$\zeta^* \mathscr{V}(\lambda_\omega) = \omega. \tag{10.5}$$

It is easy to check that the character λ_ω satisfies the conditions (i) and (ii) of Theorem 9.3 (for any value of the prime p). The first one is immediate, and for the second one, note that λ_ω is minuscule, hence for all $\alpha \in \Phi$ such that $\langle \lambda_\omega, \alpha^\vee \rangle \neq 0$, the quotient

$$\left| \frac{\langle \lambda_\omega, w\alpha^\vee \rangle}{\langle \lambda_\omega, \alpha^\vee \rangle} \right| \tag{10.6}$$

only takes the value 0 or 1 for any $w \in W$, hence is always $\leq p-1$, even for $p = 2$. Thus we may apply the theorem to the line bundle $\mathscr{V}(\lambda_\omega)$. By pulling back to S, we deduce:

Proposition 10.1. *For each isomorphism class γ of BT_1's over k, there exists $n \geq 1$ and a section $\mathrm{Ha}_\gamma \in H^0(\overline{S}_\gamma, \omega^n)$ over the Zariski closure $\overline{S}_\gamma$ which satisfies*

$$\{s \in \overline{S}_\gamma \mid \mathrm{Ha}_\gamma(s) \neq 0\} = S_\gamma. \tag{10.7}$$

§ 11. Sketch of proof

We sketch the proof of Proposition 9.2. It relies heavily on the stack of G-zip flags. It is a stack $\mathcal{Y}$ with a natural projection map $\pi : \mathcal{Y} \to \mathcal{X}$. It carries a stratification $(\mathcal{Y}_w)_{w \in W}$ indexed by the whole Weyl group. Specifically, we give the following definition.

Definition 11.1. A G-zip flag of type μ over a k-scheme S is a pair $\hat{I} = (\underline{I}, J)$ where $\underline{I} = (I, I_P, I_Q, \iota)$ is a G-zip of type μ over S, and $J \subset I_P$ is a B-torsor.

We denote by G-$\texttt{ZipFlag}^\mu(S)$ the category of G-zip flags over S. By similar arguments as for G-zips, we obtain a stack $\mathcal{Y} := G\text{-}\texttt{ZipFlag}^\mu$ over k, which we call the stack of G-zip flags of type μ. There is a natural projection $\pi : \mathcal{Y} \to \mathcal{X}$ given by forgetting the B-torsor. To stratify $\mathcal{Y}$, we need the following result. Define a subgroup $E' \subset E$ by $E' := E \cap (B \times G)$. By adapting the proof of Theorem 7.1, one can prove the following.

Theorem 11.2. *There is a commutative diagram, where the vertical maps are isomorphisms and the lower horizontal map is the natural projection:*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\pi} & \mathcal{X} \\ \simeq\downarrow & & \downarrow\simeq \\ [E'\backslash G] & \xrightarrow{\pi} & [E\backslash G]\,. \end{array}$$

We explain now how $\mathcal{Y}$ admits a natural stratification. It is based on the observation that the group E' is contained in the product $B \times {}^{z}B$, where ${}^{z}B := zBz^{-1}$ (this is an easy verification). In particular, there is a natural projection map

$$\psi : \mathcal{Y} \to [B\backslash G/{}^{z}B] \simeq [B\backslash G/B] \tag{11.1}$$

By the Bruhat decomposition of G, the latter stack has a stratification $(\mathcal{B}_w)_{w\in W}$ parametrized by the Weyl group W. The strata $(\mathcal{Y}_w)_{w\in W}$ of $\mathcal{Y}$ are defined as the fibers of the map ψ. Now we give the main steps of the proof of Proposition 9.2.

(1) First, one shows that for any $\lambda \in X^*(L)$, there exists $n \geq 1$ such that $\mathscr{V}(\lambda)^n$ admits a nonzero section $h_{\lambda,w}$ on the strata $\mathcal{X}_w$ of $\mathcal{X}$ for all $w \in W^J$. Furthermore, these sections are unique up to nonzero scalar. To prove Proposition 9.2, we need to understand for which λ the section $h_{\lambda,w}$ extends to a Hasse invariant of $\mathcal{X}_w$.

(2) In general, for $w \in W$, the image of $\mathcal{Y}_w$ by π is a union of several strata $\mathcal{X}_{w'}$. However, when $w \in W^J$, one has $\pi(\mathcal{Y}_w) = \mathcal{X}_w$. Furthermore, the map $\pi : \mathcal{Y}_w \to \mathcal{X}_w$ is finite etale.

(3) One shows that $h_{\lambda,w}$ extends to a Hasse invariant if and only if $\pi^* h_{\lambda,w}$ extends to a Hasse invariant for $\mathcal{Y}_w$. One implication is clear. The other is the following lemma:

Lemma 11.3. *Let $f : X \to Y$ a proper surjective morphism of integral schemes of finite-type over k. Let $\mathscr{L}$ be a line bundle on Y. Let $U \subset Y$ be a normal open subset and $h \in \mathscr{L}(U)$ a non-vanishing section over U. Assume that the section $f^*(h) \in H^0(f^{-1}(U), f^*\mathscr{L})$ extends to X with non-vanishing locus $f^{-1}(U)$. Then there exists $d \geq 1$ such that h^d extends to Y, with non-vanishing locus U.*

(4) One shows that the pull-back $\pi^* h_{\lambda,w}$ coincides with $\psi^* f_{\lambda,w}$ for a certain function $f_{\lambda,w}$ on the stratum $\mathcal{B}_w$ of $[B\backslash G/B]$ parametrized by w. Similarly, $\psi^* f_{\lambda,w} = \pi^* h_{\lambda,w}$ extends to a Hasse invariant for $\mathcal{Y}_w$ if and only if $f_{\lambda,w}$ extends to a Hasse invariant for $\mathcal{B}_w$.

(5) Finally, the last part is just a computation. It is based on Chevalley's formula, which makes it possible to compute the divisor of the section $f_{\lambda,w}$. The result is

that it extends to a Hasse invariant if and only if Condition (ii) of Proposition 9.2 is satisfied, because the expressions that appear (for α varying in E_w) are the multiplicities of the divisor of $f_{\lambda,w}$. This terminates the proof.

§ 12. Global sections of vector bundles

Recall that any algebraic representation $\rho : E \to GL(V)$ gives rise to a vector bundle $\mathscr{V}(\rho)$ over $\mathcal{X} = G\text{-}\mathtt{Zip}^{\mu}$. If $\rho : L \to GL(V)$ is a representation of L, we may view it as a representation of E via the map $E \to L$ defined as the composition of the first projection $E \to P$ and the Levi projection $P \to L$.

So far, we have only considered characters of L, which is the rank 1 case, and constructed Hasse invariants for those vector bundles. In what follows, we want to study higher rank vector bundles. Of particular interest are the vector bundles attached to L-dominant characters by induction. For a character $\lambda \in X^*(T)$, view λ as a character $B \to \mathbf{G}_m$ and consider the induced representation

$$V(\lambda) = \operatorname{Ind}_B^P(\lambda). \tag{12.1}$$

It is a representation of P where the unipotent radical of P acts trivially, so we may view it as a representation of L. Note that if λ is not an L-dominant character, we have $V(\lambda) = 0$. We denote by $\mathscr{V}(\lambda)$ the vector bundle over $\mathcal{X}$ attached to $V(\lambda)$. This provides an interesting family of vector bundles $(\mathscr{V}(\lambda))_\lambda$ on $\mathcal{X}$ indexed by the L-dominant characters $\lambda \in X^*(T)$.

We end this survey with a result that determines the space of global sections of $\mathscr{V}(\lambda)$ over $\mathcal{X}$. To simplify, we assume that μ is defined over $\mathbf{F}_p$. We choose again a Borel pair (B, T) defined over $\mathbf{F}_p$ and we assume also that T is split over $\mathbf{F}_p$. For a character $\lambda \in X^*(T)$, our goal is to determine the space $H^0(\mathcal{X}, \mathscr{V}(\lambda))$.

Denote by $\mathcal{U} \subset \mathcal{X}$ the unique open zip stratum. The first step is to determine the space $H^0(\mathcal{U}, \mathscr{V}(\lambda))$. This is elementary, and can be done for an arbitrary representation $\rho : L \to GL(V)$. Specifically, one has the following result.

Lemma 12.1. *For any representation $\rho : L \to GL(V)$, there is an isomorphism*

$$H^0(\mathcal{U}, \mathscr{V}(\rho)) \simeq V^{L(\mathbf{F}_p)}. \tag{12.2}$$

This is almost a tautology, given that the stack $\mathcal{U}$ can be seen to be isomorphic to $[1/L(\mathbf{F}_p)]$. In particular, this shows that $H^0(\mathcal{X}, \mathscr{V}(\rho))$ is a subspace of the $L(\mathbf{F}_p)$-invariants of V. To determine exactly which subspace demands some work.

First, we introduce some notation. For any representation $\rho : L \to GL(V)$, we may decompose V with respect to T-eigenspaces.

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi. \tag{12.3}$$

Define a subspace $V_{\leq 0} \subset V$ as follows. It is the direct sum of the T-eigenspaces V_χ for the characters $\chi \in X^*(T)$ which satisfy the condition

$$\langle \chi, \alpha^\vee \rangle \leq 0 \quad \text{for all } \alpha \in \Delta \setminus I. \tag{12.4}$$

Note that $V_{\leq 0}$ is stable under the action of T, but it is not a sub-L-representation of V. From now on we consider the L-representation $V(\lambda)$ defined previously, attached to a character $\lambda \in X^*(T)$. One has the following.

Theorem 12.2. *There is a commutative diagram where the vertical maps are the natural inclusions, and the horizontal maps are isomorphisms:*

$$\begin{array}{ccc} H^0(\mathcal{U}, \mathscr{V}(\lambda)) & \xrightarrow{\simeq} & V(\lambda)^{L(\mathbf{F}_p)} \\ \uparrow & & \uparrow \\ H^0(\mathcal{X}, \mathscr{V}(\lambda)) & \xrightarrow{\simeq} & V(\lambda)_{\leq 0} \cap V(\lambda)^{L(\mathbf{F}_p)} \end{array} \tag{12.5}$$

In a recent paper [17], we show that the above theorem also holds for an arbitrary L-representation (V, ρ) (not necessarily of the form $V(\lambda)$). Furthermore, we also determine in *loc. cit.* the space $H^0(\mathcal{X}, \mathscr{V}(\rho))$ for an arbitrary P-representation (V, ρ), and without assuming that L is defined over $\mathbf{F}_p$. It is a difficult problem in representation theory to determine for which $\lambda \in X^*(T)$, the intersection $V(\lambda)_{\leq 0} \cap V(\lambda)^{L(\mathbf{F}_p)}$ is non-zero. The set

$$C_{\text{zip}} := \left\{ \lambda \in X^*(T) \mid V(\lambda)_{\leq 0} \cap V(\lambda)^{L(\mathbf{F}_p)} \neq 0 \right\} \tag{12.6}$$

is an additive submonoid (i.e. a cone) of $X^*(T)$. The paper [9] contains some partial results on this set, but it remains quite mysterious.

Let us mention an even more difficult problem. Instead of considering one λ at a time, it is interesting to put them all together by forming the direct sum

$$R_{\text{zip}} := \bigoplus_{\lambda \in X^*(T)} H^0(\mathcal{X}, \mathscr{V}(\lambda)). \tag{12.7}$$

This group inherits a natural structure of graded k-algebra, and one can ask what the isomorphism class of R_{zip} is. The set C_{zip} is then simply the grading monoid of this graded algebra, so the question of determining R_{zip} is a refinement of the determination of C_{zip}. It is not even clear whether R_{zip} is a finite-type k-algebra. Here are some partial results:

Proposition 12.3. *One has the following:*

(i) The algebra R_{zip} *is isomorphic to a subalgebra of* $k[G]$*. In particular, it is integral.*

(ii) The field of fractions of R_{zip} *is isomorphic to the function field of* $R_u(B \cap L)$*. In particular, the scheme* $\mathrm{Spec}(R_{\mathrm{zip}})$ *is birational to an affine space.*

(iii) If $\mathrm{Pic}(G) = 0$*, then* R_{zip} *is a UFD.*

References

[1] Moonen, B., Serre-Tate theory for moduli spaces of PEL-type, *Ann. Sci. ENS*, **37** (2004), 223–269.

[2] Moonen, B. and Wedhorn, T., Discrete invariants of varieties in positive characteristic, *IMRN*, **72** (2004), 3855–3903.

[3] Pink, R. and Wedhorn, T. and Ziegler, P., Algebraic zip data, *Doc. Math.*, **16** (2011), 253–300.

[4] Pink, R. and Wedhorn, T. and Ziegler, P., F-zips with additional structure, *Pacific J. Math.*, **274** (2015), 183–236.

[5] Koskivirta, J.-S. and Wedhorn, T., Generalized μ-ordinary Hasse invariants, *J. Algebra*, **502** (2018), 98–119.

[6] Goldring, W. and Koskivirta, J.-S., Strata Hasse invariants, Hecke algebras and Galois representations, *Invent. Math. 217*, No. 3 (2019), pp. 887-984.

[7] Goldring, W. and Koskivirta, J.-S., Automorphic vector bundles with global sections on G-Zip$^{\mathcal{Z}}$-schemes, *Compositio Math.*, **154** (2018), 2586–2605.

[8] Koskivirta, J.-S., Normalization of closed Ekedahl-Oort strata, *Canad. Math. Bull.*, **61** (2018), 572–587.

[9] Koskivirta, J.-S., Automorphic forms on the stack of G-Zips, *Results in Mathematics*, 74: 91. (2019) https://doi.org/10.1007/s00025-019-1021-z

[10] Koskivirta, J.-S., Canonical sections of the Hodge bundle over Ekedahl-Oort strata of Shimura varieties of Hodge type. *Journal of Algebra*, **449** (2016), 446-459.

[11] Goldring, W. and Koskivirta, J.-S., Stratifications of flag spaces and functoriality. *International Mathematical Research Notices* (2017).

[12] Nicole, M.-H. and Vasiu, A. and Wedhorn, T., Purity of level m stratifications, *Ann. Sci. ENS*, **23** (2010), 925–955.

[13] Artin, M. and Bertin, J. E. and Demazure, M. and Gabriel, P. and Grothendieck, A. and Raynaud, M. and Serre, J.-P., SGA3: Schémas en groupes, *Institut des Hautes Études Scientifiques*, (1965/66).

[14] Demazure, M., Lectures on p-divisible groups, *Lecture Notes in Mathematics*, **302**, (1972).

[15] Berthelot, P. and Breen, L. and Messing, W., Théorie de Diedonné Cristalline II, *Lecture Notes in Mathematics*, **930**, (1982).

[16] Katz, N., Nilpotent connections and the monodromy theorem: applications of a result of Turrittin, *Publ. Math. IHÉS*, pp 175–232, **39**, (1970).

[17] Imai, N. and Koskivirta, J.-S., Automorphic vector bundles on the stack of G-zips, *preprint, arXiv:2008.02525*, (2020).