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Good reduction of hyperbolic polycurves and their fundamental groups: A survey

By

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Abstract

The goal of this manuscript is to provide a survey of good reduction criteria for hyperbolic polycurves. In particular, we give outlines of the proofs of the main theorems of the papers [19] and [20], which are details of the talk "Criteria for good reduction of hyperbolic polycurves" given at "Algebraic Number Theory and Related Topics 2018". Also, this paper contains a proof of a specialization theorem of pro-$\mathbb{L}$ fundamental groups.

§ 1. Introduction

Let K be a discrete valuation field, O_K the valuation ring of K, k the residual field of O_K, p the characteristic of k, and K^{sep} a separable closure of K. Write G_K for the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K and I_K for an inertia subgroup of K (which is uniquely determined up to conjugation). For a smooth variety X over K (i.e., a smooth separated scheme of finite type over K with geometrically connected fibers), the issue of whether or not there exists a "good" model $\mathfrak{X}$ of X (i.e., a "good" scheme over O_K such that the scheme $\mathfrak{X} \times_{\mathrm{Spec}\, O_K} \mathrm{Spec}\, K$ is isomorphic to X over K) is interesting and has been studied in arithmetic geometry. In the case where $X(= A)$ is an abelian variety over K, we may say that A has a "good" model if the Néron model of A is proper over O_K. The following criteria determine the existence of a good model of A:

Theorem 1.1 (Néron, Ogg, Shafarevich, Serre, and Tate). *For a prime number l, write $T_l A$ for the l-adic Tate module of A. Write $T_{p'} A$ for the group $\prod_{l \neq p} T_l A$ (where l ranges over the prime numbers not equal to p). The following are equivalent:*

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1. *A has good reduction (i.e., A has a proper Néron model).*

2. *$T_{p'}A$ is unramified (i.e., the action of I_K on $T_{p'}A$ is trivial).*

3. *T_lA is unramified for some prime $l \neq p$.*

4. *T_lA is unramified for any prime $l \neq p$.*

Theorem 1.2 (Fontaine, Mokrane, Coleman, Iovita, and Breuil)**.** *Suppose that K is a complete discrete valuation field of characteristic 0, k is a perfect field, and $p > 0$. Write T_pA for the p-adic Tate module of A. Then A has good reduction if and only if the p-adic G_K-representation T_pA is crystalline.*

Note that the Tate module of A is isomorphic to the dual module of the first étale cohomology of $A \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, K^{\mathrm{sep}}$. The implication $1 \Rightarrow 2$ of Theorem 1.1 follows from the proper base change theorem of étale cohomology groups and a specialization argument. Therefore, for any proper variety X over K and its étale cohomology groups, a similar implication holds in general. However, it is not sufficient to see the étale cohomology groups of X to determine whether there exists a good model of X. For example, in the case where X is a proper hyperbolic curve over K (cf. Notation-Definition 3.1), the unramifiedness of its first étale cohomology group is equivalent to the condition that the Jacobian variety of X has good reduction (cf. Theorem 1.1 and Theorem 3.6). Moreover, if we treat non-proper varieties, we need to define "good reduction" of them carefully. For example, consider the case where there exist a smooth compactification $\overline{X}$ of X over K and a semi-stable model $\overline{\mathfrak{X}}$ of $\overline{X}$ over O_K. In this case, the smooth locus $\mathfrak{U}$ of the complement of the topological closure of $\overline{X} \setminus X$ in $\overline{\mathfrak{X}}$ has a nonempty special fiber. We do not want to treat such a $\mathfrak{U}$ as a smooth model of X in general because $\overline{\mathfrak{X}}$ might not be smooth over O_K. Hence, we need to consider smooth models with their suitable compacitification.

If X is a hyperbolic curve over K, we can define the notion of good reduction of X since we have the canonical smooth compactification of X over K (cf. Section 3). Oda and Tamagawa established a good reduction criterion for hyperbolic curves. They used étale fundamental groups instead of étale cohomology groups. Andreatta, Iovita, and Kim gave a p-adic analogue of this criterion. Note that the étale fundamental groups of hyperbolic curves are not abelian. Moreover, note that, in general, we cannot define Galois actions on the étale fundamental groups similar to the Galois actions on the étale cohomology groups of hyperbolic curves. Therefore, we need to consider an outer Galois representation instead, or suppose that X has a K-rational point x and consider the Galois action on the étale fundamental group induced by x.

A hyperbolic polycurve, that is, a successive extension of hyperbolic curves, is a higher dimensional analogue of a hyperbolic curve (cf. Definition 5.1). It is natural to ask if we can determine whether a hyperbolic polycurve has good reduction or not from its outer Galois representation, which was studied in [19] and [20]. In an imitation of the proof of the criterion for hyperbolic curves, a major obstacle is that the functor of taking pro-l completion is not exact. Hence, we need to seek the condition that there exist pro-l homotopy exact sequences of étale fundamental groups. This paper gives an overview of these theories.

Contents of this paper are as follows: In Section 2, we review basic properties of profinite groups and specialization homomorphisms of étale fundamental groups. In Section 3, we overview results of Oda and Tamagawa. In Section 4, we introduce results of [8], [9], and [16]. In Section 5, we give a definition of hyperbolic polycurves and the precise statement of good reduction criteria for them. In Section 6, we describe the approach of [19]. In Section 7, we describe the approach of [20]. In Section 8, we give a proof of a pro-$\mathbb{L}$ specialization theorem.

§ 2. Basic Properties of Étale Fundamental Groups

In this section, we review basic properties of profinite groups and specialization homomorphisms of étale fundamental groups.

§ 2.1. Profinite Groups

First, we introduce some notation. Write $\mathfrak{Primes}$ for the set of prime numbers. Let G be a profinite group. For any subset $\mathbb{L} \subset \mathfrak{Primes}$, we shall write $G^{\mathbb{L}}$ for the inverse limit of the inverse system consisting of the quotient groups of G by open normal subgroups such that the prime factors of the orders of the quotient groups are in $\mathbb{L}$. Let l (resp. p) be a prime number (resp. a prime number or 0). We shall write $G^{l} = G^{\{l\}}$ (resp. $G^{p'} = G^{\mathfrak{Primes}\setminus\{p\}}$).

We shall write $\mathrm{Inn}(G)$ (resp. $\mathrm{Aut}(G)$; $\mathrm{Out}(G)$) for the group of inner automorphisms (resp. continuous automorphisms; continuous outer automorphisms) of the profinite group G. Hence, we have a natural exact sequence

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1.$$

Let N be a subgroup of G. We shall write $Z_G(N)$ for the centralizer of N in G.

Lemma 2.1. *Let*

$$1 \to N \to G \to H \to 1$$

be an exact sequence of profinite groups.

1. *The kernel of the homomorphism $N \to N^{\mathbb{L}}$ is a characteristic subgroup of N. In particular, the group $\mathrm{Ker}(N \to N^{\mathbb{L}})$ is a normal subgroup of G.*

2. *Write* $G^{(\mathbb{L})}$ *for the group* $G/\mathrm{Ker}(N \to N^{\mathbb{L}})$. *There exist an exact sequence of profinite groups*

$$1 \to N^{\mathbb{L}} \to G^{(\mathbb{L})} \to H \to 1$$

and an induced natural outer action $H \to \mathrm{Out}(N^{\mathbb{L}})$.

3. *If the outer action* $H \to \mathrm{Out}(N)$ *is trivial, we have* $G = N \cdot Z_G(N)$.

4. *If* N *is center-free, we have* $\{1\} = N \cap Z_G(N)$.

Proof. Assertion 1 holds since every automorphism of N induces a natural automorphism of $N^{\mathbb{L}}$. Assertion 2 follows from assertion 1. $H \to \mathrm{Out}(N)$ is trivial if and only if the natural homomorphism $Z_G(N) \to H$ is surjective. Therefore, assertion 3 holds. Since $N \cap Z_G(N)$ is the center of N, assertion 4 holds. □

Definition 2.2. Let g and r be integers satisfying $g \geq 0, r \geq 0$, and $2g+r-2 > 0$. We shall write $\Sigma_{g,r}$ for the group

$$\langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_r \rangle / [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g]\gamma_1 \ldots \gamma_r.$$

Here, α_i, β_i, and γ_j are letters, and "[,]" denotes the commutator. We shall write $\Pi_{g,r}$ for the profinite completion of $\Sigma_{g,r}$.

Lemma 2.3 ([21, Corollary (1.3.4)]). *The profinite groups* $\Pi_{g,r}$, $\Pi^{l}_{g,r}$, *and* $\Pi^{p'}_{g,r}$ *are center-free.*

§ 2.2. Étale Fundamental Groups and Specialization

Let K be a discrete valuation field with residual characteristic $p\,(\geq 0)$. Write O_K for the valuation ring of K and k for the residual field of O_K. Fix a separable closure K^{sep} of K and write $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Let $X \to \mathrm{Spec}\, K$ be a separated morphism of finite type with geometrically connected fibers and write $X_{K^{\mathrm{sep}}} = X \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, K^{\mathrm{sep}}$. Take a geometric point $*$ of $X_{K^{\mathrm{sep}}}$ and write Π_X (resp. Δ_X) for the étale fundamental group $\pi_1(X, *)$ (resp. $\pi_1(X_{K^{\mathrm{sep}}}, *)$). Let l be a prime number satisfying that $l \neq p$. Then we have an exact sequence of profintie groups

$$1 \to \Delta_X \to \Pi_X \to G_K \to 1, \tag{2.1}$$

which induces exact sequences of profintie groups

$$1 \to \Delta^{p'}_X \to \Pi^{(p')}_X \to G_K \to 1 \tag{2.2}$$

and

$$1 \to \Delta^{l}_X \to \Pi^{(l)}_X \to G_K \to 1 \tag{2.3}$$

by Lemma 2.1.2. Here, $\Pi_X^{(p')} = \Pi_X^{(\mathfrak{Primes}\setminus\{p\})}$ and $\Pi_X^{(l)} = \Pi_X^{(\{l\})}$. Choose an inertia subgroup I_K of G_K (which is uniquely determined up to conjugation). By using the above exact sequences, we obtain outer Galois representations

$$I_K \to \mathrm{Out}(\Delta_X^{p'}) \tag{2.4}$$

and

$$I_K \to \mathrm{Out}(\Delta_X^{l}) \tag{2.5}$$

for any prime number $l \neq p$. As in the case of étale cohomology groups, the pro-$\mathbb{L}$ specialization homomorphisms of étale fundamental groups are isomorphisms (cf. Theorem 8.3).

Proposition 2.4. *Let $\mathfrak{X} \to \operatorname{Spec} O_K$ be a morphism satisfying the following two conditions:*

- *The scheme X is isomorphic to $\mathfrak{X} \times_{\operatorname{Spec} O_K} \operatorname{Spec} K$ over K.*
- *We have a facotrization*

$$\mathfrak{X} = \mathfrak{X}_n \to \ldots \to \mathfrak{X}_0 = \operatorname{Spec} O_K$$

such that there exist a proper smooth morphism $\overline{\mathfrak{X}}_{i+1} \to \mathfrak{X}_i$ with geometrically connected fibers and a normal crossing divisor $\mathfrak{D}_{i+1} \subset \overline{\mathfrak{X}}_{i+1}$ of the scheme $\overline{\mathfrak{X}}_{i+1}$ relative to $\mathfrak{X}_i$ satisfying that the complement $\overline{\mathfrak{X}}_{i+1} \setminus \mathfrak{D}_{i+1}$ is isomorphic to $\mathfrak{X}_{i+1}$ over $\mathfrak{X}_i$ for each $0 \leq i \leq n-1$.

Then the outer action (2.4) is trivial.

Proof. We may assume that K is strictly henselian (and hence $G_K = I_K$). Take a geometric point $\bar{t}$ of $X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}$ over its generic point. Since the natural homomorphism

$$\pi_1(X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}, \bar{t})^{p'} \to \pi_1(\mathfrak{X}, \bar{t})^{p'}$$

is an isomorphism by Theorem 8.3, the exact sequence (2.2) has a retraction. Hence, the outer action (2.4) is trivial. $\square$

§ 3. Reduction of Hyperbolic Curves

In this section, we recall the precise statement of the good reduction criterion given by Oda and Tamagawa. We also see a p-adic analogue of this criterion [1].

Notation-Definition 3.1. Let S be a scheme and X a scheme over S. We shall say that X is a *hyperbolic curve* over S (or $X \to S$ is a *hyperbolic curve*) if there exists a pair of schemes $(\overline{X}, D)$ over S satisfying the following four conditions:

- The structure morphism $\overline{X} \to S$ is proper smooth of relative dimension 1 with geometrically connected fibers of genus $g_{X/S}$.
- D is a divisor of $\overline{X}$ which is finite étale of rank $r_{X/S}$ over S.
- X is isomorphic to the open subscheme $\overline{X} \setminus D$ of $\overline{X}$ over S.
- $2g_{X/S} + r_{X/S} - 2 > 0$.

Remark. If S is a connected normal scheme, the pair $(\overline{X}, D)$ in the definition of a hyperbolic curve is uniquely determined by $X \to S$. See the argument given in the discussion entitled "Curves" in [15], §0.

Let $K, O_K, k, p, K^{\mathrm{sep}}, G_K$, and I_K be as in Section 2.

Definition 3.2.

1. Let $X \to \operatorname{Spec} K$ be a proper smooth morphism with geometrically connected fibers. We shall say that X *has good reduction* if there exists a proper smooth O_K-scheme $\mathfrak{X}$ such that $\mathfrak{X} \times_{\operatorname{Spec} O_K} \operatorname{Spec} K$ is isomorphic to X over K.
2. Let $X \to \operatorname{Spec} K$ be a hyperbolic curve. We shall say that X *has good reduction* if there exists a hyperbolic curve $\mathfrak{X} \to \operatorname{Spec} O_K$ such that $\mathfrak{X} \times_{\operatorname{Spec} O_K} \operatorname{Spec} K$ is isomorphic to X over K.

Remark.

1. The morphism $\mathfrak{X} \to \operatorname{Spec} O_K$ in Definition 3.2.1 has geometrically connected fibers automatically.
2. For any proper hyperbolic curve X over K, X has good reduction in the sense of Definition 3.2.1 if and only if X has good reduction in the sense of Definition 3.2.2.
3. If X is a hyperbolic curve having good reduction, a hyperbolic curve $\mathfrak{X} \to \operatorname{Spec} O_K$ satisfying the condition in Definition 3.2.2 is unique up to canonical isomorphism (cf. [5] and [10]).
4. Suppose that X is a hyperbolic curve over K. Let L be the henselization (or, strict henselization; completion) of the field K. By using the uniqueness of good model discussed in 3 and descent theory, one can verify that X has good reduction if and only if $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$ has good reduction.

Let $X, X_{K^{\mathrm{sep}}}, *, \Delta_X$, and Π_X be as in Section 2. Suppose that X is a hyperbolic curve over K. Write $(\overline{X}, D)$ for the pair as in Notation-Definition 3.1. We have non-commutative analogues of Theorem 1.1 and Theorem 1.2.

Theorem 3.3 ([22], [23], and [24]). *The following are equivalent:*

1. *X has good reduction.*
2. *The outer action (2.4) is trivial.*
3. *The outer action (2.5) is trivial for any prime number $l \neq p$.*
4. *The outer action (2.5) is trivial for some prime number $l \neq p$.*

Theorem 3.4 ([1]). *Suppose that the assumptions on K, k, and p in Theorem 1.2 hold. Suppose that $X \to \operatorname{Spec} K$ is proper. Moreover, suppose that X has a K-rational point b_K and write $b_{K^{\mathrm{sep}}}$ for the closed point of $X_{K^{\mathrm{sep}}}$ defined by b_K. The following are equivalent:*

1. *X has good reduction.*
2. *$G^{\acute{e}t}(X_{K^{\mathrm{sep}}}, b_{K^{\mathrm{sep}}})$ is crystalline (cf. [1, THEOREM 1.6]).*

In this section, we explain the proof of Theorem 3.3 (and do not explain the proof of Theorem 3.4). Note that, in [22] and [23], outer Galois actions on truncated fundamental groups defined by lower central series were also treated. Let us describe the easy part of the proof of Theorem 3.3. The implications $2 \Rightarrow 3 \Rightarrow 4$ follow from elementary group theory, and the implication $1 \Rightarrow 2$ follows from Proposition 2.4. The nontrivial part of this theorem is the implication $4 \Rightarrow 1$. To show the implication $4 \Rightarrow 1$, we need the assumption that X is a hyperbolic curve. For example, in the case where X is a proper smooth curve of genus 1, Theorem 3.6.2 does not hold in general (and the implication $4 \Rightarrow 1$ does not hold in general (cf. the fourth item of Remark following the proof of Theorem 3.3)). We fix a prime number l satisfying that $l \neq p$ and that the outer action (2.5) is trivial for this l.

Write $J_{\overline{X}}$ for the Jacobian variety of $\overline{X}$ over K, $T_l J_{\overline{X}}$ for the l-adic Tate module of $J_{\overline{X}}$, and $\Delta_{\overline{X}}$ for $\pi_1(\overline{X}, *)$. Theorem 1.1 and the following lemma and theorem give the first step of the proof of Theorem 3.3:

Lemma 3.5. *We have an exact sequence of G_K-modules*

$$0 \to \mathbb{Z}_l(1) \to \mathbb{Z}[D(K^{\mathrm{sep}})] \otimes_{\mathbb{Z}} \mathbb{Z}_l(1) \to (\Delta_X^l)^{\mathrm{ab}} \to (\Delta_{\overline{X}}^l)^{\mathrm{ab}} \to 0.$$

Here, the superscript "ab" denotes the abelianization of the profinite groups.

Proof. One can obtain this exact sequence by taking étale cohomology groups of X. See also [24, Remark (1.3)]. □

Theorem 3.6. *Suppose that $X \to \operatorname{Spec} K$ is proper (i.e., $X = \overline{X}$).*

1. *$(\Delta_X^l)^{\mathrm{ab}} \cong T_l J_X$ as G_K-modules.*
2. *J_X has good reduction if and only if X has stable reduction and the dual graph of the special fiber of the stable model of X is a tree.*

Proof. Assertion 1 follows from the fact that the Tate module of J_X is isomorphic to the abelianization of the profinite group Δ_X as a G_K-module. Next we show assertion 2. Since J_X has stable reduction if and only if X has stable reduction by [5, Theorem (2.4)], we may assume that X has stable reduction. Then assertion 2 follows from [5, Theorem (2.5)] and [3, §9.2 Corollary 12(c)]. □

To prove the implication 4 ⇒ 1 of Theorem 3.3, we may assume that K is complete and k is separably closed by the fourth item of Remark following Definition 3.2. Moreover, we may assume that X has a stable model $\mathfrak{X}$ over $\operatorname{Spec} O_K$ by Theorem 1.1, Lemma 3.5, and Theorem 3.6. (Note that we need additional arguments in the case of $g = 0$ or 1.) It suffices to show that the morphism $\mathfrak{X} \to \operatorname{Spec} O_K$ is smooth. Since a stable model of a hyperbolic curve is stable under base extension, we may assume that k is algebraically closed.

Roughly, the rest of the proof may be regarded as a consequence of a result of [23] (see [2] for a stronger result). Precisely speaking, we can prove it by constructing a stable curve over a regular local scheme of dimension 2 having a closed subscheme which is isomorphic to $\operatorname{Spec} O_K$ (and defines the stable curve $\mathfrak{X} \to \operatorname{Spec} O_K$), and comparing $\mathfrak{X}$ with a transcendental case. Actually, a more direct proof was given in [24], which we briefly outline. Let $\mathfrak{X}'$ be the minimal semi-stable model of X obtained by applying blowing up along a closed subscheme of $\mathfrak{X}$ contained in the special fiber. Note that the dual graph of the special fiber of the scheme $\mathfrak{X}'$ over $\operatorname{Spec} O_K$ is also a tree. Let $\mathfrak{Y} \to \mathfrak{X}'$ be a $\mathbb{Z}/l\mathbb{Z}$-Galois étale covering whose restriction to each irreducible component of $\mathfrak{X}' \times_{\operatorname{Spec} O_K} \operatorname{Spec} k$ which is isomorphic to neither $\mathbb{P}^1_k$ nor $\mathbb{A}^1_k$ is also a $\mathbb{Z}/l\mathbb{Z}$-Galois étale covering. Then one can verify that the dual graph of the scheme $\mathfrak{Y} \times_{\operatorname{Spec} O_K} \operatorname{Spec} k$ is a tree. (Here, we use the assumption that the outer action (2.5) is trivial again.) Therefore, the cardinality of the set of the irreducible components of $\mathfrak{X}' \times_{\operatorname{Spec} O_K} \operatorname{Spec} k$ isomorphic to neither $\mathbb{P}^1_k$ nor $\mathbb{A}^1_k$ is 1. On the other hand, since $\mathfrak{X}'$ is the minimal semi-stable model of X, $\mathfrak{X}' \times_{\operatorname{Spec} O_K} \operatorname{Spec} k$ has no irreducible components which are isomorphic to $\mathbb{P}^1_k$ or $\mathbb{A}^1_k$ and meet the other irreducible components at precisely one point. Since the dual graph of $\mathfrak{X}' \times_{\operatorname{Spec} O_K} \operatorname{Spec} k$ is a tree, it turns out that the scheme $\mathfrak{X}' \times_{\operatorname{Spec} O_K} \operatorname{Spec} k$ is irreducible. Therefore, the morphism $\mathfrak{X}' \to \operatorname{Spec} O_K$ is smooth and X has good reduction.

Remark. We explain relations between reduction of X and its pro-l outer Galois representation for a prime number $l \neq p$ in the case where $2g + r - 2 \leq 0$. Let $(\overline{X}, D)$ be a pair defined in the same way as Notation-Definition 3.1. As in Definition 3.2.2, we say that X has good reduction if there exist a proper smooth scheme $\overline{\mathfrak{X}}$ over O_K and a reduced divisor $\mathfrak{D}$ of $\overline{\mathfrak{X}}$ finite étale over O_K such that the base change of the pair $(\overline{\mathfrak{X}}, \mathfrak{D})$ to $\operatorname{Spec} K$ is isomorphic to $(\overline{X}, D)$ over $\operatorname{Spec} K$.

1. In the case where $(g, r) = (0, 0)$, Δ_X^l is trivial and $X(= \overline{X})$ is a Severi-Brauer variety of dimension 1 over K. Therefore, we cannot determine whether X has good reduction or not from its étale fundamental group. For example, consider the case K is a finite extension field of $\mathbb{Q}_p$. Since k is a finite field, any Severi-Brauer variety over K is trivial. Therefore, X has good reduction if and only if X is isomorphic to $\mathbb{P}^1_K$ over K. On the other hand, there exists a nontrivial Severi-Brauer variety of dimension 1 over K. Here, we give an explicit construction of such a Severi-Brauer variety. Let ϖ be a uniformizer of K. Write L for the unique quadratic unramified extension field of K in K^{sep} and τ for the nontrivial element of the Galois group $\operatorname{Gal}(L/K)$. Then we define a nontrivial action of the Galois group $\operatorname{Gal}(L/K)$ on $\mathbb{P}^1_K$ such that τ induces the nontrivial involution
$$\mathbb{P}^1_K \to \mathbb{P}^1_K : z \mapsto \frac{\varpi}{z}.$$
By taking the quotient scheme of the scheme $\mathbb{P}^1_L = \mathbb{P}^1_K \times_{\operatorname{Spec} K} \operatorname{Spec} L$ by the diagonal action of $\operatorname{Gal}(L/K)$, we obtain a Severi-Brauer variety Z over K which splits over L. This Severi-Brauer variety is nontrivial. Indeed, if a closed point $a \in \mathbb{P}^1_L$ is fixed by this action, a is contained in $\mathbb{A}^1_L$ and satisfies
$$N_{L/K}(a) = \varpi,$$
where $N_{L/K}(a)$ is the norm of a. Since L is quadratic unramifeid over K, the valuation of $N_{L/K}(a)$ cannot coincides with 1. Hence, there are no K-rational points in Z.

2. Also, in the case where $(g, r) = (0, 1)$, Δ_X^l is trivial. Since $\overline{X}$ is isomorphic to $\mathbb{P}^1_K$, X has good reduction.

3. In the case where $(g, r) = (0, 2)$, we can determine whether X has good reduction or not from its étale fundamental group. Indeed, if D is not irreducible, X is isomorphic to $\mathbb{G}_{m,K}$ over K and hence has good reduction. Moreover, since Δ_X^l is isomorphic to $\mathbb{Z}_l(1)$ as G_K-module, I_K acts Δ_X^l trivially. Suppose that D is

irreducible. Write $K(D)$ for the residual field of the unique point of D. Note that $K(D)$ is a quadratic separable extension field of K. The scheme $X \times_{\operatorname{Spec} K} \operatorname{Spec} K(D)$ is isomorphic to $\mathbb{G}_{m,K(D)}$ over $K(D)$. Therefore, X has good reduction if and only if the normalization of O_K in $K(D)$ is unramified over O_K. Consider the composite homomorphism

$$G_K/G_{K(D)} \cong \{\pm 1\} \hookrightarrow \operatorname{Aut}(\mathbb{Z}_l).$$

Write $\mathbb{Z}_l(\chi)$ for the free $\mathbb{Z}_l$-module of rank 1 equipped with continuous G_K-action defined by this homomorphism. Then we have an isomorphism $\Delta^l_X \cong \mathbb{Z}_l(1) \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(\chi)$ of G_K-modules by Lemma 3.5 (note that Lemma 3.5 holds in the case where X is a (not necessarily hyperbolic) smooth curve over K). Therefore, the action of I_K on Δ^l_X is trivial if and only if the normalization of O_K in $K(D)$ is unramified over O_K. Hence, we can determine whether X has good reduction or not from its étale fundamental group.

4. In the case where $(g, r) = (1, 0)$, we cannot determine whether X has good reduction or not from its étale fundamental group. Suppose that K is strictly henselian. Let E be an elliptic curve over K having good reduction. Suppose that X is an E-torsor. Then Δ^l_X is isomorphic to Δ^l_E as a G_K-module. In particular, it follows from Theorem 1.1 that the action of $G_K (= I_K)$ on Δ^l_X is trivial. If X has good reduction, X has K-rational points and hence X is a trivial torsor over K. In the next paragraph, we show that there exists a nontrivial E-torsor. Hence, we cannot determine whether or not X has good reduction from its étale fundamental group.

   Write $\overline{k}$ for the residual field of the valuation ring of K^{sep}, $\mathfrak{E}$ for the Néron model of E, and $\mathfrak{E}_k$ for the scheme $\mathfrak{E} \times_{\operatorname{Spec} O_K} \operatorname{Spec} k$. Then $\overline{k}$ is an algebraically closed field. Moreover, we have the following commutative diagram of Galois cohomology groups:

$$\begin{array}{ccc} H^1(G_K, E(K)) & \longrightarrow & H^1(G_K, E(K^{\mathrm{sep}})) \\ \downarrow & & \downarrow \\ H^1(G_K, \mathfrak{E}_k(k)) & \hookrightarrow & H^1(G_K, \mathfrak{E}_k(\overline{k})). \end{array}$$

   Since the action of G_K on the discrete abelian group $E(K)$ (resp. $\mathfrak{E}_k(k)$; $\mathfrak{E}_k(\overline{k})$) is trivial, we have a canonical isomorphism $H^1(G_K, E(K)) \cong \operatorname{Hom}(G_K, E(K))$ (resp. $H^1(G_K, \mathfrak{E}_k(k)) \cong \operatorname{Hom}(G_K, \mathfrak{E}_k(k))$; $H^1(G_K, \mathfrak{E}_k(\overline{k})) \cong \operatorname{Hom}(G_K, \mathfrak{E}_k(\overline{k}))$). Thus, the second horizontal arrow in the diagram is injective. Let N be an integer greater than 1 and invertible in k. Since E has good reduction, and K is strictly henselian, it follows that the subgroup $E(K)_N$ (resp. $\mathfrak{E}_k(k)_N$) of $E(K)$ (resp. $\mathfrak{E}_k(k)$) consisting of all N-torsion points is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^{\oplus 2}$ as an abelian group. Moreover, the reduction homomorphism $E(K) \to \mathfrak{E}_k(k)$ induces an isomorphism $E(K)_N \to \mathfrak{E}_k(k)_N$. Choose a nontrivial continuous homomorphism $G_K(= I_K) \to E(K)_N$. Then the composite continuous homomorphism

$$G_K \to E(K)_N \hookrightarrow E(K)$$

defines an element c of $H^1(G_K, E(K))$. Since the image of c in $H^1(G_K, \mathfrak{E}_k(k))$ is also nontrivial, the image of c in $H^1(G_K, \mathfrak{E}_k(\overline{k}))$ and hence also the image of c in $H^1(G_K, E(K^{\mathrm{sep}}))$ are nontrivial. Therefore, there exists a nontrivial E-torsor.

We finish this section with another criterion which describes reduction of a proper hyperbolic curve in terms of its (geometrically) pro-p étale fundamental group. From the point of view of anabelian geometry, it is difficult to use Theorem 3.4 (see [7, Remark 3.8.2]). The following result is in line with anabelian philosophy:

Theorem 3.7 ([7]). *Suppose that K is a finite extension field of $\mathbb{Q}_p$ and the morphism $X \to \operatorname{Spec} K$ is proper. There exists a group-theoretic algorithm to determine whether or not X has ordinary good reduction from $\Pi_X^{(p)}$ (see [7] for the precise statement).*

§ 4. Further results for hyperbolic curves

In this section, we introduce some results of [8], [9], and [16], which state that the outer Galois representation (2.5) has a lot of information of stable models of hyperbolic curves.

Let $K, O_K, k, p, K^{\mathrm{sep}}, G_K, I_K, X, (\overline{X}, D), X_{K^{\mathrm{sep}}}, *, \Delta_X$, and Π_X be as in Section 3. Let $l \neq p$ be a prime number and $I_K \to \operatorname{Out}(\Delta_X^l)$ the outer representation (2.5) for this l. Suppose that X has stable reduction (or, equivalently, the action of I_K on the abelianization of Δ_X^l induced by (2.5) is unipotent). Let $(\overline{\mathfrak{X}}, \mathfrak{D})$ be a stable model of $(\overline{X}, D)$ (which is unique up to canonical isomorphism (cf. [5] and [10])).

Suppose that K is henselian. Let K^{tame} (resp. K^{unr}) be the maximal tame (resp. the maximal unramified) extension of K in K^{sep}, $O_{K^{\mathrm{tame}}}$ the integer ring of K^{tame}, k^{sep} the residual field of $O_{K^{\mathrm{tame}}}$. Note that k^{sep} is a separable closure of k. Write $(\overline{\mathfrak{X}}_{k^{\mathrm{sep}}}, \mathfrak{D}_{k^{\mathrm{sep}}})$ for the base change of $(\overline{\mathfrak{X}}, \mathfrak{D})$ to k^{sep}. Let $\mathcal{G}$ be the pro-$\{l\}$ completion of the semi-graph of anabelioids defined by $(\overline{\mathfrak{X}}_{k^{\mathrm{sep}}}, \mathfrak{D}_{k^{\mathrm{sep}}})$ as in [15, Example 2.10]. Note that the underlying semi-graph $\mathbb{G}$ of $\mathcal{G}$ is the dual semi-graph of $(\overline{\mathfrak{X}}_{k^{\mathrm{sep}}}, \mathfrak{D}_{k^{\mathrm{sep}}})$. See [15, Section 1] (resp. [15, Section 2]) for the definition of a semi-graph (resp. the definition of a semi-graph of anabelioids). Each element of $D(K^{\mathrm{sep}})$ defines an inertia subgroup of Δ_X^l (well-defined up to conjugation), which we refer to as a cuspidal subgroup. The main theorem of this section is the following:

Theorem 4.1 (cf. [8, Corollary 4.2], [9, Theorem 1.9 (ii)], and [16, Corollary 2.7 (iii)]). *The data of the outer representation $I_K \to \mathrm{Out}\,(\Delta_X^l)$ and the cuspidal subgroups of Δ_X^l determine the isomorphism class of $\mathcal{G}$, and hence the isomorphism class of the dual semi-graph $\mathbb{G}$. More precisely, the following assertion holds:*
Let X' be another hyperbolic curve over K. Suppose that X' has stable reduction. We define $I_K \to \mathrm{Out}\,(\Delta_{X'}^l)$, $\mathcal{G}'$, and $\mathbb{G}'$ in the same way to define $I_K \to \mathrm{Out}\,(\Delta_X^l)$, $\mathcal{G}$, and $\mathbb{G}$, respectively. Let $\alpha : \Delta_X \cong \Delta'_X$ be an isomorphism of profinite groups such that α preserves the cuspidal subgroups (i.e., the image of every cuspidal subgroup of Δ_X^l is a cuspidal subgroup of $\Delta_{X'}^l$, and any cuspidal subgroup of $\Delta_{X'}^l$ is the image of a cuspidal subgroup of Δ_X^l) and α is compatible with the outer representations $I_K \to \mathrm{Out}\,(\Delta_X^l)$ and $I_K \to \mathrm{Out}\,(\Delta_{X'}^l)$. Then there exists an isomorphism $\mathcal{G} \cong \mathcal{G}'$ inducing α (and an isomorphism $\mathbb{G} \cong \mathbb{G}'$).

The outer representation $I_K \to \mathrm{Out}\,(\Delta_X^l)$ factors through the quotient group $\mathrm{Gal}(K^{\mathrm{tame}}/K^{\mathrm{unr}})$ of I_K, which is isomorphic to the group $I_{K^{\log}}$ as we shall see later. In Theorem 4.2, we rewrite the statement of Theorem 4.1 in terms of the induced outer action of the quotient group. To state Theorem 4.2, we use logarithmic geometry. We consider the log structure of the scheme $\mathrm{Spec}\, O_K$ (resp. $\mathrm{Spec}\, O_{K^{\mathrm{tame}}}$) defined by the direct image of the structure sheaf of $\mathrm{Spec}\, K$ (resp. $\mathrm{Spec}\, K^{\mathrm{tame}}$) and write $\mathrm{Spec}\, O_K^{\log}$ (resp. $\mathrm{Spec}\, O_{K^{\mathrm{tame}}}^{\log}$) for the resulting log scheme. As in [16, Section 0], we consider a natural log structure on $\overline{\mathfrak{X}}$ defined by the fact that $(\overline{\mathfrak{X}}, \mathfrak{D})$ is a stable curve and write $\overline{\mathfrak{X}}^{\log}$ for the resulting log scheme. Write $\overline{\mathfrak{X}}_k$ for the scheme $\overline{\mathfrak{X}} \times_{\mathrm{Spec}\, O_K} \mathrm{Spec}\, k$. Consider the log structure on $\overline{\mathfrak{X}}_k$ (resp. $\mathrm{Spec}\, k$; $\mathrm{Spec}\, k^{\mathrm{sep}}$) defined as the inverse image of the log structure on $\overline{\mathfrak{X}}$ (resp. $\mathrm{Spec}\, O_K$; $\mathrm{Spec}\, O_{K^{\mathrm{tame}}}$) and write $\overline{\mathfrak{X}}_k^{\log}$ (resp. $(\mathrm{Spec}\, k)^{\log}$; $(\mathrm{Spec}\, k^{\mathrm{sep}})^{\log}$) for the resulting log scheme. We have a commutative diagram of profinite groups with exact horizontal lines and three vertical isomorphisms

$$\text{(4.1)} \quad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Gal}(K^{\mathrm{tame}}/K^{\mathrm{unr}}) & \longrightarrow & \mathrm{Gal}(K^{\mathrm{tame}}/K) & \longrightarrow & \mathrm{Gal}(K^{\mathrm{unr}}/K) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & I_{k^{\log}} & \longrightarrow & G_{k^{\log}} & \longrightarrow & \mathrm{Gal}(k^{\mathrm{sep}}/k) & \longrightarrow & 1, \end{array}$$

where $G_{k^{\log}}$ is the automorphism group $\mathrm{Aut}\,((\mathrm{Spec}\, k^{\mathrm{sep}})^{\log}/(\mathrm{Spec}\, k)^{\log})$ of $(\mathrm{Spec}\, k^{\mathrm{sep}})^{\log}$ over $(\mathrm{Spec}\, k)^{\log}$ and $I_{k^{\log}} = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, (k^{\mathrm{sep}})^{\times})$.

As in [16, Example 2.5], we consider the admissible fundamental group $\Pi_{\overline{\mathfrak{X}}^{\log}}$ (resp. $\Delta_{\overline{\mathfrak{X}}^{\log}}$) associated with $\overline{\mathfrak{X}}^{\log}$ (resp. the stable curve $(\overline{\mathfrak{X}}_{k^{\mathrm{sep}}}, \mathfrak{D}_{k^{\mathrm{sep}}})$ over k^{sep}). See [12, §3], [13, §2], and [14, §2 and Appendix] for admissible coverings of stable curves. Then we have a commutative diagram of profinite groups with exact horizontal lines and three vertical surjections

(4.2)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & G_K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_{\overline{\mathfrak{X}}^{\log}} & \longrightarrow & \Pi_{\overline{\mathfrak{X}}^{\log}} & \longrightarrow & G_{k^{\log}} & \longrightarrow & 1. \end{array}$$

Here, the third vertical homomorphism of the diagram (4.2) is the composite of the second vertical homomorphism of the diagram (4.1) and the natural surjective homomorphism $G_K \to \mathrm{Gal}(K^{\mathrm{tame}}/K)$. Then the first vertical homomorphism $\Delta_X \to \Delta_{\overline{\mathfrak{X}}^{\log}}$ induces an isomorphism $\Delta_X^l \cong \Delta_{\overline{\mathfrak{X}}^{\log}}^l$. As explained in [16, Example 2.5], $\Delta_{\overline{\mathfrak{X}}^{\log}}^l$ is isomorphic to the fundamental group $\Pi_{\mathcal{G}}$ of $\mathcal{G}$. In particular, we obtain an outer action

$$G_{k^{\log}} \to \mathrm{Out}(\Delta_X^l)$$

and

(4.3)
$$I_{k^{\log}} \to \mathrm{Out}(\Delta_X^l)$$

Theorem 4.2 (cf. [8, Corollary 4.2], [9, Theorem 1.9 (ii)], and [16, Corollary 2.7 (iii)]). *The data of the outer action (4.3) and the cuspidal subgroups of Δ_X^l determine the isomorphism class of $\mathcal{G}$, and hence the isomorphism class of the dual semi-graph $\mathbb{G}$ (i.e., a statement similar to that of Theorem 4.1 for the outer action (4.3) holds).*

Remark. In this section, we start with a hyperbolic curve X over a discrete valuation field K, and then determine the reduction type of X from the outer Galois representation $I_K \to \mathrm{Out}\,(\Delta_X^l)$. On the other hand, in [8], [9], and [16], the authors start with the fundamental group Π of some sort of semi-graphs of anabelioids and an outer action of a profinite group on Π. This type of study is important in anabelian geometry and brings deeper results. (For example, Theorem 4.2 holds even if the coefficient field of X' is another henselian discrete valuation field K' and we have an isomorphism $\iota : I_K \cong I_{K'}$ of profinite groups which is compatible with α and the outer Galois representations.) In this paper, we just apply the results of [8], [9], and [16], and avoid explanations from a viewpoint of anabelian geometry.

Proof of Theorem 4.1. Theorem 4.1 follows from Theorem 4.2. □

Proof of Theorem 4.2. Here, we outline the proof. To prove that α arises from an isomorphism between $\mathcal{G}$ and $\mathcal{G}'$, it suffices to show that α is graphically filtration preserving (cf. [16, Definition 1.4 (iii)]) by [16, Theorem 1.6 (ii)] (and [17, Comments on [16]]). Since α preserves the cuspidal subgroups, it suffices to show that α is group-theoretically verticial (cf. [16, Definition 1.4 (iv)]) by [8, Proposition 1.13]. By [8, Theorem 4.1], it suffices to show that there exists a verticial subgroup A of $\Pi_{\mathcal{G}}$ (cf. [16, Definition 1.1 (ii)]) such that $\alpha(A)$ is also a verticial subgroup of $\Pi_{\mathcal{G}'}$. This follows from [9, Theorem 1.9 (ii)]. (See also [16, Proposition 2.6] (and [16, Remark 1.4.1]) (resp. [8, Corollary 4.2]), which treats the case where X is proper (resp. not proper).) $\square$

§ 5. Reduction of Hyperbolic Polycurves

In this section, we give a precise definition of a hyperbolic polycurve and we state the main theorems of [20] which improve those of [19].

Definition 5.1. Let S be a scheme, X a scheme over S, and n a positive integer. We shall say that X is a *hyperbolic polycurve of relative dimension* n over S if there exists a sequence of schemes

$$\mathcal{S} : X = X_n \to X_{n-1} \to \ldots \to X_0 = S$$

such that $X_{i+1} \to X_i$ is a hyperbolic curve for each $0 \leq i \leq n-1$. We shall refer to such a sequence as a *sequence of parametrizing morphisms*. We write $g_{\mathcal{S}}$ for the number $\max_{1 \leq i \leq n} g_{X_i/X_{i-1}}$ and $g_{X/S}$ for the number $\min_{\mathcal{S}} g_{\mathcal{S}}$ (where $\mathcal{S}$ ranges over the sequences satisfying the conditions in the definition of a hyperbolic polycurve).

Remark. Note that, for a hyperbolic polycurve $X \to S$, the following three conditions are equivalent:

- The morphism $X \to S$ is proper.
- There exists a sequence of parametrizing morphisms $X = X_n \to \ldots \to X_0 = S$ such that the morphism $X_i \to X_{i-1}$ is proper for each $1 \leq i \leq n$.
- For any sequence of parametrizing morphisms $X = X_n \to \ldots \to X_0 = S$ and any $1 \leq i \leq n$, the morphism $X_i \to X_{i-1}$ is proper.

Hence, if these equivalent conditions are satisfied, we say that the hyperbolic polycurve $X \to S$ is proper.

Let $K, p, O_K, K^{\mathrm{sep}}, G_K$, and I_K be as in Section 2.

Definition 5.2. Let

$$\mathcal{S} : X = X_n \to \ldots \to X_0 = \operatorname{Spec} K$$

be a sequence of parametrizing morphisms of a hyperbolic polycurve over K. We shall say that X *has good reduction with respect to* $\mathcal{S}$ if there exists a sequence of parametrizing morphisms

$$\mathfrak{X} = \mathfrak{X}_n \to \ldots \to \mathfrak{X}_0 = \operatorname{Spec} O_K$$

of a hyperbolic polycurve over O_K such that the sequence

$$\mathfrak{X} \times_{\operatorname{Spec} O_K} \operatorname{Spec} K = \mathfrak{X}_n \times_{\operatorname{Spec} O_K} \operatorname{Spec} K \to \ldots \to \mathfrak{X}_0 \times_{\operatorname{Spec} O_K} \operatorname{Spec} K = \operatorname{Spec} K$$

is isomorphic to $\mathcal{S}$.

Remark. Let X be a proper hyperbolic polycurve over K. If X has good reduction with respect to some sequence of parametrizing morphisms, X has good reduction in the sense of Definition 3.2.1. By using the proofs of Theorem 5.3 and Theorem 5.4, we can show that the converse to this statement holds under some assumptions. In the case where X has good reduction in the sense of Definition 3.2.1 and $p = 0$, X has good reduction with respect to any sequence of parametrizing morphisms of the hyperbolic polycurve $X \to \operatorname{Spec} K$. Furthermore, in the case where X has good reduction in the sense of Definition 3.2.1 and at least one of the following three conditions: $p > 2g_{X/\mathrm{Spec}K} + 1$ and $\dim X = 2$; $p > 2g_{X/\mathrm{Spec}K} + 1$ and X has a K-rational point; $p \gg 0$ (cf. [20, Theorem 1.3] for the precise bound), is satisfied, then X has good reduction with respect to some sequence of parametrizing morphisms of $X \to \operatorname{Spec} K$. At the time of writing, the author does not know whether or not the statement "if X has good reduction in the sense of Definition 3.2.1, then X has good reduction with respect to any (or some) sequence of parameterizing morphisms of $X \to \operatorname{Spec} K$" holds.

Let X, $*$, Δ_X, and Π_X be as in Section 2.

Theorem 5.3. *Suppose that X is a proper hyperbolic polycurve over K of dimension n. Consider the following two conditions:*

(A) *X has good reduction.*

(B) *The outer Galois action $I_K \to \operatorname{Out}(\Delta_X^{p'})$ is trivial.*

Then the following hold:

1. *(A) $\Rightarrow$ (B).*
2. *If $p = 0$, (B) $\Rightarrow$ (A).*
3. *If $p > 2g_{X/\operatorname{Spec} K} + 1$ and $n = 2$, (B) $\Rightarrow$ (A).*
4. *Suppose that $p > 2g_{X/\operatorname{Spec} K} + 1$. Moreover, suppose that X has a K-rational point x and the Galois action $I_K \to \operatorname{Aut}(\Delta_X^{p'})$ defined by x is trivial. Then (A) holds.*

Theorem 5.4. *Let*

$$\mathcal{S} : X = X_n \to \ldots \to X_0 = \operatorname{Spec} K$$

be a sequence of parametrizing morphisms of a hyperbolic polycurve X over K. Consider the following two conditions:

(A') *X has good reduction with respect to $\mathcal{S}$.*

(B) *The outer Galois action $I_K \to \operatorname{Out}(\Delta_X^{p'})$ is trivial.*

Then the following hold:

1. *(A') ⇒ (B).*
2. *If $p = 0$, (B) ⇒ (A').*
3. *If $p > 2g_{X/\operatorname{Spec} K} + 1$ and $\dim X = 2$, (B) ⇒ (A').*
4. *If $p \gg 0$ (cf. [20, Theorem 1.3] for the precise bound), (B) ⇒ (A').*

Remark. For any prime number l_0, there exists a proper hyperbolic polycurve $X \to \operatorname{Spec} \mathbb{C}((t))$ satisfying the following conditions (cf. [20, Example 8.2]):

- X has bad reduction.
- For any prime number $l \neq l_0$, the outer action $I_{\mathbb{C}((t))} \to \operatorname{Out}(\Delta_X^l)$ is trivial.

Therefore, we cannot expect that a naive analogue for proper hyperbolic polycurves of the pro-l good reduction criterion given by Oda and Tamagawa holds.

Remark. There exist a finite extension field L of $\mathbb{Q}_p$ and a sequence of parameterizing morphisms $X_2 \to X_1 \to \operatorname{Spec} L$ of a proper hyperbolic polycurve $X_2 \to \operatorname{Spec} L$ satisfying the condition that $\Delta_{X_2}^p \cong \Delta_{X_1}^p$, where Δ_{X_1} (resp. Δ_{X_2}) for the étale fundamental group of X_1 (resp. X_2) (cf. [20, Example 8.4]). Hence, we cannot expect the existence of pro-p good reduction criterion for hyperbolic polycurves unlike in the case of abelian varieties or hyperbolic curves.

First, we reduce the main theorems to Claim 5.5.

Proof. The implications (A) ⇒ (B) and (A') ⇒ (B) follow from Proposition 2.4. Suppose that condition (B) holds. If $n = 1$, the main theorems follow from Theorem 3.3. Suppose that $n \geq 2$ and the main theorems hold for hyperbolic polycurves of dimension $n - 1$. If we work under the setting of Theorem 5.3, take a sequence of parametrizing morphisms

$$\mathcal{S} : X = X_n \to \ldots \to X_0 = \operatorname{Spec} K$$

such that $g_{X/S} = g_{\mathcal{S}}$. Thus, by applying the main theorems to the hyperbolic polycurve X_{n-1} of dimension $n-1$, we can take a hyperbolic polycurve $\mathfrak{X}_{n-1}$ over O_K such that the scheme $\mathfrak{X}_{n-1} \times_{\operatorname{Spec} O_K} \operatorname{Spec} K$ is isomorphic to X_{n-1} over K. Write Π_n (resp. Π_{n-1}) for the étale fundamental group of X_n (resp. X_{n-1}) and Δ_n (resp. Δ_{n-1}) for the étale fundamental group of $X_n \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}$ (resp. $X_{n-1} \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}$). Note that $\Delta_n = \Delta_X$ and $\Pi_n = \Pi_X$. Let $K(X_{n-1})$ be the function field of X_{n-1} and $K(X_{n-1})^{\mathrm{sep}}$ a separable closure of $K(X_{n-1})$. We may assume that the morphism $* \to X_n \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}$ factors through the morphism

$$X_n \times_{X_{n-1}} \operatorname{Spec} K(X_{n-1})^{\mathrm{sep}} \to X_n \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}.$$

Write G_{n-1} (resp. Δ) for the profinite group $\operatorname{Gal}(K(X_{n-1})^{\mathrm{sep}}/K(X_{n-1}))$ (resp. $\pi_1(X_n \times_{X_{n-1}} \operatorname{Spec} K(X_{n-1})^{\mathrm{sep}}, *)$). We have a diagram of profinite groups with exact horizontal lines

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \pi_1(X_n \times_{X_{n-1}} \operatorname{Spec} K(X_{n-1}), *) & \longrightarrow & G_{n-1} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ & & \Delta & \longrightarrow & \Pi_n & \longrightarrow & \Pi_{n-1} & \longrightarrow & 1. \end{array} \tag{5.1}$$

Here, all the vertical homomorphisms are surjective. Write ξ for the generic point of the special fiber of the morphism $\mathfrak{X}_{n-1} \to \operatorname{Spec} O_K$ and I for an inertia subgroup of ξ in Π_{n-1} (uniquely determined up to conjugation). By using Theorem 8.3 and the diagram (5.1), we obtain natural outer actions

$$I \to \operatorname{Out}(\Delta^{p'}) \quad \text{and} \quad I \to \operatorname{Out}(\Delta^{l}) \quad (l \neq p : \text{prime number}). \tag{5.2}$$

Claim 5.5. There exists a prime number $l \neq p$ such that the outer action $I \to \operatorname{Out}(\Delta^l)$ in (5.2) is trivial.

In the rest of this section, we prove the main theorems under the assumption that this claim holds. It suffices to show that there exists a hyperbolic curve $\mathfrak{X}_n \to \mathfrak{X}_{n-1}$ such that the scheme $\mathfrak{X}_n \times_{\mathfrak{X}_{n-1}} X_{n-1}$ is isomorphic to X_n over X_{n-1}. By [18], it suffices to show that $X_n \times_{X_{n-1}} \operatorname{Spec} K(X_{n-1}) \to \operatorname{Spec} K(X_{n-1})$ has good reduction at the local ring $O_{\mathfrak{X}_{n-1},\xi}$ of ξ. Let $\widetilde{I}$ be an inertia subgroup of $O_{\mathfrak{X}_{n-1},\xi}$ in G_{n-1} and

$$\widetilde{I} \to \operatorname{Out}(\Delta^l) \tag{5.3}$$

the outer Galois action defined by applying the construction of the Galois action (2.5) to the first horizontal exact sequence of the diagram (5.1). By Theorem 3.3, it suffices to show that the outer Galois action (5.3) is trivial. This follows from Claim 5.5. □

§ 6. Approach of [19]

In Section 6 and Section 7, we give overviews of the proofs of Claim 5.5. In this section, we describe the strategy to show Claim 5.5 given in [19].

To prove Claim 5.5, we need to describe the difference between the outer action of the inertia group I_K of the base field K and the outer action of the inertia group I. In [19], we used the actions defined by all the closed points of X_n to compare these two outer actions. Namely, we assume that the following condition:

Assumption 6.1. Let x be a closed point of X_n, $K(x)$ the residual field of x, and $K(x)^{\mathrm{sep}}$ a separable closure of $K(x)$. Take a geometric point $\overline{x}$ of $X \times_{\operatorname{Spec} K} \operatorname{Spec} K(x)^{\mathrm{sep}}$ over x and an inertia subgroup $I_{K(x)}$ of the Galois group $\operatorname{Gal}(K(x)^{\mathrm{sep}}/K(x))$ of a valuation ring of $K(x)$ dominating O_K. Then the Galois action

$$I_{K(x)} \subset \operatorname{Gal}(K(x)^{\mathrm{sep}}/K(x)) \to \operatorname{Aut}(\pi_1(X_n \times_{\operatorname{Spec} K} \operatorname{Spec} K(x)^{\mathrm{sep}}, \overline{x})^{p'})$$

is trivial. Here, this Galois action is defined by the section of the exact sequence (2.1) for the variety $X \times_{\operatorname{Spec} K} \operatorname{Spec} K(x)$ over $K(x)$ determined by the $K(x)$-rational point determined by x.

Moreover, as in the case of [19], we also assume that the morphism $f : X_n \to X_{n-1}$ is proper and has a section σ. Note that we do not need the assumption on all the closed points of X_n. We only need the assumption on all the closed points in $\sigma(X_{n-1})$. To compare the inertia subgroups associated to closed points of X_{n-1} with the inertia subgroup I, we use the following Proposition:

Proposition 6.2 (cf. [19, Lemma 5.1, Corollary 5.2, and Theorem 6.4])**.** *Let $\mathfrak{U}$ be a regular scheme surjective, flat, separated, and of finite type over* $\operatorname{Spec} O_K$*. Write U for the generic fiber of $\mathfrak{U}$. Suppose that $\mathfrak{U}$ is connected (or, equivalently, U is connected (cf. Remark 1 following this proposition)). Write $\pi_1(U)$ for the étale fundamental group of U (cf. Remark 2 following this proposition). Moreover, suppose that the special fiber of $\mathfrak{U}$ is integral and write η for the generic point of the special fiber. We define two closed normal subgroups of $\pi_1(U)$ as follows:*

- *Let I_1 be the normal closed subgroup generated by the inertia subgroups of η in $\pi_1(U)$.*
- *For any closed point x of U, we write $K(x)$ for the residual field of x and $O_{K(x)}$ for the normalization of O_K in $K(x)$. For such a closed point x and any closed point y of* $\operatorname{Spec} O_{K(x)}$*, we write O_y for the local ring of y in* $\operatorname{Spec} O_{K(x)}$*. For such a closed point y, any separable closure $K(x)^{\mathrm{sep}}$ of $K(x)$, and any inertia subgroup I_y of* $\operatorname{Gal}(K(x)^{\mathrm{sep}}/K(x))$ *of O_y, we write $\overline{I}_y$ for the image of I_y via the homomorphism*

$\mathrm{Gal}(K(x)^{\mathrm{sep}}/K(x)) \to \pi_1(U)$ *induced by the morphism* $\mathrm{Spec}\, K(x) \to U$*. We define* I_2 *to be the normal closed subgroup of* $\pi_1(U)$ *generated by the subgroups* $\overline{I}_y$ *defined by all such closed points* x*, closed points* y*, and inertia subgroups* I_y *satisfying that the natural morphisms* $\mathrm{Spec}\, K(x) \to U$ *induce canonical morphisms* $\mathrm{Spec}\, O_y \to \mathfrak{U}$*.*

Then $I_1 = I_2$*.*

Remark.

1. Since $\mathfrak{U}$ in Proposition 6.2 is regular and flat over $\mathrm{Spec}\, O_K$, the following are equivalent: (i) $\mathfrak{U}$ is irreducible. (ii) $\mathfrak{U}$ is connected. (iii) U is irreducible. (iv) U is connected.

2. In Proposition 6.2, we only consider normal subgroups of $\pi_1(U)$. Hence, we do not fix a geometric point of U and consider $\pi_1(U)$ as the étale fundamental group of the scheme U up to inner isomorphism in Proposition 6.2.

3. In [19], the exact same statement as that in Proposition 6.2 is not given.

Proof of Proposition 6.2. Note that $I_1 = \mathrm{Ker}(\pi_1(U) \to \pi_1(\mathfrak{U}))$ by the Zariski-Nagata purity theorem. First we show that $I_2 \subset I_1$. Let x, y, and I_y be as in Proposition 6.2. Since the morphism $\mathrm{Spec}\, K(x) \to U$ induces $\mathrm{Spec}\, O_y \to \mathfrak{U}$, the image of I_y in $\pi_1(\mathfrak{U})$ is trivial. Next we show $I_1 \subset I_2$. By the Zariski-Nagata purity, it suffices to show the following claim:

Claim 6.3 (cf. [19, Lemma 5.1])**.** Let G be a finite group and Z a G-torsor over U which does not extends to a G-torsor over $\mathfrak{U}$. There exist x and y as in the definition of I_2 satisfying the following property:
Take a geometric point $\overline{y}$ over y. Let $\mathfrak{U}_{\overline{y}}$ be the strict henselization of $\mathfrak{U}$ with respect to $\overline{y}$. Then the pull-back of the G-torsor Z to $\mathrm{Spec}\, K(x)$ does not extends to a G-torsor over $\mathrm{Spec}\, O_y$.

This claim follows from [19, Lemma 5.1 and Claim in the proof of Theorem 6.4]. □

There is another remarkable difference between condition (A) or (A') and Claim 5.5. We need to compare the outer action of the inertia group on Δ and that on Δ_n. To overcome this problem, in [19], a homotopy exact sequence of Tannakian fundamental groups was constructed.

Notation-Definition 6.4. Let Y be a connected Noetherian scheme, l a prime number invertible on Y, and m an integer.

1. We shall write $\acute{\mathrm{E}}\mathrm{t}_l(Y)$ for the category of smooth $\mathbb{Q}_l$-sheaves on Y, which is a Tannakian category over $\mathbb{Q}_l$.

2. We define a Tannakian subcategory $\acute{\mathrm{E}}\mathrm{t}_l^{\le m}(Y)$ (resp. $U\acute{\mathrm{E}}\mathrm{t}_l(Y)$) of $\acute{\mathrm{E}}\mathrm{t}_l(Y)$ to be the minimal one which contains all the smooth $\mathbb{Q}_l$-sheaves of rank $\le m$ (resp. the trivial smooth $\mathbb{Q}_l$-sheaf $\mathbb{Q}_l$) and which is closed under taking subquotients, tensor products, duals, and extensions.

3. Let $g : Z \to Y$ be a proper smooth morphism with geometrically connected fibers. We define a Tannakian subcategory $U_g\acute{\mathrm{E}}\mathrm{t}_l^{\le m}(Z)$ of $\acute{\mathrm{E}}\mathrm{t}_l(Z)$ to be the minimal one which contains the essential image of $g^* : \acute{\mathrm{E}}\mathrm{t}_l^{\le m}(Y) \to \acute{\mathrm{E}}\mathrm{t}_l^{\le m}(Z)$ and which is closed under taking subquotients, tensor products, duals, and extensions.

Fix a geometric point $s \to X_{n-1}$. Write X_s for the scheme $X_n \times_{X_{n-1}} s$ and i_s for the projection $X_s \to X_n$. We have functors of Tannakian categories

$$\acute{\mathrm{E}}\mathrm{t}_l^{\le m}(X_{n-1}) \underset{\sigma^*}{\overset{f^*}{\rightleftarrows}} U_f\acute{\mathrm{E}}\mathrm{t}_l^{\le m}(X_n) \overset{i_s^*}{\to} U\acute{\mathrm{E}}\mathrm{t}_l(X_s), \tag{6.1}$$

which induce homomorphisms

$$\pi_1(X_s, s)^{l\text{-unip}} \overset{i_{s*}}{\to} \pi_1(X_n, s)^{l\text{-rel-unip},m} \underset{\sigma_*}{\overset{f_*}{\rightleftarrows}} \pi_1(X_{n-1}, s)^{l\text{-alg},m} \tag{6.2}$$

between their Tannaka duals.

By [4], the homomorphisms of affine group schemes in (6.2) are induced from homomorphisms between the fundamental groups

$$\pi(U_f\acute{\mathrm{E}}\mathrm{t}_l^{\le m}(X_n)) \overset{f_*}{\to} f^*\pi(\acute{\mathrm{E}}\mathrm{t}_l^{\le m}(X_{n-1})), \tag{6.3}$$

$$\sigma^*\pi(U_f\acute{\mathrm{E}}\mathrm{t}_l^{\le m}(X_n)) \overset{\sigma_*}{\leftarrow} \pi(\acute{\mathrm{E}}\mathrm{t}_l^{\le m}(X_{n-1})), \tag{6.4}$$

and

$$\pi(U\acute{\mathrm{E}}\mathrm{t}_l(X_s)) \overset{i_{s*}}{\to} i_s^*\pi(U_f\acute{\mathrm{E}}\mathrm{t}_l^{\le m}(X_n)). \tag{6.5}$$

Write $K_{r,\sigma}$ for the kernel of the homomorphism (6.4). The morphisms of schemes $X_s \overset{i_s}{\to} X_n \overset{f}{\to} X_{n-1}$ induce homomorphisms

$$\pi(U\acute{\mathrm{E}}\mathrm{t}_l(X_s)) \overset{i_{s*}}{\to} i_s^*\pi(U_f\acute{\mathrm{E}}\mathrm{t}_l^{\le m}(X_n)) \overset{i_s^*f_*}{\to} i_s^*f^*\pi(\acute{\mathrm{E}}\mathrm{t}_l^{\le m}(X_{n-1})) \tag{6.6}$$

of affine group schemes over $U\acute{\mathrm{E}}\mathrm{t}_l(X_s)$. By taking fibers at s, we obtain homomorphisms

$$\pi_1(X_s, s)^{l\text{-unip}} \overset{i_{s*}}{\to} \pi_1(X_n, s)^{l\text{-rel-unip},m} \overset{f_*}{\to} \pi_1(X_{n-1}, s)^{l\text{-alg},m}. \tag{6.7}$$

We have a canonical morphism

$$\pi_1(X_s, s)^{l\text{-unip}} = s^*\pi(U\acute{\mathrm{E}}\mathrm{t}_l(X_s)) \to s^*K_{r,\sigma}. \tag{6.8}$$

The following theorem is an l-adic étale analogue of [11, Theorem 1.6]:

Theorem 6.5 ([19, Theorem 4.3]). *Suppose that* $\operatorname{rank} R^1 f_* \mathbb{Q}_l \leq m$. *Then the homomorphism (6.8) is an isomorphism.*

Since the homomorphism $\pi_1(X_n, s)^{l\text{-rel-unip},m} \to \pi_1(X_{n-1}, s)^{l\text{-alg},m}$ is surjective, Theorem 6.5 shows that the sequence

(6.9)
$$1 \to \pi_1(X_s, s)^{l\text{-unip}} \overset{i_{s*}}{\to} \pi_1(X_n, s)^{l\text{-rel-unip},2g_{X/\operatorname{Spec} K}} \overset{f_*}{\to} \pi_1(X_{n-1}, s)^{l\text{-alg},2g_{X/\operatorname{Spec} K}} \to 1$$

is an exact sequence of affine group schemes over $\mathbb{Q}_l$. By using this exact sequence, we can see that there exists an ind-étale sheaf K on X_{n-1} such that the outer action $I \to \operatorname{Out}(\Delta^l)$ of (5.2) is trivial if and only if K extends to $\mathfrak{X}_n$. Then we can apply Proposition 6.2 and prove Claim 5.5.

§ 7. Approach of [20]

In this section, we describe the strategy to show Claim 5.5 given in [20].

As explained in the beginning of Section 6, we need to compare the inertia subgroups to prove Claim 5.5. To send the information of the outer actions of the inertia subgroups back and forth, we use centralizer subgroups. It is convenient to discuss the case $p = 0$ and the case $p > 0$ separately. Indeed, if $\operatorname{char} K = p > 0$, the sequence

$$1 \to \Delta \to \Delta_n \to \Delta_{n-1} \to 1 \tag{7.1}$$

which we can construct by using (5.1) and (2.1) for X_{n-1} and X_n may not be exact in general. Moreover, if $p > 0$, the sequence

$$\Delta^{p'} \to \Delta_n^{p'} \to \Delta_{n-1}^{p'} \to 1 \tag{7.2}$$

induced by the sequence (7.1) may not be exact in general.

For simplicity, we assume that K is strictly henselian (and hence $I_K = G_K$) in this section.

§ 7.1. The Case of Residual Characteristic $p = 0$

Suppose that $p = 0$ in this subsection. By [6, PROPOSITION 2.4], the sequence (7.1) is exact. To prove Claim 5.5, it suffices to show the following claim:

Claim 7.1. The outer action $I \to \operatorname{Out}(\Delta)(= \operatorname{Out}(\Delta^{p'}))$ in (5.2) is trivial.

It is easy to see that Claim 7.1 is equivalent to the following claim:

Claim 7.2. The image of the composite homomorphism

$$Z_{\Pi_n}(\Delta) \subset \Pi_n \to \Pi_{n-1}$$

contains I.

Hence, it suffices to show the following two lemmas:

Lemma 7.3 ([20, Lemma 4.1]). *We have* $\Pi_n = \Delta_n \times Z_{\Pi_n}(\Delta_n)$ *and* $\Pi_{n-1} = \Delta_{n-1} \times Z_{\Pi_{n-1}}(\Delta_{n-1})$.

Lemma 7.4 ([20, Lemma 4.2]). $I \subset Z_{\Pi_{n-1}}(\Delta_{n-1})$. *(In fact,* $I = Z_{\Pi_{n-1}}(\Delta_{n-1})$.*)*

Proof of Lemma 7.3. The assertions follow from the assumptions that $I_K = G_K$ and that the outer Galois action $I_K \to \mathrm{Out}(\Delta)$ is trivial, Lemma 2.1.3, Lemma 2.1.4, and Lemma 2.3. □

Proof of Lemma 7.4. Since the composite homomorphism

$$\Delta_{n-1} \to \Pi_{n-1} \to \pi_1(\mathfrak{X}_{n-1}, *)$$

is an isomorphism by Theorem 8.3, we have

$$I \subset \mathrm{Ker}\,(\Pi_{n-1} \to \pi_1(\mathfrak{X}_{n-1}, *)) \subset Z_{\Pi_{n-1}}(\Delta_{n-1}).$$

□

§ 7.2. The Case of Residual Characteristic $p > 0$

Suppose that $p > 0$ in this subsection. To apply a similar argument to the argument of the case of $p = 0$, we need to see that there exist a prime number $l \neq p$ and a quotient group of the group Δ_n such that the quotient group Δ^l of Δ injects into it. We use the assumptions on p and g to find such an l and a quotient group. Indeed, we have a prime number $l \neq p$ and a commutative diagram with exact horizontal lines

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^l & \longrightarrow & \Delta_n^{(l,p')} & \longrightarrow & \Delta_{n-1}^{p'} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^l & \longrightarrow & \Pi_n^{(l,p')} & \longrightarrow & \Pi_{n-1}^{(p')} & \longrightarrow & 1 \end{array} \tag{7.3}$$

induced by the diagram (5.1). Here, $\Delta_n^{(l,p')}$ is a quotient group of Δ_n which makes the first line exact, and $\Pi_n^{(l,p')}$ (resp. $\Pi_{n-1}^{(p')}$) is a quotient group of Π_n (resp. Π_{n-1}). Note that the existence of the first line is equivalent to the fact that the homomorphism $\Delta_{n-1} \to \mathrm{Out}\,(\Delta^l)$ factors through the canonical homomorphism $\Delta_{n-1} \to \Delta_{n-1}^{p'}$. It is easy to see that Claim 5.5 follows from Claim 7.5.

Claim 7.5 (cf. Claim 7.2). The image of the composite homomorphism

$$Z_{\Pi_n^{(l,p')}}(\Delta_n^{(l,p')}) \hookrightarrow \Pi_n^{(l,p')} \to \Pi_{n-1}^{(p')}$$

contains the image of the composite homomorphism $I \hookrightarrow \Pi_{n-1} \to \Pi_{n-1}^{(p')}$.

Theorem 5.3.4 follows from the diagram (7.3) and the assumption of Theorem 5.3.4. To prove the other assertions, we need an analogue of Lemma 7.3.

Lemma 7.6 (cf. [20, Section 5 and Corollary 6.2]). *If $\Delta_{n-1}^{p'}$ is center-free, $\Delta_n^{(l,p')}$ is also center-free. In this case, we have decompositions of $\Pi_n^{(l,p')}$ and $\Pi_{n-1}^{(p')}$ similar to those of Lemma 7.3.*

Remark. In [20], the exact same statement as that in Lemma 7.6 is not given.

Proof of Lemma 7.6. The assertion follows from the same argument as that in the proof of Lemma 7.3. □

If $n = 2$, $\Delta_{2-1}^{p'}$ is center-free by Lemma 2.3. Hence, Theorem 5.3.3 and Theorem 5.4.3 follow from Lemma 7.6.

Next, we explain a main ingredient of the proof of Theorem 5.4.4 given in [20]. In the proof, we constructed a center-free quotient group of $\Delta_n^{p'}$ in the following way: For any open normal subgroup H of $\Pi_{g,r}$ (cf. Definition 2.2), the group H^l is center-free by Lemma 2.3, and moreover, the group $\Pi_{g,r}/(\mathrm{Ker}\,(H \to H^l))$ is also center-free (cf. [20, Lemma 6.3.2]). If the quotient group $\Pi_{g,r}/H$ is of order prime to p, the group $\Pi_{g,r}/(\mathrm{Ker}\,(H \to H^l))$ is a quotient group of $\Pi_{g,r}^{p'}$. Similarly, one can take an open subgroup Δ'_{n-1} of $\Delta_{n-1}^{p'}$ such that the quotient group $\Delta_{n-1}^{p'}/(\mathrm{Ker}\,(\Delta'_{n-1} \to (\Delta'_{n-1})^l))$ is center-free. (Indeed, in [20], we take such an open normal subgroup of $\Delta_{n-1}^{p'}$ by [20, Lemma 6.3], [20, Proposition 6.4], and the assumption that $p \gg 0$. Then we prove Theorem 5.4.4 by applying [20, Proposition 6.2].)

§ 8. Appendix : A Specialization Theorem for Pro-$\mathbb{L}$ Étale Fundamental Groups

In this section, we prove that a sort of specialization homomorphism of pro-$\mathbb{L}$ étale fundamental groups is an isomorphism. This fact (Theorem 8.3) seems to be known to experts, but the author cannot find it in the literature.

Let K be a discrete valuation field, O_K the valuation ring of K, p (≥ 0) the residual characteristic of O_K, k the residual field of O_K, O_K^{h} a henselization of O_K, O_K^{sh} a strict henselization of O_K^{h}, K^{h} the field of fractions of O_K^{h}, K^{sh} the field of fractions of O_K^{sh}, and

K^{sep} a separable closure of K^{sh}. Let $\mathbb{L} \subset \mathfrak{Primes}$ be a nonempty subset not containing p. For any scheme Z geometrically connected over K and any geometric point $*$ of the scheme $Z \times_{\text{Spec}K} \text{Spec}K^{\text{sep}}$, we write $\pi_1(Z,*)^{(\mathbb{L})}$ for the group

$$\pi_1(Z,*)/\text{Ker}(\pi_1(Z \times_{\text{Spec}\,K} \text{Spec}\,K^{\text{sep}},*) \to \pi_1(Z \times_{\text{Spec}\,K} \text{Spec}\,K^{\text{sep}},*)^{\mathbb{L}}).$$

Lemma 8.1. *Let $K \subset K_1 \subset K_2$ be finite extensions of fields. Suppose that K_2 and K_1 are Galois over K. Write O_{K_1} (resp. O_{K_2}) for the normalization of O_K in K_1 (resp. K_2). Suppose that O_{K_1} is a discrete valuation ring totally ramified over O_K and O_{K_2} is a discrete valuation ring unramified over O_{K_1}. Then the maximal unramified extension K_3 of K in K_2 is Galois over K and $K_2 \cong K_1 \otimes_K K_3$.*

Proof. Since O_{K_2} is a discrete valuation ring, we may assume that K is complete. In this case, one can verify the assertion easily. □

Lemma 8.2.

1. *Let $\overline{\mathfrak{X}} \to \text{Spec}\,O_K$ be a proper smooth morphism with geometrically connected fibers. Let $\mathfrak{D} \subset \overline{\mathfrak{X}}$ be a normal crossing divisor of the scheme $\overline{\mathfrak{X}}$ relative to $\text{Spec}\,O_K$. Write $\mathfrak{X}$ (resp. $\overline{X}$; X; $\mathfrak{X}_k$) for the scheme $\overline{\mathfrak{X}} \setminus \mathfrak{D}$ (resp. $\overline{\mathfrak{X}} \times_{\text{Spec}\,O_K} \text{Spec}\,K$; $\mathfrak{X} \times_{\text{Spec}\,O_K} \text{Spec}\,K$; $\mathfrak{X} \times_{\text{Spec}\,O_K} \text{Spec}\,k$). Since the scheme $\mathfrak{X}_k$ is a dense open subscheme of a connected regular scheme, $\mathfrak{X}_k$ is irreducible. We write ξ for the generic point of $\mathfrak{X}_k$. Take a geometric point $\bar{t}$ of $X \times_{\text{Spec}\,K} \text{Spec}\,K^{\text{sep}}$. Consider a finite Galois étale covering $Y \to X$ corresponding to an open subgroup of $\pi_1(X,\bar{t})^{(\mathbb{L})}$. Suppose that the coefficient field of Y is K. Then the normalization $\text{Spec}\,O(\mathfrak{Y},\xi)$ of the spectrum of the local ring $O_{\mathfrak{X},\xi}$ in Y is the spectrum of a discrete valuation ring.*

2. *Suppose that $O_K = O_K^{\text{h}}$. Write $\mathfrak{X}_0$ (resp. X_0) for the spectrum of the ring O_K (resp. K). Let*

$$\mathfrak{X}_n \to \ldots \to \mathfrak{X}_0$$

be morphisms such that there exist a proper smooth morphism $\overline{\mathfrak{X}}_{i+1} \to \mathfrak{X}_i$ with geometrically connected fibers and a normal crossing divisor $\mathfrak{D}_{i+1} \subset \overline{\mathfrak{X}}_{i+1}$ of the scheme $\overline{\mathfrak{X}}_{i+1}$ relative to $\mathfrak{X}_i$ satisfying that the complement $\overline{\mathfrak{X}}_{i+1} \setminus \mathfrak{D}_{i+1}$ is isomorphic to $\mathfrak{X}_{i+1}$ for each $0 \le i \le n-1$. Write $\overline{X}_i$ (resp. X_i) for the scheme $\overline{\mathfrak{X}}_i \times_{\text{Spec}\,O_K} \text{Spec}\,K$ (resp. $\mathfrak{X}_i \times_{\text{Spec}\,O_K} \text{Spec}\,K$). Since the scheme $\mathfrak{X}_{i,k} = \mathfrak{X}_i \times_{\text{Spec}\,O_K} \text{Spec}\,k$ is a dense open subscheme of a connected regular scheme, $\mathfrak{X}_{i,k}$ is irreducible for each $0 \le i \le n$. We write ξ_i for the generic point of $\mathfrak{X}_{i,k}$ for each $0 \le i \le n$. Consider a finite Galois étale covering $Y_n \to X_n$ corresponding to an open subgroup of $\pi_1(X_n,\bar{t})^{(\mathbb{L})}$. Then the normalization $\text{Spec}\,O(\mathfrak{Y}_n,\xi_n)$ of the spectrum of the local ring $O_{\mathfrak{X}_n,\xi_n}$ in Y_n is the spectrum of a discrete valuation ring.

Proof. To show assertion 1, we may assume that O_K is strictly henselian. Write G for the opposite group of the automorphism group of Y over X. Then Y is a G-torsor over X. Since the coefficient field of Y is K, G is a finite pro-$\mathbb{L}$ group. By [25, Exposé XIII, Corollary 2.9], there exists a finite Galois étale morphism $\mathfrak{Z} \to \mathfrak{X}$ such that the pull-back $\mathfrak{Z} \times_{\operatorname{Spec} O_K} \operatorname{Spec} K^{\mathrm{sep}}$ is isomorphic to $Y \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}$ over $X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}$. Therefore, it suffices to show that the normalization $\operatorname{Spec} O(\mathfrak{Z}, \xi)$ of the spectrum of the local ring $O_{\mathfrak{X},\xi}$ in $\mathfrak{Z}$ is the spectrum of a discrete valuation ring. Let $\overline{t'}$ be a geometric point of $\mathfrak{X}_k$. The induced homomorphism $\pi_1(\mathfrak{X}_k, \overline{t'})^{\mathbb{L}} \to \pi_1(\mathfrak{X}, \overline{t'})^{\mathbb{L}}$ is an isomorphism again by [25, Exposé XIII, 2.10 and Corollaire 2.9]. Therefore, the scheme $\mathfrak{Z} \times_{\operatorname{Spec} O_K} \operatorname{Spec} k$ is irreducible, and hence the scheme $\operatorname{Spec} O(\mathfrak{Z}, \xi)$ is local.

Next, we show assertion 2. Let Y_i (resp. $\mathfrak{Y}_i$) be the normalization of X_i (resp. $\mathfrak{X}_i$) in Y_n. Since the morphism $X_n \to X_i$ is smooth and generically geometrically connected, the scheme $X_n \times_{X_i} Y_i$ is connected and normal. Moreover, since the finite étale morphism $Y_n \to X_n$ factors through the projection $X_n \times_{X_i} Y_i \to X_n$, this projection is also finite étale. Therefore, it holds that the morphism $Y_i \to X_i$ is finite étale for each $0 \leq i \leq n$ since the morphism $X_n \to X_i$ is faithfully flat. We will show assertion 2 by induction on n. Note that the normalization $\operatorname{Spec} O(\mathfrak{Y}_0, \xi_0)$ of the spectrum of the local ring $O_{\mathfrak{X}_0,\xi_0} = \operatorname{Spec} O_K$ in Y_0 is the spectrum of a discrete valuation ring. Therefore, we may assume that $Y_0 = X_0$.

If $n = 1$, the normalization $\operatorname{Spec} O(\mathfrak{Y}_1, \xi_1)$ of the spectrum of the local ring $O_{\mathfrak{X}_1,\xi_1}$ in Y_1 is the spectrum of a discrete valuation ring by assertion 1. Assume that the normalization $\operatorname{Spec} O(\mathfrak{Y}_{n-1}, \xi_{n-1})$ of the spectrum of the local ring $O_{\mathfrak{X}_{n-1},\xi_{n-1}}$ in Y_{n-1} is the spectrum of a discrete valuation ring. Write $K(Y_{n-1})$ for the field of fractions of the scheme Y_{n-1}. By applying assertion 1 to the pair $Y_n \times_{Y_{n-1}} \operatorname{Spec} K(Y_{n-1}) \to X_n \times_{X_{n-1}} \operatorname{Spec} K(Y_{n-1}) \to \operatorname{Spec} K(Y_{n-1})$ and $\mathfrak{X}_n \times_{\mathfrak{X}_{n-1}} \operatorname{Spec} O(\mathfrak{Y}_{n-1}, \xi_{n-1}) \to \operatorname{Spec} O(\mathfrak{Y}_{n-1}, \xi_{n-1})$, we can show that the scheme $\operatorname{Spec} O(\mathfrak{Y}_n, \xi_n)$ is local. □

Theorem 8.3. *Let $\mathfrak{X} \to \operatorname{Spec} O_K$ be a morphism satisfying the following condition: There exists a facotrization*

$$\mathfrak{X} = \mathfrak{X}_n \to \ldots \to \mathfrak{X}_0 = \operatorname{Spec} O_K$$

such that there exist a proper smooth morphism $\overline{\mathfrak{X}}_{i+1} \to \mathfrak{X}_i$ with geometrically connected fibers and a normal crossing divisor $\mathfrak{D}_{i+1} \subset \overline{\mathfrak{X}}_{i+1}$ of the scheme $\overline{\mathfrak{X}}_{i+1}$ relative to $\mathfrak{X}_i$ satisfying that the complement $\overline{\mathfrak{X}}_{i+1} \setminus \mathfrak{D}_{i+1}$ is isomorphic to $\mathfrak{X}_{i+1}$ for each $0 \leq i \leq n-1$. Write X for the scheme $\mathfrak{X} \times_{\operatorname{Spec} O_K} \operatorname{Spec} K$ and take a geometric point $\bar{t}$ of $X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}$ over its generic point. Then the natural homomorphism

$$\alpha : \pi_1(X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}, \bar{t})^{\mathbb{L}} \to \pi_1(\mathfrak{X} \times_{\operatorname{Spec} O_K} \operatorname{Spec} O_K^{\mathrm{sh}}, \bar{t})^{\mathbb{L}}$$

is an isomorphism.

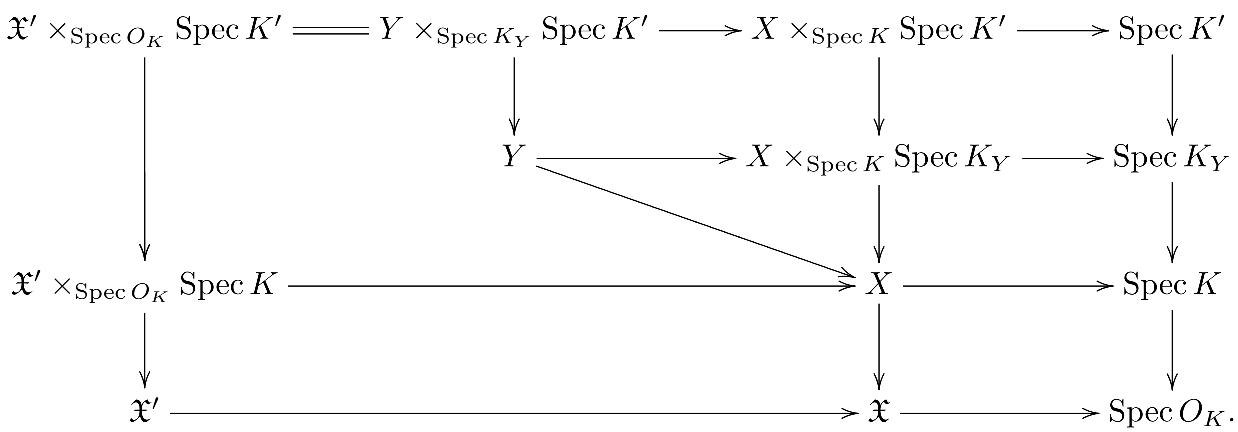
Proof. We may and do assume $K = K^{\mathrm{sh}}$. Since the scheme $\mathfrak{X}$ is smooth over $\operatorname{Spec} O_K$, $\mathfrak{X}$ is normal. Since the étale fundamental group of the scheme $\operatorname{Spec} O_K (= \operatorname{Spec} O_K^{\mathrm{sh}})$ is trivial, for any Galois étale covering space $\mathfrak{Y} \to \mathfrak{X}$, the coefficient field of $\mathfrak{Y} \times_{\operatorname{Spec} O_K} \operatorname{Spec} K$ is equal to K. Hence, the homomorphism

$$\pi_1(X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}, \bar{t}) \to \pi_1(\mathfrak{X} \times_{\operatorname{Spec} O_K} \operatorname{Spec} O_K^{\mathrm{sh}}, \bar{t})$$

is surjective. Therefore, α is also surjective.

We prove that the homomorphism α is injective. It suffices to show that each étale covering space of $X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}$ corresponding to an open subgroup of $\pi_1(X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}, \bar{t})^{\mathbb{L}}$ is isomorphic to the pull-back of an étale covering space of $\mathfrak{X}$ over $X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}$. Since each open subgroup of $\pi_1(X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}, \bar{t})^{\mathbb{L}}$ includes the intersection of $\pi_1(X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}, \bar{t})^{\mathbb{L}}$ and an open normal subgroup of the group $\pi_1(X, \bar{t})^{(\mathbb{L})}$, Theorem 8.3 follows from the next lemma. □

Lemma 8.4. *Let $X, \mathfrak{X}$, and $\bar{t}$ be as in Theorem 8.3. Suppose that $O_K = O_K^{\mathrm{sh}}$. Let Y be a Galois étale covering space of X corresponding to an open subgroup of $\pi_1(X, \bar{t})^{(\mathbb{L})}$. Write $K_Y (\subset K^{\mathrm{sep}})$ for the coefficient field of Y and e for the extension degree of $Y \times_{\operatorname{Spec} K_Y} \operatorname{Spec} K^{\mathrm{sep}} \to X \times_{\operatorname{Spec} K} \operatorname{Spec} K^{\mathrm{sep}}$, which is prime to p. Let K' be the tame extension of K_Y of degree e in K^{sep}. Then there exists a Galois étale covering space $\mathfrak{X}'$ of $\mathfrak{X}$ such that the scheme $Y \times_{\operatorname{Spec} K_Y} \operatorname{Spec} K'$ is isomorphic to the scheme $\mathfrak{X}' \times_{\operatorname{Spec} O_K} \operatorname{Spec} K'$ over $X \times_{\operatorname{Spec} K} \operatorname{Spec} K'$.*



Proof. Note that the field extension $K' \supset K$ is Galois, and hence the étale covering $Y \times_{\operatorname{Spec} K_Y} \operatorname{Spec} K' \to X$ is also Galois. By Abhyankar's lemma and the Zariski-Nagata purity, the normalization $\mathfrak{Y}'$ of $\mathfrak{X} \times_{\operatorname{Spec} O_K} \operatorname{Spec} O_{K'}$ in the field of fractions of $Y \times_{\operatorname{Spec} K_Y} \operatorname{Spec} K'$ is étale over $\mathfrak{X} \times_{\operatorname{Spec} O_K} \operatorname{Spec} O_{K'}$. Let ξ_X (resp. $\xi_{X,K'}$) be the generic point of the special fiber of $\mathfrak{X}$ (resp. $\mathfrak{X} \times_{\operatorname{Spec} O_K} \operatorname{Spec} O_{K'}$). Then the extension of the discrete valuation rings $O_{\mathfrak{X}\times_{\operatorname{Spec} O_K}\operatorname{Spec} O_{K'},\, \xi_{X,K'}} \supset O_{\mathfrak{X},\, \xi_X}$ is totally ramified and the extension of their fields of fractions is Galois. Note that the normalization $\operatorname{Spec} O(\mathfrak{Y}', \xi_X)$ of the scheme $\operatorname{Spec} O_{\mathfrak{X},\xi_X}$ in the field of fractions of $Y \times_{\operatorname{Spec} K_Y} \operatorname{Spec} K'$ is the spectrum of a discrete valuation ring by Lemma 8.2.2. By Lemma 8.1, the field of fractions of the maximal unramified extension of $O_{\mathfrak{X},\xi_X}$ in $O(\mathfrak{Y}', \xi_X)$ is Galois over the function field of $\mathfrak{X}$. We write $\mathfrak{X}'$ for the normalization of $\mathfrak{X}$ in this field. Then the morphism $\mathfrak{X}' \to \mathfrak{X}$ is étale over X and ξ_X. By the Zariski-Nagata purity theorem, the morphism $\mathfrak{X}' \to \mathfrak{X}$ is étale. □

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