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A zig-zag conjecture and local constancy for Galois representations

By

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Abstract

We make a zig-zag conjecture describing the reductions of irreducible crystalline two-dimensional representations of $G_{\mathbb{Q}_p}$ of half-integral slopes and exceptional weights. Such weights are two more than twice the slope mod $(p-1)$. We explain how zig-zag can be deduced from known results for half-integral slopes at most $\frac{3}{2}$. We then explore the connection between zig-zag and local constancy results in the weight. We show that known cases of zig-zag force local constancy to fail for small weights, and explain how local constancy forces zig-zag to fail for some small weights and half-integral slopes at least 2. However, we expect zig-zag to be qualitatively true in general. We end with some compatibility results between zig-zag and other results.

§ 1. Introduction

Let p be an odd prime. This paper is concerned with understanding the reductions of certain crystalline two-dimensional representations of the local Galois group $G_{\mathbb{Q}_p}$. This problem is classical, and important in view of its applications to Galois representations attached to modular forms.

Let V_{k,a_p} be the irreducible two-dimensional crystalline representation of $G_{\mathbb{Q}_p}$ defined over a finite extension E of $\mathbb{Q}_p$, of Hodge-Tate weights $(0, k-1)$ with $k \geq 2$, and positive slope $v(a_p) > 0$, for $a_p \in E$, and v the p-adic valuation of $\bar{\mathbb{Q}}_p$ normalized so that $v(p) = 1$. The (dual of the) representation V_{k,a_p} can be described explicitly in terms of Fontaine's functor D_{cris} which sets up an equivalence of categories between crystalline

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representations of $G_{\mathbb{Q}_p}$ over E and weakly admissible filtered φ-modules over E. We have $D_{\mathrm{cris}}(V_{k,a_p}^*) = D_{k,a_p}$, where $D_{k,a_p} = Ee_1 \oplus Ee_2$ is the filtered φ-module given by

$$\varphi(e_1) = p^{k-1}e_2 \ \text{ and } \ \varphi(e_2) = -e_1 + a_p e_2,$$

and

$$\mathrm{Fil}^i D_{k,a_p} = \begin{cases} D_{k,a_p} & \text{if } i \leq 0, \\ Ee_1 & \text{if } 1 \leq i \leq k-1, \\ 0 & \text{if } i \geq k. \end{cases}$$

Let $\bar{V}_{k,a_p}$ be the *semisimplification* of the reduction of V_{k,a_p} modulo the maximal ideal of the ring of integers of E. It is a two-dimensional semisimple representation of $G_{\mathbb{Q}_p}$ defined over $\bar{\mathbb{F}}_p$, and is independent of the choice of lattice used to define the reduction. If $f = \sum_{n=1}^{\infty} a_n q^n$ is a primitive cusp form of weight $k \geq 2$, level coprime to p, and (for simplicity) trivial nebentypus character, then the local Galois representation $\rho_f|_{G_{\mathbb{Q}_p}}$ attached to f and p is isomorphic to V_{k,a_p}, at least if $a_p^2 \neq 4p^{k-1}$, so the reduction $\bar{\rho}_f|_{G_{\mathbb{Q}_p}}^{ss}$ is isomorphic to $\bar{V}_{k,a_p}$. Thus the structure of the reductions $\bar{V}_{k,a_p}$ has important applications to the study of various objects attached to modular forms such as their motives and Galois representations.

It is an outstanding problem to understand the shape of the reduction $\bar{V}_{k,a_p}$. The reduction $\bar{V}_{k,a_p}$ was computed classically by Fontaine and Edixhoven [Edi92] for all small weights $2 \leq k \leq p+1$. In a remarkable breakthrough using representation theoretic techniques (Langlands correspondences, as developed by Breuil, Berger, Colmez, Paškūnas, Dospinescu [Bre03a], [Bre03b], [Ber10], [Col10], [Pas13], [CDP14], and others), this range of weights was extended by Breuil [Bre03b] to all $k \leq 2p+1$, at least if p is odd. Using different techniques (the theory of (φ, Γ)-modules), Yamashita-Yasuda [YY] have announced a further extension to weights up to $\frac{p^2+1}{2}$.

For simplicity, let us write v for the slope $v(a_p)$. Thus, v denotes both the p-adic valuation and the slope, depending on the context. The reduction $\bar{V}_{k,a_p}$ is also known for all large slopes $v > \lfloor \frac{k-2}{p-1} \rfloor$ by Berger-Li-Zhu [BLZ04]. There has been a spate of recent work computing the reduction $\bar{V}_{k,a_p}$ for small slopes v. Buzzard-Gee [BG09], [BG13] treated the case of slopes v in $(0,1)$. The case of slopes v in $(1,2)$ was treated in [GG15], [BG15], under an assumption when $v = \frac{3}{2}$. The case of slope $v = 1$ was treated in [BGR18].

The first goal of this paper is to fill a gap in the literature computing the reduction for all slopes at most 2 by describing the complete recent treatment of the case of slope $v = \frac{3}{2}$ carried out in [GR19]. A more general second aim of this paper is as follows. Let us say that a weight k is *exceptional* for a particular half-integral (and possibly integral) slope $v \in \frac{1}{2}\mathbb{Z}$ with $0 < v \leq \frac{p-1}{2}$ if

$$k \equiv 2v + 2 \mod (p-1).$$

With hindsight, it has emerged that these weights are the hardest to treat. In this paper, we make a general zig-zag conjecture which describes the reduction $\bar{V}_{k,a_p}$ for all exceptional weights for all half-integral slopes in a qualitative way. We also make some refinements for small half-integral slopes that specify the reduction precisely. Our refined conjecture specializes to known theorems when $v = \frac{1}{2}$ [BG13], $v = 1$ [BGR18], which is not surprising since it was modelled on these results, and is now known to be true for $v = \frac{3}{2}$ in view of our recent work [GR19].

As far as we are aware, the reduction problem for slope 2 is still open, though partial results for small slopes larger than 2 have been announced by Arsovski [Ars18] and Nagel-Pande [NP18]. In related parallel developments on the reduction problem, we remark that certain crystabelian cases of weight 2 and slope at most 1 had been treated earlier by Savitt [Sav05], and that similarly, semistable (non-crystalline) cases of small even weights $k \leq p+1$ had earlier been treated by Breuil-Mézard [BM02], and more recently by Guerberoff-Park [GP19] for the remaining odd weights in this range, using the theory of strongly divisible modules.

§ 1.1. History

Before we state the zig-zag conjecture, and as motivation for it, let us describe in more detail some recent history, which was also described in [GR19, §1.1], showing how the exceptional congruences classes of weights have emerged as the most difficult to treat. To this end, we recall some standard notation. Let ω and ω_2 be the mod p fundamental characters of levels 1 and 2. Let $\mathrm{ind}(\omega_2^c)$ be the (irreducible) mod p representation of $G_{\mathbb{Q}_p}$ obtained by inducing the c-th power of ω_2 from the index 2 subgroup $G_{\mathbb{Q}_{p^2}}$ of $G_{\mathbb{Q}_p}$ to $G_{\mathbb{Q}_p}$ (for $p+1 \nmid c$), normalized so that $\det \mathrm{ind}(\omega_2^c) = \omega^c$. Let μ_λ be the unramified character of $G_{\mathbb{Q}_p}$ mapping a (geometric) Frobenius at p to $\lambda \in \bar{\mathbb{F}}_p^\times$. Let $r := k-2$ and let $b \in \{1, 2, \ldots, p-1\}$ represent the congruence class of r mod $(p-1)$. Then $b = 2v$ is a representative for the exceptional congruence class of weights r mod $(p-1)$. In particular, $b = 1, 2, 3, \ldots,$ represents the exceptional congruence classes of weights r mod $(p-1)$ for the half-integral slopes $\frac{1}{2}, 1, \frac{3}{2}, \ldots,$ respectively.

In [BG09], Buzzard-Gee showed that the reduction $\bar{V}_{k,a_p}$ is always *irreducible* for slopes v in $(0,1)$ (and isomorphic to $\mathrm{ind}(\omega_2^{b+1})$), except possibly in the exceptional case $v = \frac{1}{2}$ and $b = 1$. This case was only treated completely in [BG13], where the authors show that when a certain parameter, which we call τ, is larger than another parameter, which we call t, a reducible possibility (namely $\omega \oplus \omega$ on inertia) occurs instead. More precisely, setting

$$\tau = v\left(\frac{a_p^2 - rp}{pa_p}\right),$$
$$t = v(1-r),$$

it is shown in [BG13, Theorem A] that in the exceptional case $v = \frac{1}{2}$ and $b = 1$, there is a *dichotomy*:

$$\bar{V}_{k,a_p} \sim \begin{cases} \mathrm{ind}(\omega_2^{b+1}), & \text{if } \tau < t \\ \mu_\lambda \cdot \omega^b \oplus \mu_{\lambda^{-1}} \cdot \omega, & \text{if } \tau \geq t, \end{cases}$$

for $r > 1$, where λ is a root of the quadratic equation

$$\lambda + \frac{1}{\lambda} = \overline{\frac{1}{1-r} \cdot \frac{a_p^2 - rp}{pa_p}}.$$

Note that the representation $\mu_\lambda \cdot \omega^b \oplus \mu_{\lambda^{-1}} \cdot \omega$ is independent of the choice of the root λ since $b = 1$.

In [BGR18], the reduction $\bar{V}_{k,a_p}$ was completely determined for $p > 3$ on the boundary $v = 1$ of the Buzzard-Gee annulus $(0, 1)$, and was shown to be generically *reducible* instead. In the difficult exceptional case $v = 1$ and $b = 2$, the authors were able (after some previous iterations of the paper on the `arXiv`) to establish, for $r > 2$, a *trichotomy*:

$$\bar{V}_{k,a_p} \sim \begin{cases} \mathrm{ind}(\omega_2^{b+1}), & \text{if } \tau < t \\ \mu_\lambda \cdot \omega^b \oplus \mu_{\lambda^{-1}} \cdot \omega, & \text{if } \tau = t \\ \mathrm{ind}(\omega_2^{b+p}), & \text{if } \tau > t, \end{cases}$$

where

$$\tau = v\left(\frac{a_p^2 - \binom{r}{2}p^2}{pa_p}\right),$$
$$t = v(2-r),$$

and where λ is a constant given by

$$\lambda = \overline{\frac{2}{2-r} \cdot \frac{a_p^2 - \binom{r}{2}p^2}{pa_p}}.$$

§ 1.2. A conjecture

Based on these results for slopes $\frac{1}{2}$ and 1, and some computations of Rozensztajn [Roz18] for some small half-integral slopes, one might guess that in the general exceptional case $v \in \frac{1}{2}\mathbb{Z}$ and $b = 2v$, there are $b+1$ possibilities for $\bar{V}_{k,a_p}$, with various *irreducible and reducible cases occurring alternately.* More precisely, we make the following qualitative conjecture.

Conjecture 1.1 (Zig-Zag Conjecture)**.** *Say that* $r = k-2 \equiv b = 2v \mod (p-1)$ *is an exceptional congruence class of weights for a particular half-integral slope* $0 < v \in \frac{1}{2}\mathbb{Z} \leq \frac{p-1}{2}$*. Set* $t = v(r-b)$*. Then, for all weights* $r > b$*, except possibly some* r *that are* p*-adically close to some small weights, there is a rational parameter* $\tau = v(c)$*, for some* $c = c(r, a_p)$*, such that as* τ *varies through the rational line, the (semisimplification of the) reduction* $\bar{V}_{k,a_p}$ *of the crystalline representation* V_{k,a_p} *satisfies a* $(b+1)$*-fold-chotomy consisting of alternating irreducible and reducible cases:*

$$\bar{V}_{k,a_p} \sim \begin{cases} \mathrm{ind}(\omega_2^{b+1}), & \text{if } \tau < t \\ \mu_{\lambda_1} \cdot \omega^b \oplus \mu_{\lambda_1^{-1}} \cdot \omega, & \text{if } \tau = t \\ \mathrm{ind}(\omega_2^{b+p}), & \text{if } t < \tau < t+1 \\ \mu_{\lambda_2} \cdot \omega^{b-1} \oplus \mu_{\lambda_2^{-1}} \cdot \omega^2, & \text{if } \tau = t+1 \\ \mathrm{ind}(\omega_2^{b+2p-1}), & \text{if } t+1 < \tau < t+2 \\ \mu_{\lambda_3} \cdot \omega^{b-2} \oplus \mu_{\lambda_3^{-1}} \cdot \omega^3, & \text{if } \tau = t+2 \\ \qquad \vdots & \qquad \vdots \\ \mu_{\lambda_n} \cdot \omega^n \oplus \mu_{\lambda_n^{-1}} \cdot \omega^n, & \text{if } \tau \geq t+(n-1), \quad \text{and } b = 2n-1 \text{ is odd,} \\ \qquad \text{or} & \\ \left.\begin{array}{ll} \mu_{\lambda_n} \cdot \omega^{n+1} \oplus \mu_{\lambda_n^{-1}} \cdot \omega^n, & \text{if } \tau = t+(n-1) \\ \mathrm{ind}(\omega_2^{b+1+n(p-1)}), & \text{if } \tau > t+(n-1), \end{array}\right\} & \text{and } b = 2n \text{ is even,} \end{cases}$$

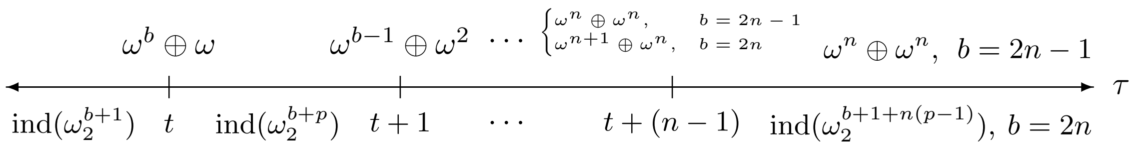
where the λ_i *for all* $1 \leq i \leq n$ *are constants given by*

$$\lambda_i = \overline{*_i \cdot \frac{c}{p^{i-1}}}$$

for some 'fudge factors' $*_i$*, except if* $b = 2n-1$ *is odd and* $i = n$*, in which case* λ_n *satisfies*

$$\lambda_n + \frac{1}{\lambda_n} = \overline{*_n \cdot \frac{c}{p^{n-1}}}.$$

Thus, on the inertia subgroup $I_{\mathbb{Q}_p}$, Conjecture 1.1 predicts the reduction is given by the picture:



The alternating occurrence of irreducible and reducible representations in Conjecture 1.1 is connected to a zig-zag pattern among the irreducible factors of a certain quotient of the symmetric power representation of $\mathrm{GL}_2(\mathbb{F}_p)$ and will be explained in more details in Section 3.5. One might further refine the conjecture by giving a formula for the explicit parameter c. One possibility for c that works for small half-integral slopes is as follows:

$$c = \frac{a_p^2 - \binom{r-v_-}{v_+}\binom{r-v_+}{v_-}p^b}{pa_p}, \tag{1.1}$$

where v_- and v_+ are the largest, respectively smallest, integers not equal to $v \in \frac{1}{2}\mathbb{Z}$ such that v lies in the interval (v_-, v_+). Moreover, we expect that the fudge factor $*_i$ in the conjecture is a simple rational expression in r, though we have not been able to guess a precise formula for it in general. However, we expect that $*_1 = \dfrac{b}{b-r}$, and we will soon give a formula for $*_2$ when $v = \frac{3}{2}$. Finally, the caveat about possibly avoiding some weights r that are p-adically close to some small weights r will be explained in Section 2.4.

§ 1.3. Evidence

As mentioned earlier, the conjecture is true for $v = \frac{1}{2}$ [BG13] and $v = 1$ [BGR18] for all weights $r > b$, including the refinement involving the shape of c above and the constants λ_i. As further evidence towards the refined form of the conjecture, we present [GR19, Theorem 1.1]:

Theorem 1.2. *If $v = \frac{3}{2}$, then the zig-zag conjecture is true. More precisely, if $p \geq 5$, $v = \frac{3}{2}$, $r > b = 3$, and*

$$c = \frac{a_p^2 - (r-2)\binom{r-1}{2}p^3}{pa_p},$$

and we set

$$\tau = v(c),$$
$$t = v(b-r),$$

then the reduction $\bar{V}_{k,a_p}$ enjoys the following tetrachotomy:

$$\bar{V}_{k,a_p} \sim \begin{cases} \mathrm{ind}(\omega_2^{b+1}), & \text{if } \tau < t \\ \mu_{\lambda_1} \cdot \omega^b \oplus \mu_{\lambda_1^{-1}} \cdot \omega, & \text{if } \tau = t \\ \mathrm{ind}(\omega_2^{b+p}), & \text{if } t < \tau < t+1 \\ \mu_{\lambda_2} \cdot \omega^{b-1} \oplus \mu_{\lambda_2^{-1}} \cdot \omega^2, & \text{if } \tau \geq t+1, \end{cases}$$

where the λ_i are constants given by

$$\lambda_1 = \overline{\frac{b}{b-r} \cdot c},$$
$$\lambda_2 + \frac{1}{\lambda_2} = \overline{\frac{b-1}{(b-1-r)(b-r)} \cdot \frac{c}{p}}.$$

Since $\bar{V}_{k,a_p}$ was completely determined in [BG15] for all other slopes v in $(1,2)$ not equal to $\frac{3}{2}$, (even for $p \geq 3$), and in [BGR18] for slope $v = 1$ (for $p \geq 5$), we obtain the following corollary.

Corollary 1.3. *If $p \geq 5$, then the reduction $\bar{V}_{k,a_p}$ is known for all slopes $v = v(a_p) < 2$.*

§ 2. Local Constancy vs Zig-Zag

Let us now explain why we have included the caveat 'except possibly some weights r that are p-adically close to some small weights' in the qualitative statement of the zig-zag conjecture. As Theorem 1.2 above and the preceding historical discussion shows, there is no such caveat for the first few half-integral slopes $\frac{1}{2}$, 1 and $\frac{3}{2}$. More precisely, for these slopes, it suffices to take $r > b$.[1] However, this caveat is required for slopes at least 2.

In order to explain this, we recall the following local constancy result of Berger [Ber12, Theorem B], whose proof uses the families of trianguline representations constructed in [Col08], [Che13].

Theorem 2.1 (Local constancy in the weight). *Suppose $a_p \neq 0$ and $k > 3 \cdot v + \alpha(k-1) + 1$, where $\alpha(n) = \sum_{j \geq 1} \lfloor \frac{n}{p^{j-1}(p-1)} \rfloor$. Then there is*[2] *a positive integer $m = m(k, a_p)$ such that*

$$\bar{V}_{k',a_p} \sim \bar{V}_{k,a_p},$$

for all $k' > k$ with $k' \equiv k \bmod p^{m-1}(p-1)$.

There is an interesting interplay between local constancy and the zig-zag conjecture. In most cases both are compatible. However, sometimes they are not. As we shall see below, in some cases zig-zag holds forcing local constancy to fail. And in others, local

[1] We exclude $r = b$ in order to exclude the degenerate case $t = \infty$ and $\tau = v(c) = \infty$ which happens for $a_p^2 = p^b$ when the explicit parameter c in equation (1.1) vanishes. In fact, we could include $r = b$ if we exclude $a_p = \pm p^v$, since both zig-zag (now τ is finite so less than $t = \infty$) and Fontaine-Edixhoven predict the same answer, namely $\mathrm{ind}(\omega_2^{b+1})$.

[2] The theorem asserts the existence of m. For some bounds on its size for small weights, see Bhattacharya [Bha20].

constancy holds forcing zig-zag to fail. It is these last kinds of cases that we exclude from the statement of the zig-zag conjecture, whence the caveat in the statement of the conjecture. Let us elaborate.

§ 2.1. Fontaine-Edixhoven weights

First consider weights in the Fontaine-Edixhoven range, namely $k \leq p+1$ or $r = b \leq p-1$. It is known that $\bar{V}_{k,a_p} \sim \mathrm{ind}(\omega_2^{b+1})$ for such weights. The only exceptional weight in this range satisfying the hypothesis of Theorem 2.1 is $r = b = 1$ for $v = \frac{1}{2}$, since

$$k = 2v + 2 \not> 3 \cdot v + \alpha(b+1) + 1,$$

unless $v < 1$ and $p \geq 5$. So assume $r = 1$ and $v = \frac{1}{2}$ and $p \geq 5$. By local constancy, if $r' = k' - 2 \equiv 1 \bmod p^{t'}(p-1)$, for t' sufficiently large, we must have $\bar{V}_{k',a_p} \sim \bar{V}_{k,a_p} \sim \mathrm{ind}(\omega_2^2)$. But if t' is larger than $\tau' = v\left(\frac{a_p^2 - r'p}{pa_p}\right)$, then zig-zag (theorem of [BG13]) also predicts $\bar{V}_{k',a_p} \sim \mathrm{ind}(\omega_2^2)$, so there is no apparent incompatibility with local constancy at $r = 1$.

However, as remarked above, the next exceptional weight, $r = 2$ for $v = 1$ does not satisfy the bound in Theorem 2.1, since, for example, for $p \geq 5$, we have

$$4 \not> 3 \cdot 1 + 0 + 1. \tag{2.1}$$

In fact, we can use zig-zag to show that *local constancy does not hold* in this case! Indeed, if it did, then for $r' = k' - 2 \equiv 2 \bmod p^{t'}(p-1)$, with t' sufficiently large, we would have $\bar{V}_{k',a_p} \sim \bar{V}_{k,a_p} \sim \mathrm{ind}(\omega_2^3)$ is irreducible. But if we take $a_p = p \geq 5$ say, then $\tau' = v\left(\frac{a_p^2 - \binom{r'}{2}p^2}{pa_p}\right) = t'$, so by zig-zag (theorem of [BGR18]) we must have $\bar{V}_{k',a_p}$ is reducible ($\sim \omega^2 \oplus \omega$ on inertia), a contradiction.

Similarly, we can use the recent work [GR19] to show that *local constancy fails* when $r = 3$ and say $a_p = p^{\frac{3}{2}}$ for $p \geq 7$ (so $v = \frac{3}{2}$ and the bound on k in Theorem 2.1 is not satisfied). If local constancy were to hold, then for $r' = k' - 2$ as above we would have $\bar{V}_{k',a_p} \sim \bar{V}_{k,a_p} \sim \mathrm{ind}(\omega_2^4)$. But by (1.1) we have, $\tau' = \left(\frac{a_p^2 - \binom{r'-1}{2}(r'-2)p^3}{pa_p}\right) = t' + \frac{1}{2}$, so by zig-zag (Theorem 1.2), it is now known that $\bar{V}_{k',a_p} \sim \mathrm{ind}(\omega_2^{3+p})$, the next irreducible possibility, a contradiction!

Let us record these observations formally now, since they do seem to have been noticed before.

Theorem 2.2. *Local constancy in the weight for the reductions $\bar{V}_{k,a_p}$ may fail for small weights when $a_p \neq 0$, e.g., it fails for $(k, a_p) = (4, p)$ and $(5, p^{\frac{3}{2}})$.*

It has been known for some time (see [Ber12]) that local constancy in the weight fails when $a_p = 0$, in view of the main result of [BLZ04]. Also, the above discussion answers the second question below [Bha20, Theorem 1.1] in the negative, namely one *cannot always* improve the lower bound $3v + \alpha(k-1) + 1$ in Theorem 2.1, since (2.1) shows this lower bound is sharp when $k = 4$ and $v = 1$. However, this bound can be improved for other small weights k and slopes v, see [Bha20, Thm. 1.2].

§ 2.2. Breuil weights

We now consider the next range of small weights, namely $p \leq r = k - 2 = b + (p-1) \leq 2p - 2$ which we refer to as Breuil weights, since the reductions $\bar{V}_{k,a_p}$ at these weights were investigated completely in [Bre03b] for p odd. Every exceptional Breuil weight satisfies the hypothesis of Theorem 2.1, since

$$k = b + p + 1 = 2v + p + 1 > 3v + 1 + 1$$

(or even $3v + 2 + 1$, for $p > 3$). So if $r' = k' - 2 \equiv b + (p-1) \bmod p^{t'}(p-1)$ for t' sufficiently large, we must have $\bar{V}_{k',a_p} \sim \bar{V}_{k,a_p}$. Now Breuil showed (see [Ber11, Theorem 5.2.1, parts 2 and 3]) that, for $k = p+2$, $p+3$ and $p+4$ and slopes $v = \frac{1}{2}$, 1 and $\frac{3}{2}$, respectively, the reduction $\bar{V}_{k,a_p}$ is isomorphic to $\mathrm{ind}(\omega_2^2)$, or $\mu_\lambda \cdot \omega^2 \oplus \mu_{\lambda^{-1}} \cdot \omega$ with $\lambda = \overline{2 \cdot \frac{a_p}{p}}$, or $\mathrm{ind}(\omega_2^{p+3})$, respectively. Now the parameter $\tau' = v(c)$, with $c = c(r', a_p)$ as in (1.1), is easily checked in these cases to be minimal, namely $\tau' = -\frac{1}{2}$, 0 and $\frac{1}{2}$, respectively, whereas the other parameter in zig-zag namely $v(b - r') = 0$ vanishes in all three cases. An easy check using zig-zag (a theorem in all three cases, including slope $\frac{3}{2}$, by Theorem 1.2), shows that $\bar{V}_{k',a_p}$ is exactly isomorphic to one of the above three representations, respectively (even the formula for λ matches well). In other words, local constancy at the Breuil weights is compatible with zig-zag for slopes $v = \frac{1}{2}, 1$ and $\frac{3}{2}$.

However, things begin to go wrong at the next exceptional Breuil weight $k = p+5$, where $b = 4$ and $v = 2$. Breuil proves $\bar{V}_{k,a_p} \sim \mathrm{ind}(\omega_2^{p+4})$ ([Ber11, Theorem 5.2.1, part 2]) and so is irreducible. However, with notation as above, one checks $\tau' = 1$ which is one more than $v(b - r') = 0$, so zig-zag says $\bar{V}_{k',a_p} \sim \omega^3 \oplus \omega^2$ (on inertia), which is reducible! In fact the first 'counterexamples' found by Rozensztajn to the refined form of zig-zag for $v = 2$ and $r > b$ were of this kind - they arise as just explained from Breuil's results and local constancy. Subsequently, the author and Rai found other Breuil weights (for $v > 2$) for which zig-zag doesn't hold, so when $v \geq 2$, we need to exclude such small weights $r = b + p - 1$ (and large weights that are p-adically close to them) from the zig-zag conjecture. This explains the caveat in the statement of zig-zag, at least for Breuil weights.

§ 2.3. Higher small weights

Let us now turn to general small weights $r = b + m(p-1)$, for an integer $m > 0$. When $m \geq 2$, we have $r \geq 2p - 1$ and we call such weights super-Breuil weights. As far as we are aware, the reductions $\bar{V}_{k,a_p}$ have not been determined for super-Breuil weights,[3] except in the smallest case $r = 2p - 1$, where $b = 1$ and $m = 2$ (again by Breuil, though the answer is only stated in [Ber11, Theorem 5.2.1, part 4]). Local constancy (Theorem 2.1) holds for this weight since

$$k = 2p + 1 > 3 \cdot \frac{1}{2} + (4 \text{ or } 2) + 1,$$

for all $p \geq 3$, so for $r' = k' - 2 \equiv 2p - 1 \bmod p^{t'}(p - 1)$, for t' sufficiently large, we obtain

$$\bar{V}_{k',a_p} \sim \bar{V}_{k,a_p} \sim \begin{cases} \mathrm{ind}(\omega_2^2) & \text{if } v(a_p^2 + p) < \frac{3}{2}, \\ \mu_\lambda \cdot \omega \oplus \mu_{\lambda^{-1}} \cdot \omega & \text{if } v(a_p^2 + p) \geq \frac{3}{2}, \end{cases} \tag{2.2}$$

for an explicit λ. On the other hand $v(b - r') = 0$, and by (1.1) we have

$$\tau' = v\left(\frac{a_p^2 - (2p-1)p}{pa_p}\right) \text{ which is } \begin{cases} < 0 & \text{if } v(a_p^2 + p) < \frac{3}{2}, \\ \geq 0 & \text{if } v(a_p^2 + p) \geq \frac{3}{2}, \end{cases}$$

so zig-zag (theorem of [BG13]) predicts exactly the same answers as in (2.2) (with the same λ). Thus local constancy at the first super-Breuil weight $r = 2p - 1$ is compatible with zig-zag for $v = \frac{1}{2}$.

Since, as far as we are aware, there are no general theorems yet giving $\bar{V}_{k,a_p}$ for other super-Breuil weights (for all slopes), it is not possible to directly compare local constancy at these points with zig-zag. However, recently the author and Rai found some further numerical 'counterexamples' to the refined version of zig-zag at super-Breuil points, showing that for slopes $v \geq 2$, such weights (and large weights p-adically close to them) need also to be excluded.

§ 2.4. A bound

The question now arises as to which small weights one needs to exclude from the zig-zag pattern. To this end we recall that the main result of [BLZ04] specifies the reduction $\bar{V}_{k,a_p}$ *for slopes that are large compared to the weight*. This result immediately gives us an explicit bound on which small weights (and large weights that are p-adically close to them) one must exclude from zig-zag. We describe this now.

The main theorem of [BLZ04] says that if $v > \lfloor \frac{k-2}{p-1} \rfloor$, then $\bar{V}_{k,a_p} \sim \mathrm{ind}(\omega_2^{k-1})$. For $r = b + m(p - 1)$, where we take $b \neq p - 1$ for simplicity, this translates to saying that if $v > m$, then $\bar{V}_{k,a_p} \sim \mathrm{ind}(\omega_2^{b+1+m(p-1)})$. Let us compare this result with the refined form of zig-zag which uses the explicit form of c in (1.1). Take $v = m + \frac{1}{2}$, which is the

[3] for all slopes, though the reductions have been determined in [Bha20] for $m = 2$ and 3 and some small slopes.

smallest half-integral slope larger than m. Then $v_- = m$ and $v_+ = m+1$, and p divides the binomial coefficient $\binom{r-(m+1)}{m}$, so $\tau = (2m+1) - (m+\frac{3}{2}) = m - \frac{1}{2}$ and we obtain that $m-1 < \tau < m$. Since $t = 0$ (we are taking m small, so prime to p), zig-zag also predicts $\bar{V}_{k,a_p} \sim \mathrm{ind}(\omega_2^{b+1+m(p-1)})$, so there is no inconsistency between [BLZ04] and zig-zag for $r = b + m(p-1)$ and $v = m + \frac{1}{2}$.

However, if we take $v = m+1$, then $v_- = m$, $v_+ = m+2$, and again p divides $\binom{r-(m+2)}{m}$, so $\tau = (2m+2) - (m+2) = m = t + m$, since $t = 0$, so τ is integral and zig-zag predicts a reducible answer. Thus we see that the refined form of zig-zag is not compatible with the main result of [BLZ04] if $v \geq \lfloor \frac{k-2}{p-1} \rfloor + 1$. For a fixed v, this says that zig-zag may not hold for small weights $r = b + m(p-1)$ with $0 < m \leq v-1$ (or possibly only for $0 < m < v-1$, for $b = p-1$), and by local constancy, for large weights which are p-adically close to these small weights.

Thus the caveat 'except possibly some weights r that are p-adically close to some small weights' in the qualitative statement of zig-zag should be taken to mean that zig-zag should hold for all exceptional weights k, for all $k-2 > b$, except possibly for some k lying in small p-adic disks around Breuil and super-Breuil weights of the form $r = b + m(p-1)$ with $0 < m \leq v-1$. Since we are excluding the case $m = 0$, this is only a caveat for slopes $v \geq 2$!

§ 3. Proof of Theorem 1.2 and beyond

The proof of Theorem 1.2 uses the compatibility between the p-adic and mod p Local Langlands Correspondences first introduced in [Bre03b], with respect to the process of reduction [Ber10]. This compatibility allows one to reduce the reduction problem to a representation theoretic one, namely, to computing the reduction of a $\mathrm{GL}_2(\mathbb{Q}_p)$-stable lattice in a certain unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-Banach space. We recall the key ingredients of the argument here for slope $v = \frac{3}{2}$ (see also [GR19, §2.1-2.3]) and discuss how the details of the argument should generalize to larger half-integral slopes v. In doing this, we will see how the zig-zag conjecture acquired its name.

§ 3.1. Hecke operator

Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$ be the standard maximal compact subgroup of G and $Z = \mathbb{Q}_p^\times$ be the center of G. Let R be a $\mathbb{Z}_p$-algebra and let $V = \mathrm{Sym}^r R^2 \otimes D^s$ be the usual symmetric power representation of KZ twisted by a power of the determinant character D, modeled on homogeneous polynomials of degree r in the variables X and Y over R. We denote compact induction by ind_{KZ}^G. Thus $\mathrm{ind}_{KZ}^G V$ consists of functions $f : G \to V$ such that $f(hg) = h \cdot f(g)$, for all $h \in KZ$ and $g \in G$, and f is compactly supported mod KZ. For $g \in G$, $v \in V$, let $[g, v] \in \mathrm{ind}_{KZ}^G V$ be the function with support in KZg^{-1} given by

$$g' \mapsto \begin{cases} g'g \cdot v, & \text{if } g' \in KZg^{-1} \\ 0, & \text{otherwise.} \end{cases}$$

Any function in $\mathrm{ind}_{KZ}^G V$ is a finite linear combination of functions of the form $[g, v]$, for $g \in G$ and $v \in V$. The Hecke operator T is defined by its action on these elementary functions via

$$T([g, v(X,Y)]) = \sum_{\lambda \in \mathbb{F}_p} \left[g \left(\begin{smallmatrix} p & [\lambda] \\ 0 & 1 \end{smallmatrix} \right), \, v\left(X, -[\lambda]X + pY\right) \right] + \left[g \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right), \, v(pX, Y) \right], \tag{3.1}$$

where $[\lambda]$ denotes the Teichmüller representative of $\lambda \in \mathbb{F}_p$.

§ 3.2. The mod p Local Langlands Correspondence

For $0 \le r \le p-1$, $\lambda \in \bar{\mathbb{F}}_p$ and $\eta : \mathbb{Q}_p^\times \to \bar{\mathbb{F}}_p^\times$ a smooth character, let

$$\pi(r, \lambda, \eta) := \frac{\mathrm{ind}_{KZ}^G \, \mathrm{Sym}^r \bar{\mathbb{F}}_p^2}{T - \lambda} \otimes (\eta \circ \det)$$

be the smooth admissible representation of G, known to be irreducible unless $(r, \lambda) = (0, \pm 1)$ or $(p-1, \pm 1)$, by the classification of irreducible representations of G in characteristic p in [BL94, BL95, Bre03a]. For $(r, \lambda) = (0, \pm 1)$, we have exact sequences

$$0 \to \mathrm{St} \to \pi(0, 1, 1) \to \bar{\mathbb{F}}_p \to 0,$$

$$0 \to \mathrm{St} \otimes \mu_{-1} \to \pi(0, -1, 1) \to \bar{\mathbb{F}}_p(\mu_{-1}) \to 0,$$

where the last surjective maps are induced by 'total sum', respectively 'alternating sum', of the values of a compactly supported function on the tree, and the (irreducible) Steinberg representation St can be taken to be defined by the first sequence. Similar exact sequences exist for $(r, \lambda) = (p-1, \pm 1)$. With this notation, Breuil's semisimple mod p Local Langlands Correspondence (mod p LLC) is given by [Bre03b, Def. 1.1]:

- $\lambda = 0$: $\quad \mathrm{ind}(\omega_2^{r+1}) \otimes \eta \overset{LL}{\longmapsto} \pi(r, 0, \eta)$,
- $\lambda \neq 0$: $\quad \left(\mu_\lambda \cdot \omega^{r+1} \oplus \mu_{\lambda^{-1}}\right) \otimes \eta \overset{LL}{\longmapsto} \pi(r, \lambda, \eta)^{ss} \oplus \pi([p-3-r], \lambda^{-1}, \eta\omega^{r+1})^{ss}$,

where $\{0, 1, \ldots, p-2\} \ni [p-3-r] \equiv p-3-r \mod (p-1)$.

§ 3.3. Standard lattice

Recall that $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let $B(V_{k,a_p})$ be the unitary G-Banach space associated to V_{k,a_p} by the p-adic Local Langlands Correspondence. The reduction $\overline{B(V_{k,a_p})}^{ss}$ of a lattice in this space coincides with the image of $\bar{V}_{k,a_p}$ under the (semisimple) mod p LLC defined above. Since the mod p LLC is by definition injective, it suffices to compute the reduction $\overline{B(V_{k,a_p})}^{ss}$.

Recall $K = \mathrm{GL}_2(\mathbb{Z}_p)$ and $Z = \mathbb{Q}_p^\times$. Let $X = KZ\backslash G$ be the (vertices of the) Bruhat-Tits tree attached to G. The module $\mathrm{Sym}^r \bar{\mathbb{Q}}_p^2$, for $r = k-2$, carries a natural action of KZ, and the projection

$$KZ\backslash(G \times \mathrm{Sym}^{k-2}\bar{\mathbb{Q}}_p^2) \to KZ\backslash G = X$$

defines a local system on X. The space

$$\mathrm{ind}_{KZ}^G \mathrm{Sym}^{k-2}\bar{\mathbb{Q}}_p^2$$

consisting of all sections $f : G \to \mathrm{Sym}^{k-2}\bar{\mathbb{Q}}_p^2$ of this local system which are compactly supported mod KZ is a representation for G, equipped with a G-equivariant Hecke operator T. Let Π_{k,a_p} be the locally algebraic representation of G defined by taking the cokernel of $T - a_p$ acting on the above space of sections.[4] Let Θ_{k,a_p} be the image of the integral sections $\mathrm{ind}_{KZ}^G \mathrm{Sym}^{k-2}\bar{\mathbb{Z}}_p^2$ in Π_{k,a_p}. Then $B(V_{k,a_p})$ is the completion $\hat{\Pi}_{k,a_p}$ of Π_{k,a_p} with respect to the lattice Θ_{k,a_p}. The completion $\hat{\Theta}_{k,a_p}$, and sometimes by abuse of notation Θ_{k,a_p} itself, is called the standard lattice in $B(V_{k,a_p})$. We have $\overline{B(V_{k,a_p})}^{ss} \cong \bar{\hat{\Theta}}_{k,a_p}^{ss} \cong \bar{\Theta}_{k,a_p}^{ss}$. Thus, to compute $\bar{V}_{k,a_p}$, it suffices to compute the reduction $\bar{\Theta}_{k,a_p}^{ss}$ of the standard lattice Θ_{k,a_p}.

§ 3.4. A filtration

We introduce some further notation. Let $V_r = \mathrm{Sym}^r \bar{\mathbb{F}}_p^2$ (this is just V from §3.1 with $R = \bar{\mathbb{F}}_p$ and $s = 0$) denote the $(r+1)$-dimensional $\bar{\mathbb{F}}_p$-vector space of homogeneous polynomials $P(X,Y)$ in two variables X and Y of degree r over $\bar{\mathbb{F}}_p$. The group $\Gamma = \mathrm{GL}_2(\mathbb{F}_p)$ acts on V_r by the formula $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot P(X,Y) = P(aX+cY, bX+dY)$, and KZ acts on V_r via projection to Γ (with $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right) \in Z$ acting trivially). By definition of the lattice Θ_{k,a_p}, there is a surjection $\mathrm{ind}_{KZ}^G \mathrm{Sym}^{k-2}\bar{\mathbb{Z}}_p^2 \twoheadrightarrow \Theta_{k,a_p}$, which induces a surjective map

$$\mathrm{ind}_{KZ}^G V_r \twoheadrightarrow \bar{\Theta}_{k,a_p}. \tag{3.2}$$

Thus, in order to compute $\bar{\Theta}_{k,a_p}$, it suffices to investigate the kernel of the map (3.2). Let $\theta(X,Y) = X^pY - XY^p$. The action of Γ on θ is via the determinant $D : \Gamma \to \mathbb{F}_p^\times$. For each $i \geq 0$, consider the Γ- hence KZ-submodule $V_r^{(i)} = \{P(X,Y) \in V_r : \theta^i | P\}$ of

[4] Π_{k,a_p} is the tensor product of a smooth representation and an algebraic representation of G and is generically irreducible; in a more general setting, D. Prasad [Pra01] has shown that irreducible locally algebraic representations always have this form.

V_r consisting of all polynomials divisible by the i-th power of the θ-polynomial. Clearly $V_r^{(0)} = V_r$ and $V_r^{(1)} = V_r^*$ is the largest singular[5] submodule of V_r and

$$V_r^{(i)} \sim \begin{cases} 0, & \text{if } r < i(p+1) \\ V_{r-i(p+1)} \otimes D^i, & \text{if } r \geq i(p+1), \end{cases}$$

for $i \geq 0$. The submodules $V_r^{(i)}$ are important in computing the kernel of (3.2) because of the following useful fact [BG09, Rem. 4.4]: if the slope $v = v(a_p) < i$ and $r \geq i(p+1)$, then $\mathrm{ind}_{KZ}^G V_r^{(i)}$ lies in the kernel of (3.2), i.e., the surjection (3.2) factors through the map

$$\mathrm{ind}_{KZ}^G \frac{V_r}{V_r^{(i)}} \twoheadrightarrow \bar{\Theta}_{k,a_p}.$$

In the setting of the zig-zag conjecture, the smallest i we can choose is as follows. Recall $b = 2v$ and $b = 2n-1$ is odd or $b = 2n$ is even, with $n \geq 1$. In the former case, $v = n-\frac{1}{2}$, so we can take $i = n$, whereas in the latter case, we have $v = n$, and so we can take $i = n+1$.

In general, the submodules $V_r^{(i)}$ for $i \geq 0$ define a filtration on V_r

$$V_r \supset V_r^{(1)} \supset \cdots \supset V_r^{(i)} \supset V_r^{(i+1)} \supset \cdots,$$

with each subquotient $V_r^{(i)}/V_r^{(i+1)}$ containing two explicit Jordan-Hölder (JH) factors J_{2i} and J_{2i+1} fitting into the exact sequence

$$0 \to J_{2i} \to V_r^{(i)}/V_r^{(i+1)} \to J_{2i+1} \to 0. \tag{3.3}$$

In the setting of the zig-zag conjecture, these JH factors can be described explicitly as follows:

$$J_{2i} = V_{b-2i} \otimes D^i \quad \text{and} \quad J_{2i+1} = V_{p-1-b+2i} \otimes D^{b-i},$$

for $0 \leq i \leq n-1$ if $b = 2n-1$ is odd, and $0 \leq i \leq n$ if $b = 2n$ is even. We remark that when $b = 2n$ is even, the last JH-factor $J_{2n+1} = V_{p-1} \otimes D^n$ is projective and the exact sequence (3.3) for $i = n$ splits, so we can flip the places of J_{2n+1} and J_{2n} in the exact sequence (3.3) above.

[5] Submodules of V_r are called singular if one can extend the action of matrices in Γ to those in the multiplicative semigroup $M = \mathrm{M}_2(\mathbb{F}_p)$, and the singular matrices in $M \setminus \Gamma$ annihilate the submodule.

Thus the associated graded of the relevant part of the filtration above fits into the following picture:

$$\begin{tikzcd}
J_0 \arrow[d] & J_2 \arrow[d] & J_4 \arrow[d] & & & J_{2n-2} \arrow[d] & J_{2n} \text{ or } J_{2n+1} \arrow[d] \\
\dfrac{V_r}{V_r^{(1)}} \arrow[d] & \dfrac{V_r^{(1)}}{V_r^{(2)}} \arrow[d] & \dfrac{V_r^{(2)}}{V_r^{(3)}} \arrow[d] & \cdots & \cdots & \dfrac{V_r^{(n-1)}}{V_r^{(n)}} \arrow[d] & \dfrac{V_r^{(n)}}{V_r^{(n+1)}}, \arrow[d] \\
J_1 \arrow[uur, dashed, leftrightarrow, "d"] & J_3 \arrow[uur, dashed, leftrightarrow, "d"] & J_5 \arrow[uur, dashed, leftrightarrow, "d"] & & \arrow[uur, dashed, leftrightarrow, "d"] & J_{2n-1} \arrow[uur, dashed, leftrightarrow, "d"] \arrow[loop below, dashed, "d"] & J_{2n+1} \text{ or } J_{2n}
\end{tikzcd}$$

A zig-zag pattern among the JH factors

where the picture is taken to end at the column for $V_r^{(n-1)}/V_r^{(n)}$ for $b = 2n-1$ odd, and at the column for $V_r^{(n)}/V_r^{(n+1)}$ for $b = 2n$ even.

§ 3.5. Proof

Recall that the mod p LLC essentially says that irreducible Galois representations $\bar{V}_{k,a_p}$ correspond to supersingular representations of the form $\frac{\mathrm{ind}_{KZ}^{G} J}{T}$, for some irreducible Γ-modules J, whereas reducible $\bar{V}_{k,a_p}$ correspond (generically) to a sum of two principal series representations of the form $\frac{\mathrm{ind}_{KZ}^{G} J}{T-\lambda}$ and $\frac{\mathrm{ind}_{KZ}^{G} J'}{T-\lambda^{-1}}$, for some 'dual' irreducible (possibly equal) Γ-modules J, J', the sum of whose dimensions is $p-1$ mod $(p-1)$, and some $\lambda \in \bar{\mathbb{F}}_p^{\times}$. Thus, in the picture above, exactly one, or possibly exactly two, of the above JH factors contribute to $\bar{\Theta}_{k,a_p}$. Moreover:

- the sum of the dimensions of J_{2i} and J_{2i+1} in each vertical column above is $p+1$, and by the mod p LLC each of J_{2i} and J_{2i+1}, if it occurs as the sole contributing factor to $\bar{\Theta}_{k,a_p}$, gives the same *irreducible* Galois representation for $\bar{V}_{k,a_p}$.

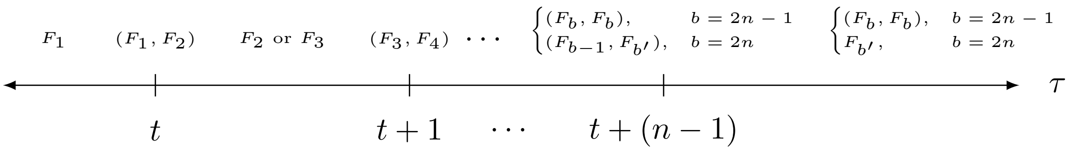
- the sum of the dimensions in each adjacent diagonal pair (J_{2i+1}, J_{2i+2}) is $p-1$ and a potential 'duality' occurs (indicated by a 'd' on the diagonal dotted arrows), since these two JH factors may contribute together to $\bar{\Theta}_{k,a_p}$, giving a *reducible* Galois representation $\bar{V}_{k,a_p}$.

- when $b = 2n-1$ is odd, the last JH factor $J_{2n-1} = V_{p-2} \otimes D^n$ is potentially 'self-dual' (indicated by a 'd' on the looped dotted arrow), since twice its dimension is 0 mod $(p-1)$.

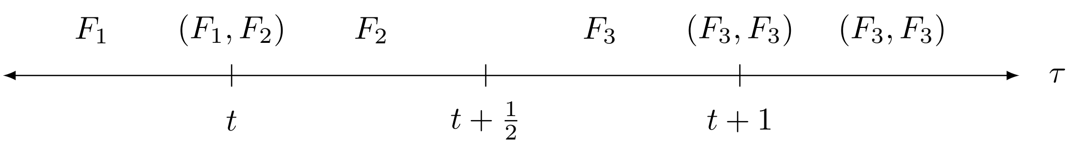
- when $b = 2n$ is even, the JH factor $J_{2n-1} = V_{p-3} \otimes D^{n+1}$ has two options with which to set up a 'duality', namely with $J_{2n} = V_0 \otimes D^n$ or with $J_{2n+1} = V_{p-1} \otimes D^n$, and in practice it does indeed 'break rank' and pair with J_{2n+1} sometimes (which is why we have allowed either JH factor in the last column to be the target of the last dotted diagonal arrow 'd').

We are now ready to make the key observation of this section. We claim that as τ varies through the rational line, the JH factors that contribute to $\bar{\Theta}_{k,a_p}$ actually occur in a *zig-zag fashion* in the picture above, first down starting from J_1 (it is known that J_0 never contributes), then diagonally up and across to J_2 (where both J_1 and J_2 contribute together), then only J_2 contributes, then down again to J_3 where it only contributes, then diagonally across and up to J_4 (where both J_3 and J_4 contribute), then J_4 only contributes, then down again to J_5 where it only contributes, and so on and so forth, as displayed by the zig-zag pattern in the diagram above. Moreover, we claim that the diagonal jumps occur exactly when τ takes on *integer* values between t and $t + (n-1)$ inclusive. This claim is rather remarkable considering that *a priori* there is no reason to expect that there should be any patterns in the way $\bar{\Theta}_{k,a_p}$ 'selects' JH factors in the picture above. It also explains why the *zig-zag conjecture* (Conjecture 1.1) has been christened as such. Finally, it explains why the conjecture predicts that the reduction $\bar{V}_{k,a_p}$ alternates between irreducible and reducible possibilities, with the reducible possibilities occurring exactly at the aforementioned integer points (and for $\tau \geq n-1$, when $b = 2n-1$ is odd).

All of this is best summarized with another picture. Let F_i for $i \geq 0$ be the subquotient of $\bar{\Theta}_{k,a_p}$ occurring as the image of $\mathrm{ind}_{KZ}^{G} J_i$ for $i \geq 0$. Then we expect that all $F_i = 0$ vanish in $\bar{\Theta}_{k,a_p}$, *except* for the following F_i, occurring exactly when τ is in the following regions ($b' = b$ or $b+1$ if $b = 2n$):



In particular, when $v = \frac{3}{2}$ and $b = 3$, zig-zag predicts that the F_i, for $i = 1, 2, 3$, of $\bar{\Theta}_{k,a_p}$ occur in the above order. Indeed, one might even expect the following (slightly more refined) picture holds:



This is proved in [GR19]. More precisely, using delicate computations with the Hecke operator T, the following nine symmetric statements are established in [GR19, Proposition 1.3]:

1. Around t:
   - $\tau > t \implies F_1 = 0$
   - $\tau = t \implies F_1 \twoheadleftarrow \dfrac{\operatorname{ind} J_1}{T - \lambda_1^{-1}}$ and $F_2 \twoheadleftarrow \dfrac{\operatorname{ind} J_2}{T - \lambda_1}$, with $\lambda_1 = \overline{\dfrac{b}{b-r} \cdot c}$
   - $\tau < t \implies F_2 = 0$,
2. Around $t + \frac{1}{2}$:
   - $\tau > t + \frac{1}{2} \implies F_2 = 0$
   - $\tau = t + \frac{1}{2} \implies F_2 \twoheadleftarrow \dfrac{\operatorname{ind} J_2}{T}$ and $F_3 \twoheadleftarrow \dfrac{\operatorname{ind} J_3}{T}$
   - $\tau < t + \frac{1}{2} \implies F_3 = 0$,[6]
3. Around $t + 1$:
   - $\tau > t + 1 \implies F_3 \twoheadleftarrow \dfrac{\operatorname{ind} J_3}{T^2 + 1}$
   - $\tau = t + 1 \implies F_3 \twoheadleftarrow \dfrac{\operatorname{ind} J_3}{T^2 - dT + 1}$, with $d = \overline{\dfrac{b-1}{(b-1-r)(b-r)} \cdot \dfrac{c}{p}}$.
   - $\tau < t + 1 \implies F_3 \twoheadleftarrow \dfrac{\operatorname{ind} J_3}{T}$,

where we have written 'ind' for '$\operatorname{ind}_{KZ}^{G}$' for simplicity. Theorem 1.2 now follows immediately from these nine statements and the mod p LLC, proving zig-zag holds for all $r > b$ when the slope $v = \frac{3}{2}$.

While higher cases of the zig-zag conjecture (Conjecture 1.1) have yet to be attempted (it required a long paper just to write up all the details of the proof of the case of slope $\frac{3}{2}$), we expect that the selection of the JH factors that go into the proof will follow the general zig-zag pattern outlined above.

§ 4. Compatibility

We check the compatibility of the zig-zag conjecture with other conjectures and recent results.

§ 4.1. Irreducibility conjecture

There is a general conjecture, attributed to Buzzard, Breuil and Emerton [BG16, Conjecture 4.1.1], which says that $\bar{V}_{k,a_p}$ is irreducible if k is even and $v = v(a_p)$ is fractional, i.e., non-integral. All computations of $\bar{V}_{k,a_p}$ so far support this conjecture. We

[6] In fact, we only prove this for $\tau \leq t$, but this suffices.

remark that the zig-zag conjecture is also (vacuously!) consistent with this conjecture, since the weight k is even exactly when the residue class $b = 2v$ is even, and this happens only if v is integral.

§ 4.2. Theta operator

The author has suspected for some time (e.g., see the slides of his talk in the Fields symposium in 2016 in honor of Bhargava) that some of the reductions $\bar{V}_{k,a_p}$ in slope $v+1$ should be related to the reductions in slope $v > 0$ by twisting by ω. Some evidence for this would be provided if for a given normalized cuspidal eigenform f of level N coprime to p and slope v, the twisted (global) representation $\bar{\rho}_f \otimes \omega$ is isomorphic to $\bar{\rho}_g$ for some normalized cuspidal eigenform g of level N coprime to p and slope $v' = v + 1$. That this is indeed true (under some assumptions) was recently proved[7] using the θ-operator acting on overconvergent p-adic modular forms and the theory of Hida-Coleman families in [GK19, Corollary 1.2]. Let us compare this result with the refined version of zig-zag. Generically for f, we have $t = 0$. Assume that τ takes its minimal value which by (1.1) is $2v - (v + 1) = v - 1$. So by zig-zag (cf. Conjecture 1.1), we have, for $n \geq 1$,

$$\bar{\rho}_f|_{I_{\mathbb{Q}_p}} \simeq \begin{cases} \mathrm{ind}(\omega_2^{b+1+(n-1)(p-1)}), & \text{if } v = n - \frac{1}{2} \\ \omega^{b-n+1} \oplus \omega^n, & \text{if } v = n, \end{cases} \tag{4.1}$$

depending on whether $v > 0$ is half-integral or integral. It turns out that the weight l of g satisfies $l \equiv k + 2 \mod (p - 1)$, so l is an exceptional weight for g since $l \equiv 2v' + 2 \mod (p - 1)$ if and only if $k \equiv 2v + 2 \mod (p - 1)$. Let $b' = b + 2 \in \{1, 2, \ldots, p - 1\}$ be the residue class of $l \mod (p - 1)$. We apply zig-zag to g. Generically, one of the parameters in zig-zag, namely $v(b' - (l - 2)) = 0$ vanishes. Assume that the other parameter $\tau' = v' - 1 = v$ is minimal again. By Conjecture 1.1, we have

$$\bar{\rho}_g|_{I_{\mathbb{Q}_p}} \simeq \begin{cases} \mathrm{ind}(\omega_2^{b'+1+n(p-1)}), & \text{if } v' = n + \frac{1}{2} \\ \omega^{b'-n} \oplus \omega^{n+1}, & \text{if } v' = n + 1, \end{cases} \tag{4.2}$$

again depending on whether the slope v' of g is half-integral or integral. But as the reader may easily check, the expressions (4.1) and (4.2) for $\bar{\rho}_f|_{I_{\mathbb{Q}_p}}$ and $\bar{\rho}_g|_{I_{\mathbb{Q}_p}}$ are exactly compatible with the fact that the local representation $\bar{\rho}_g|_{I_{\mathbb{Q}_p}}$ is isomorphic to $\bar{\rho}_f|_{I_{\mathbb{Q}_p}}$ twisted by ω. There is a similar compatibility with the zig-zag conjecture if τ and τ' take values other than their minimal values, which we leave to the reader to check.

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[7] That a cusp form g of *some slope* exists follows from the seminal work of Khare-Wintenberger [KW09] and Kisin [Kis09] on Serre's modularity conjecture. However, the forms g of minimal Serre weight arising in Serre's conjecture do not necessarily have slope $v + 1$.

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