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Absolute zeta functions and multiple sine functions: logarithmic derivatives

By

Nobushige KUROKAWA* and Hidekazu TANAKA**

Abstract

In this paper we investigate logarithmic derivatives of absolute zeta functions and multiple sine functions.

§ 1. Introduction

First we recall from [6] that a function $f(x)$ on $\mathbb{R}_{>0}$ is an absolute automorphic form when it satisfies the absolute automorphy

$$f(\frac{1}{x}) = Cx^{-D} f(x)$$

for constants C and D. From an absolute automorphic form $f(x)$ we define the absolute zeta function $\zeta_f(s)$ and the absolute ε-function $\varepsilon_f(s)$ by

$$\zeta_f(s) := \exp\left(\frac{\partial}{\partial w} Z_f(w,s) \bigg|_{w=0} \right)$$

and

$$\varepsilon_f(s) := \frac{\zeta_{f^*}(-s)}{\zeta_f(s)}$$

respectively, where

$$Z_f(w,s) := \frac{1}{\Gamma(w)} \int_1^\infty f(x) x^{-s-1} (\log x)^{w-1} dx$$

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*Department of Mathematics, Tokyo Institute of Technology, Tokyo 152-8551, Japan.
e-mail: `kurokawa@math.titech.ac.jp`
**Department of Mathematics, Shibaura Institute of Technology, Saitama 337-8570, Japan.
e-mail: `htanaka@sic.shibaura-it.ac.jp`

and

$$f^*(x) := f(\frac{1}{x}).$$

We refer to Kurokawa–Tanaka [6] and Kurokawa–Tanaka [7] for a general theory of $\zeta_f(s)$ and $\varepsilon_f(s)$ (see also Kurokawa–Ochiai [4] and Kurokawa–Tanaka [8]). From the viewpoint of absolute zeta functions the theory of multiple sine functions is considered as the ε–function theory: see Kurokawa–Tanaka [9].

In this paper we study logarithmic derivatives

$$\begin{aligned}\gamma_f(s) &:= \frac{d}{ds}\log\zeta_f(s)\\ &= \frac{\zeta_f'(s)}{\zeta_f(s)}\end{aligned}$$

for absolute automorphic forms $f(x)$. The following theorem is our first result.

Theorem 1. *Let $f(x) = \sum_k a(k)x^k \in \mathbb{Z}[x]$, $\zeta_f(s) = \prod_k (s-k)^{-a(k)}$ and $\gamma_f(s) = -\sum_k \frac{a(k)}{s-k}$. Then the following conditions (1), (2) and (3) are equivalent:*

(1)

$$f(\frac{1}{x}) = Cx^{-D}f(x).$$

(2)

$$\zeta_f(D-s)^C = (-1)^{f(1)}\zeta_f(s).$$

Here we remark that $\varepsilon_f(s) = (-1)^{f(1)}$.

(3)

$$C\gamma_f(D-s) = -\gamma_f(s).$$

Hereafter, we look at the zeros and the poles of $\gamma_f(s)$ for various $f(x)$. For a scheme X we define

$$\begin{aligned}\zeta_{X/\mathbb{F}_1}(s) &= \zeta_f(s),\\ \gamma_{X/\mathbb{F}_1}(s) &= \gamma_f(s),\end{aligned}$$

when there exists $f(x)$ satisfying $|X(\mathbb{F}_q)| = f(q)$ for all prime powers q.

Theorem 2. *Let $f(x) = x^n + x^{n-1} + \cdots + 1$. Then we have*

$$\gamma_f(s) = \gamma_{\mathbb{P}^n/\mathbb{F}_1}(s) = -(\frac{1}{s} + \frac{1}{s-1} + \cdots + \frac{1}{s-n})$$

and

$$\begin{cases}\text{zeros of } \gamma_f(s)\text{: } n\text{-distinct zeros each in } (0,1), (1,2), \cdots, (n-1,n) \text{ (all real)},\\ \text{poles of } \gamma_f(s)\text{: } \qquad 0,1,...,n \text{ (all real)}.\end{cases}$$

Next result shows the existence of imaginary zeros for a simple $\gamma_f(s)$, which is a particular case of $\gamma_{\Phi_{2l}}(s)$ for the $2l$-th cyclotomic polynomial with a prime $l \equiv 3 \bmod 4$.

Theorem 3. *Let $f(x) = \Phi_6(x) = x^2 - x + 1$, where $\Phi_6(x)$ is the 6-th cyclotomic polynomial. Then*

$$\gamma_f(s) = -\frac{(s-1)^2+1}{s(s-1)(s-2)}$$

and

$$\begin{cases} \text{zeros of } \gamma_f(s)\text{: } 1 \pm \sqrt{-1} \text{ (imaginary)}, \\ \text{poles of } \gamma_f(s)\text{: } \quad 0, 1, 2. \end{cases}$$

Theorem 4. *Let $f(x) = (x^a - 1)(x^b + 1)$, where a and b are integers with $0 < a < b$. Then*

$$\gamma_f(s) = -\frac{2a\left((s - \frac{a+b}{2})^2 + \frac{b^2-a^2}{4}\right)}{s(s-a)(s-b)(s-(a+b))}$$

and

$$\begin{cases} \text{zeros of } \gamma_f(s)\text{: } \frac{a+b\pm\sqrt{a^2-b^2}}{2} \text{ (imaginary)}, \\ \text{poles of } \gamma_f(s) : \quad 0, a, b, a+b. \end{cases}$$

To explain our next result, let $f(x) = (x-1)^n$. We remark that the absolute zeta function $\zeta_{\mathbb{G}_m^n/\mathbb{F}_1}(s)$ of $\mathbb{G}_m^n = \mathrm{GL}(1)^n$ $(n \geq 1)$ is defined as

$$\begin{aligned} \zeta_{\mathbb{G}_m^n/\mathbb{F}_1}(s) &:= \zeta_f(s) \\ &= \lim_{p \to 1} \zeta_{\mathbb{G}_m^n/\mathbb{F}_p}(s), \end{aligned}$$

where the last zeta function is the congruence zeta function; see Soulé [11], Connes–Consani [2] and Kurokawa–Taguchi–Tanaka [5]. We recall that $\zeta_{\mathbb{G}_m^n/\mathbb{F}_p}(s)$ is defined by

$$\begin{aligned} \zeta_{\mathbb{G}_m^n/\mathbb{F}_p}(s) &:= \exp\left(\sum_{m=1}^{\infty} \frac{|\mathbb{G}_m^n(\mathbb{F}_{p^m})|}{m} p^{-ms}\right) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{f(p^m)}{m} p^{-ms}\right). \end{aligned}$$

This construction gives

$$\begin{aligned} \zeta_{\mathbb{G}_m^n/\mathbb{F}_1}(s) &= \zeta_f(s) \\ &= \prod_{k=0}^{n} (s-k)^{(-1)^{n+1-k}\binom{n}{k}}. \end{aligned}$$

Theorem 5. *Let $f(x) = (x-1)^n$. Then we have*

$$\gamma_f(s) = \gamma_{\mathbb{G}_m^n/\mathbb{F}_1}(s) = \frac{\zeta'_{\mathbb{G}_m^n/\mathbb{F}_1}(s)}{\zeta_{\mathbb{G}_m^n/\mathbb{F}_1}(s)}.$$

Then

$$\gamma_f(s) = -\frac{n!}{s(s-1)\cdots(s-n)}.$$

and

$$\begin{cases} \text{zeros of } \gamma_f(s)\text{: None,} \\ \text{poles of } \gamma_f(s)\text{: } 0,1,\ldots,n. \end{cases}$$

Here we remark that

$$\zeta_{\mathbb{G}_m^n/\mathbb{F}_1}(s) = -n!\zeta_{\mathbb{P}^n/\mathbb{F}_1}(s).$$

Our third result concerns the regularized multiple sine function constructed in Shintani [10] ($r = 2$) and Kurokawa [3] (general r):

$$S_r(s,(\omega_1,...,\omega_r)) := \Gamma_r(s,(\omega_1,...,\omega_r))^{-1}\Gamma_r(\omega_1+\cdots+\omega_r-s,(\omega_1,...,\omega_r))^{(-1)^r},$$

where $\Gamma_r(s,(\omega_1,...,\omega_r))$ is the regularized version of the multiple gamma function introduced by Barnes [1]. Here we explain the construction when $\mathrm{Re}(\omega_1)$, ..., $\mathrm{Re}(\omega_r) > 0$ and we put $\boldsymbol{\omega} = (\omega_1,...,\omega_r)$ for simplicity. The multiple Hurwitz zeta function $\zeta_r(w,s,\boldsymbol{\omega})$ is defined as

$$\zeta_r(w,s,\boldsymbol{\omega}) = \sum_{n_1,\ldots,n_r \geq 0} (s+n_1\omega_1+\cdots+n_r\omega_r)^{-w}$$

for $\mathrm{Re}(w) > r$. It has an analytic continuation to all $w \in \mathbb{C}$ and it is holomorphic at $w = 0$. Then we obtain the regularized multiple gamma function

$$\Gamma_r(s,\boldsymbol{\omega}) = \exp\left(\frac{\partial}{\partial w}\zeta_r(w,s,\boldsymbol{\omega})\bigg|_{w=0}\right).$$

It is a meromorphic function in $s \in \mathbb{C}$.

A defect of this construction is the difficulty to treat the general case $\boldsymbol{\omega} \in (\mathbb{C}-\{0\})^r$. For example

$$\zeta_2(w,s,(1,-1)) = \sum_{n_1,n_2 \geq 0} (s+n_1-n_2)^{-w}$$

is meaningless, so we do not have $\Gamma_2(s,(1,-1))$ nor $S_2(s,(1,-1))$ in this way.

Now, we construct the absolute multiple sine function from the absolute automorphic form

$$f_{\boldsymbol{\omega}}(x) := \prod_{k=1}^{r}(1-x^{-\omega_k})^{-1}$$

satisfying

$$f_{\boldsymbol{\omega}}(\frac{1}{x}) = (-1)^r x^{-|\boldsymbol{\omega}|} f_{\boldsymbol{\omega}}(x),$$

where $\boldsymbol{\omega} := (\omega_1, ..., \omega_r)$, $|\boldsymbol{\omega}| = \omega_1 + \cdots + \omega_r$ and $x > 0$. For $\boldsymbol{\omega} = (\omega_1, ..., \omega_r) \in (\mathbb{C} - \sqrt{-1}\mathbb{R})^r$ we define the absolute multiple gamma function

$$\lneqq_r(s, \boldsymbol{\omega}) := \zeta_{f_{\boldsymbol{\omega}}}(s)$$

and the absolute multiple sine function

$$\mathbb{S}_r(s, \boldsymbol{\omega}) := \varepsilon_{f_{\boldsymbol{\omega}}}(s).$$

Then we have the following results (see Kurokawa–Tanaka [9]).

Fact 1. *For* $\boldsymbol{\omega} \in (\mathbb{C} - \sqrt{-1}\mathbb{R})^r$, $\lneqq_r(s, \boldsymbol{\omega})$ *and* $\mathbb{S}_r(s, \boldsymbol{\omega})$ *are meromorphic functions in* $s \in \mathbb{C}$.

Fact 2. *When* $\mathrm{Re}(\omega_k) > 0$ $(k = 1, ..., r)$, *we have*

$$\lneqq_r(s, \boldsymbol{\omega}) = \Gamma_r(s, \boldsymbol{\omega})$$

and

$$\mathbb{S}_r(s, \boldsymbol{\omega}) = S_r(s, \boldsymbol{\omega}).$$

Fact 3. *Let*

$$\mathbb{S}_{2n}(s, \pm) = \mathbb{S}_{2n}(s, (\overbrace{1, ..., 1}^{n}, \overbrace{-1, ..., -1}^{n})).$$

Then

$$\mathbb{S}_{2n}(s, \pm) = \exp\left(\frac{(-1)^{n-1}}{(2n-1)!} \int_0^s \prod_{k=1}^{n-1} (u^2 - k^2) \pi u \cot(\pi u) du \right).$$

Fact 4. *Let*

$$\mathbb{S}_{2n+1}(s, \pm) = \frac{\mathbb{S}_{2n+1}(s, (\overbrace{1, ..., 1}^{n}, \overbrace{-1, ..., -1}^{n+1}))}{\mathbb{S}_{2n+1}(s, (\underbrace{1, ..., 1}_{n+1}, \underbrace{-1, ..., -1}_{n}))}.$$

Then

$$\mathbb{S}_{2n+1}(s, \pm) = C_n \exp\left(\frac{(-1)^{n-1} 2}{(2n)!} \int_0^s \prod_{k=0}^{n-1} (u^2 - k^2) \pi \cot(\pi u) du \right),$$

where

$$C_n = \frac{(-1)^n 4}{(2n)!} \sum_{m=1}^{n} c(n, m) \zeta'(-2m)$$

and $c(m,n)$ is defined by

$$\prod_{k=0}^{n-1}(X-k^2)=\sum_{m=1}^{n}c(n,m)X^m\in\mathbb{Z}[X]:$$

$c(n,n)=1$ *and* $c(n,1)=(-1)^{n-1}((n-1)!)^2$.

Theorem 6. *For $n\geq 1$, let*

$$f_n(x)=\begin{cases}\frac{1}{(1-x)^m(1-x^{-1})^m} & if\ n=2m,\\ \frac{1}{(1-x)^{m+1}(1-x^{-1})^m}-\frac{1}{(1-x)^m(1-x^{-1})^{m+1}} & if\ n=2m+1\end{cases}$$

and

$$\mathbb{C}\mathrm{ot}_n(s,\pm)=\frac{\mathbb{S}'_n(s,\pm)}{\mathbb{S}_n(s,\pm)}.$$

Then we have

$$\begin{aligned}\gamma_{f_n}(s)+(-1)^n\gamma_{f_n}(-s)&=-\mathbb{C}\mathrm{ot}_n(s,\pm)\\ &=\begin{cases}\frac{(-1)^m}{(2m-1)!}\prod_{k=1}^{m-1}(s^2-k^2)\pi s\cot(\pi s) & if\ n=2m,\\ \frac{(-1)^m 2}{(2m)!}\prod_{k=0}^{m-1}(s^2-k^2)\pi\cot(\pi s) & if\ n=2m+1\end{cases}\end{aligned}$$

and

$$\begin{cases}\textit{zeros of } \gamma_{f_n}(s)+(-1)^n\gamma_{f_n}(-s): & \mathbb{Z}+\frac{1}{2} & \textit{if } n=2m,\\ \textit{zeros of } \gamma_{f_n}(s)+(-1)^n\gamma_{f_n}(-s): & \mathbb{Z}+\frac{1}{2} & \textit{if } n=1,\\ \textit{zeros of } \gamma_{f_n}(s)+(-1)^n\gamma_{f_n}(-s): & \left(\mathbb{Z}+\frac{1}{2}\right)\cup\{0\} & \textit{if } n=2m+1\ (m\geq 1),\\ \textit{poles of } \gamma_{f_n}(s)+(-1)^n\gamma_{f_n}(-s): & \mathbb{Z}-\{0,\pm 1,\dots,\pm(m-1)\} & \textit{if } n=2m,\\ \textit{poles of } \gamma_{f_n}(s)+(-1)^n\gamma_{f_n}(-s): & \mathbb{Z} & \textit{if } n=1,\\ \textit{poles of } \gamma_{f_n}(s)+(-1)^n\gamma_{f_n}(-s): & \mathbb{Z}-\{0,\pm 1,\dots,\pm(m-1)\} & \textit{if } n=2m+1.\end{cases}$$

§ 2. Proof of Theorem 1

Consider the following condition (0) on the coefficients $a(k)$ of the polynomial $f(x)$:
(0)

$$a(D-k)=Ca(k)\ \ (k\in\mathbb{Z}).$$

Then

$$
\begin{aligned}
(1) &\Leftrightarrow \sum_k a(k)x^{-k} = Cx^{-D}\sum_k a(k)x^k.\\
&\Leftrightarrow \sum_k a(D-k)x^{k-D} = \sum_k Ca(k)x^{k-D}.\\
&\Leftrightarrow a(D-k) = Ca(k) \ \ (k\in\mathbb{Z}).\\
&\Leftrightarrow (0).\\
(2) &\Leftrightarrow \prod_k (D-s-k)^{-Ca(k)} = (-1)^{f(1)}\prod_k (s-k)^{-a(k)}.\\
&\Leftrightarrow \prod_k \Big((D-k)-s\Big)^{-Ca(k)} = \prod_k (k-s)^{-a(k)}.\\
&\Leftrightarrow \prod_k \Big((D-k)-s\Big)^{-Ca(k)} = \prod_k \Big((D-k)-s\Big)^{-a(D-k)}.\\
&\Leftrightarrow Ca(k) = a(D-k) \ \ (k\in\mathbb{Z}).\\
&\Leftrightarrow (0).\\
(3) &\Leftrightarrow -\sum_k \frac{Ca(k)}{D-s-k} = \sum_k \frac{a(k)}{s-k}.\\
&\Leftrightarrow -\sum_k \frac{Ca(k)}{(D-k)-s} = -\sum_k \frac{a(D-k)}{(D-k)-s}.\\
&\Leftrightarrow Ca(k) = a(D-k) \ \ (k\in\mathbb{Z}).\\
&\Leftrightarrow (0).
\end{aligned}
$$

Q.E.D.

§ 3. Proof of Theorem 2

From $f(x) = x^n + x^{n-1} + \cdots + 1$ we have

$$\zeta_f(s) = \frac{1}{s(s-1)\cdots(s-n)}.$$

This gives

$$
\begin{aligned}
\gamma_f(s) &= -\Big(\frac{1}{s} + \frac{1}{s-1} + \cdots + \frac{1}{s-n}\Big)\\
&= -\frac{P_n(s)}{s(s-1)\cdots(s-n)}
\end{aligned}
$$

for a polynomial $P_n(s) = (n+1)s^n + \cdots + (-1)^n n!$. Note that

$$\gamma_f'(s) = \frac{1}{s^2} + \frac{1}{(s-1)^2} + \cdots + \frac{1}{(s-n)^2} > 0$$

for $s \in (k-1,k)$, where $k=1,...,n$. Hence we see that $\gamma_f(s)$ takes values from $-\infty$ to $+\infty$ on each $(k-1,k)$. Thus we have distinct n real zeros. The complex zeros are exhausted by these zeros. Q.E.D.

§ 4. Proof of Theorem 3

Since $f(x)=x^2-x+1$, we have

$$\begin{aligned}\zeta_f(s) &= \frac{s-1}{s(s-2)},\\ \gamma_f(s) &= -\frac{1}{s-2}+\frac{1}{s-1}-\frac{1}{s}\\ &= -\frac{(s-1)^2+1}{s(s-1)(s-2)}.\end{aligned}$$

Thus, we obtain Theorem 3. Q.E.D.

§ 5. Proof of Theorem 4

From $f(x)=x^{a+b}-x^b+x^a-1$ we have

$$\zeta_f(s)=\frac{s(s-b)}{(s-a)(s-(a+b))}$$

and

$$\begin{aligned}\gamma_f(s) &= -\frac{1}{s-(a+b)}+\frac{1}{s-b}-\frac{1}{s-a}+\frac{1}{s}\\ &= -\frac{2a\left((s-\frac{a+b}{2})^2+\frac{b^2-a^2}{4}\right)}{s(s-a)(s-b)(s-(a+b))}.\end{aligned}$$

This gives Theorem 4. Q.E.D.

§ 6. Proof of Theorem 5

We have

$$\begin{aligned}\zeta_{\mathbb{G}_m^n/\mathbb{F}_1}(s) &= \zeta_f(s)\\ &= \prod_{k=0}^{n}(s-k)^{(-1)^{n+1-k}\binom{n}{k}}.\end{aligned}$$

We have

$$\begin{aligned}\gamma_{\mathbb{G}_m^n/\mathbb{F}_1}(s) &= \left(\log \zeta_{\mathbb{G}_m^n/\mathbb{F}_1}(s)\right)' \\ &= \left(\sum_{k=0}^{n}(-1)^{n+1-k}\binom{n}{k}\log(s-k)\right)' \\ &= \sum_{k=0}^{n}\frac{(-1)^{n+1-k}\binom{n}{k}}{s-k}.\end{aligned}$$

Hence, it remains to show that

$$\sum_{k=0}^{n}\frac{(-1)^{n+1-k}\binom{n}{k}}{s-k} = -\frac{n!}{s(s-1)\cdots(s-n)}.$$

Let

$$\frac{1}{s(s-1)\cdots(s-n)} = \sum_{k=0}^{n}\frac{a(k)}{s-k}.$$

Since

$$\begin{aligned}a(k) &= \lim_{s\to k}\frac{1}{s(s-1)\cdots(s-k+1)(s-k-1)\cdots(s-n)} \\ &= \frac{1}{k(k-1)\cdots 1\cdot(-1)\cdots(-(n-k))} \\ &= \frac{(-1)^{n-k}}{k!(n-k)!} \\ &= \frac{(-1)^{n-k}\binom{n}{k}}{n!},\end{aligned}$$

we obtain

$$\sum_{k=0}^{n}\frac{(-1)^{n+1-k}\binom{n}{k}}{s-k} = -\frac{n!}{s(s-1)\cdots(s-n)}.$$

This gives

$$\gamma_{\mathbb{G}_m^n/\mathbb{F}_1}(s) = -\frac{n!}{s(s-1)\cdots(s-n)}.$$

Lastly we notice that our calculation implies the following results:

$$\begin{aligned}\zeta'_{\mathbb{G}_m^n/\mathbb{F}_1}(s) &= -n!\prod_{k=0}^{n}(s-k)^{(-1)^{n+1-k}\binom{n}{k}-1} \\ &= -n!\frac{\zeta_{\mathbb{G}_m^n/\mathbb{F}_1}(s)}{s(s-1)\cdots(s-n)}.\end{aligned}$$

Q.E.D.

§ 7. Proof of Theorem 6

As noted in [9, §6] we have

$$\mathbb{S}_{2m+1}(s,\pm) = \zeta_{f_n}(s)^{-1}\zeta_{f_n}(-s)^{(-1)^{n-1}}.$$

Hence we obtain

$$\gamma_{f_n}(s) + (-1)^n\gamma_{f_n}(-s) = -\mathbb{C}\mathrm{ot}_n(s,\pm)$$

by the logarithmic differentiation. From Fact 3 and Fact 4, we have

$$\log \mathbb{S}_{2m}(s,\pm) = \frac{(-1)^{m-1}}{(2m-1)!}\int_0^s \prod_{k=1}^{m-1}(u^2-k^2)\pi u\cot(\pi u)du$$

and

$$\log \mathbb{S}_{2m+1}(s,\pm) = \log C_m + \frac{(-1)^{m-1}2}{(2m)!}\int_0^s \prod_{k=0}^{m-1}(u^2-k^2)\pi\cot(\pi u)du$$

respectively. Hence the logarithmic differentiation gives Theorem 6. Q.E.D.

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