RIMS Kôkyûroku Bessatsu
B86 (2021), 279–286

Lehto–Virtanen-type and big Picard-type theorems for Berkovich analytic spaces

By

Yûsuke Okuyama *

Abstract

In non-archimedean setting, we establish a Lehto–Virtanen-type theorem for a morphism from the punctured Berkovich closed unit disk $\overline{\mathsf{D}} \setminus \{0\}$ in the Berkovich affine line to the Berkovich projective line P^1 having an isolated essential singularity at the origin, and then establish a big Picard-type theorem for such an open subset Ω in the Berkovich projective space P^N of any dimension N that the family of all morphisms from $\overline{\mathsf{D}} \setminus \{0\}$ to Ω is normal in a non-archimedean Montel's sense. As an application of the latter theorem, we see a big Brody-type hyperbolicity of the Berkovich harmonic Fatou set of an endomorphism of P^N of degree > 1.

§ 1. Introduction

Let K be a field of any characteristic that is complete with respect to a *non-archimedean* absolute value $|\cdot|$. Let $\overline{\mathsf{D}}$ be the Berkovich closed unit disk in the Berkovich affine line $\mathsf{A}^1 = \mathsf{A}^1(K) = \mathbb{A}^1(K)^{\mathrm{an}}$. We note that

$$\overline{\mathsf{D}} \cap K = \mathcal{O}_K = \{z \in K : |z| \leq 1\},$$

and that $\overline{\mathsf{D}} = \mathsf{P}^1 \setminus U(\overrightarrow{\mathcal{S}_{\mathrm{can}}\infty})$ in the notation in §2.1. We say a morphism f from $\overline{\mathsf{D}} \setminus \{0\}$ to a Berkovich K-analytic space X *has an isolated essential singularity at the origin* if f does not extend to a morphism from $\overline{\mathsf{D}}$ to any Berkovich K-analytic space. One of our aims is to see the following non-archimedean analog of Lehto–Virtanen and Lehto [10, 9] (see also [12, 13]).

Received March 26, 2019. Revised October 3, 2019.
2020 Mathematics Subject Classification(s): 2010 Mathematics Subject Classification(s): Primary 32P05; Secondary 37P40, 37P50
Key Words: *Berkovich analytic space, isolated essential singularity, Lehto–Virtanen-type theorem, big Picard-type theorem, big Brody-type hyperbolicity, Berkovich harmonic Fatou set*
Partially supported by JSPS Grant-in-Aid for Scientific Research (C), 15K04924
*Division of Mathematics, Kyoto Institute of Technology, Sakyo-ku, Kyoto 606-8585, Japan.
e-mail: `okuyama@kit.ac.jp`

Theorem 1 (a Lehto–Virtanen-type theorem). *Let K be a field of any characteristic that is complete with respect to a non-archimedean absolute value $|\cdot|$. Then for every morphism f from $\overline{\mathsf{D}}\setminus\{0\}$ to the Berkovich projective line $\mathsf{P}^1=\mathsf{P}^1(K)$ having an isolated essential singularity at the origin, we have*

$$\limsup_{r\searrow 0}\operatorname{diam}_\#\bigl(f(\{z\in K:|z|=r\})\bigr)=\operatorname{diam}_\#(\mathbb{P}^1). \tag{1.1}$$

Here, $\operatorname{diam}_\#$ *is the chordal diameter function on $(2^{\mathbb{P}^1})\setminus\{\emptyset\}$ with respect to an equipped chordal metric on $\mathbb{P}^1=\mathbb{P}^1(K)$.*

In the proof of Theorem 1, we will normalize the equipped chordal metric on $\mathbb{P}^1$ as $\operatorname{diam}_\#(\mathbb{P}^1)=1$. The proof of Theorem 1 is an improvement of some argument in the proof of Rodríguez Vázquez [14, Proposition 7.17], which was a little Picard-type theorem. An argument similar to that in the proof of Theorem 1 also yields the following big version of [14, Proposition 7.17]. For the definition of the non-archimedean Montel-type normality appearing in Theorem 2, see [14, §1, §7] and [4, Introduction].

Theorem 2 (a big Picard-type theorem). *Let K be a field of any characteristic that is complete with respect to a non-archimedean absolute value, and let Ω be an open subset in the Berkovich projective space $\mathsf{P}^N=\mathsf{P}^N(K)$ of any dimension N such that the family* $\operatorname{Mor}(\overline{\mathsf{D}}\setminus\{0\},\Omega)$ *of all morphisms from $\overline{\mathsf{D}}\setminus\{0\}$ to Ω is normal. Then any morphism from $\overline{\mathsf{D}}\setminus\{0\}$ to Ω extends to a morphism $\overline{\mathsf{D}}\to\mathsf{P}^N$.*

By Rodríguez Vázquez [14, Theorem C], an example of such an Ω as in Theorem 2 is a component of the (Berkovich) *harmonic Fatou set* $\mathsf{F}_{\mathrm{harm}}(f)$ of an endomorphism f of P^N of degree >1; see [14, Definition 7.9] for the definition of the (Berkovich) harmonic Fatou set of f. Hence we conclude the following.

Corollary 1 (a big Brody-type hyperbolicity of the harmonic Fatou set). *Let K be a field of any characteristic that is complete with respect to a non-archimedean absolute value, and let f be an endomorphism of $\mathsf{P}^N=\mathsf{P}^N(K)$ of any dimension N of degree >1. Then any morphism from $\overline{\mathsf{D}}\setminus\{0\}$ to the* (*Berkovich*) *harmonic Fatou set $\mathsf{F}_{\mathrm{harm}}(f)$ of f extends to a morphism $\overline{\mathsf{D}}\to\mathsf{P}^N$.*

The proof of [14, Proposition 7.17] invoked the non-archimedean *little* Picard theorem, which asserts that *any K-analytic mapping $f:\mathbb{A}^1\to\mathbb{P}^1$ satisfying $\#(\mathbb{P}^1\setminus f(\mathbb{A}^1))\ge 2$ is constant* (see, e.g., [16, (1.3) Proposition]). Our argument in the proofs of Theorems 1 and 2 instead requires a Riemann-type extension theorem (near an isolated singularity) below, which is almost straightforward from the Laurent expansion of a K-analytic function around an isolated singularity of it and the strong triangle inequality, and which can also be adopted to give a more elementary proof of [14, Proposition 7.17].

Proposition 1.1 (a Riemann-type extension theorem). *Let f be a K-analytic function on $\mathcal{O}_K \setminus \{0\}$. If $|f|$ is bounded near 0, then f extends to a K-analytic function on $\mathcal{O}_K$. In particular, f extends to a morphism $\overline{\mathsf{D}} \to \mathsf{P}^1$.*

§ 2. Background

Let K be a field of any characteristic that is complete with respect to a *non-archimedean* absolute value $|\cdot|$. Recall that the absolute value $|\cdot|$ is said to be non-archimedean if the *strong* triangle inequality $|z+w| \le \max\{|z|,|w|\}$ holds for any $z, w \in K$. Let $\pi = \pi_N : K^{N+1} \setminus \{(0,\dots,0)\} \to \mathbb{P}^N = \mathbb{P}^N(K)$ be the canonical projection associated to $\mathbb{P}^N$ of any dimension N, and let $\|\cdot\| = \|\cdot\|_\ell$ be the *maximal* norm $\|(z_1,\dots,z_\ell)\| = \max\{|z_1|,\dots,|z_\ell|\}$ on K^ℓ of any dimension ℓ. Noting that $\bigwedge^2 K^{N+1} \cong K^{\binom{N+1}{2}}$ as K-linear spaces (cf. [8, §8.1]), the (normalized) *chordal metric* on $\mathbb{P}^N$ is defined as

$$[z,w]_{\mathbb{P}^N} := \frac{\|Z \wedge W\|_{\binom{N+1}{2}}}{\|Z\|_{N+1} \cdot \|W\|_{N+1}}, \quad z, w \in \mathbb{P}^N,$$

where $Z \in \pi^{-1}(z), W \in \pi^{-1}(w)$ (the notation is adopted from Nevanlinna's and Tsuji's books [11, 15]), so that $\operatorname{diam}_\#(\mathbb{P}^N) = 1$. We equip $\mathbb{P}^1 = K \cup \{\infty\}$ with this normalized $[z,w]_{\mathbb{P}^1}$ in this section. The topology of $\mathbb{P}^1$ coincides with the metric topology of $(\mathbb{P}^1, [z,w]_{\mathbb{P}^1})$.

§ 2.1. Berkovich projective line as a tree

For the details on P^1, see [1, 5]. For simplicity, we also assume that K is algebraically closed and $|\cdot|$ is non-trivial. As a set, the Berkovich affine line $\mathsf{A}^1 = \mathsf{A}^1(K)$ is the set of all multiplicative seminorms on $K[z]$ extending $|\cdot|$. An element of A^1 is denoted by $\mathcal{S}$, and also by $[\cdot]_{\mathcal{S}}$ as a multiplicative seminorm on $K[z]$. The topology of A^1 is the weakest topology such that for any $\phi \in K[z]$, the function $\mathsf{A}^1 \ni \mathcal{S} \mapsto [\phi]_{\mathcal{S}} \in \mathbb{R}_{\ge 0}$ is continuous, and then A^1 is a locally compact, uniquely arcwise connected, Hausdorff topological space. A subset B in K is called a *K-closed disk* if

$$B = B(a,r) := \{z \in K : |z-a| \le r\}$$

for some $a \in K$ and some $r \ge 0$. For any K-closed disks B, B', *if $B \cap B' \ne \emptyset$, then either $B \subset B'$ or $B' \subset B$* (by the strong triangle inequality). The Berkovich representation [3] asserts that *any element $\mathcal{S} \in \mathsf{A}^1$ is induced by a non-increasing and nesting sequence (B_n) of K-closed disks in that $[\phi]_{\mathcal{S}} = \inf_{n\in\mathbb{N}} \sup_{z \in B_n} |\phi(z)|$ for any $\phi \in K[z]$*; a point $\mathcal{S} \in \mathsf{A}^1$ is said to be of *type* I, II, III, and IV if $\mathcal{S}$ can be induced by a (constant sequence of a singleton $B(a,0) = \{a\}$ consisting of a unique) point $a \in \mathbb{A}^1$, a (constant sequence of a) K-closed disk $B(a,r)$ satisfying $r \in |K^*|$, a (constant sequence of a) K-closed disk $B(a,r)$ satisfying $r \in \mathbb{R}_{>0} \setminus |K^*|$, and any other case holds, respectively. We identify, as a set, K with the set of all type I points in A^1.

Any $[\cdot]_{\mathcal{S}} \in \mathsf{A}^1$ extends to the function $K(z) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that, for any $\phi = \phi_1/\phi_2 \in K(z)$ where $\phi_1, \phi_2 \in K[z]$ are coprime, $[\phi]_{\mathcal{S}} = [\phi_1]_{\mathcal{S}}/[\phi_2]_{\mathcal{S}} \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$, and we also regard $\infty \in \mathbb{P}^1$ as the function $[\cdot]_\infty : K(z) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that for every $\phi \in K(z)$, $[\phi]_\infty = |\phi(\infty)| \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$. As a set, the Berkovich projective line $\mathsf{P}^1 = \mathsf{P}^1(K)$ is nothing but $\mathsf{A}^1 \cup \{\infty\}$. The point ∞ is also said to be of type I.

Set $\mathsf{H}^1 := \mathsf{P}^1 \setminus \mathbb{P}^1$, and let $\mathsf{H}^1_{\mathrm{II}}$ (resp. $\mathsf{H}^1_{\mathrm{III}}$) be the set of all type II (resp. type III) points in P^1. The *Gauss* (*or canonical*) *point*

$$\mathcal{S}_{\mathrm{can}} \in \mathsf{H}^1_{\mathrm{II}}$$

is induced by the (constant sequence of the) K-closed disk $\mathcal{O}_K = B(0,1)$, that is, the ring of K-integers. Let $\mathcal{M}_K$ be the unique maximal ideal of $\mathcal{O}_K$ and k be the residue field $\mathcal{O}_K/\mathcal{M}_K$ of K.

An ordering $\leq_\infty$ on P^1 is defined so that for any $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$, $\mathcal{S} \leq_\infty \mathcal{S}'$ if and only if $[\cdot]_{\mathcal{S}} \leq_\infty [\cdot]_{\mathcal{S}'}$ on $K[z]$. For any $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$, if $\mathcal{S} \leq_\infty \mathcal{S}'$, then set $[\mathcal{S}, \mathcal{S}'] = [\mathcal{S}', \mathcal{S}] := \{\mathcal{S}'' \in \mathsf{P}^1 : \mathcal{S} \leq_\infty \mathcal{S}'' \leq_\infty \mathcal{S}'\}$, and in general, there is the unique point, say, $\mathcal{S} \wedge_\infty \mathcal{S}' \in \mathsf{P}^1$ such that $[\mathcal{S}, \infty] \cap [\mathcal{S}', \infty] = [\mathcal{S} \wedge_\infty \mathcal{S}', \infty]$, and set

$$[\mathcal{S}, \mathcal{S}'] := [\mathcal{S}, \mathcal{S} \wedge_\infty \mathcal{S}'] \cup [\mathcal{S} \wedge_\infty \mathcal{S}', \mathcal{S}'].$$

These *closed intervals* $[\mathcal{S}, \mathcal{S}'] \subset \mathsf{P}^1$ make P^1 an "$\mathbb{R}$-"tree in the sense of Jonsson [7, Definition 2.2]. For any $\mathcal{S} \in \mathsf{P}^1$, the equivalence class $T_{\mathcal{S}}\mathsf{P}^1 := (\mathsf{P}^1 \setminus \{\mathcal{S}\})/\sim$ is defined so that for any $\mathcal{S}', \mathcal{S}'' \in \mathsf{P}^1 \setminus \{\mathcal{S}\}$, $\mathcal{S}' \sim \mathcal{S}''$ if $[\mathcal{S}, \mathcal{S}'] \cap [\mathcal{S}, \mathcal{S}''] = [\mathcal{S}, \mathcal{S}' \wedge_{\mathcal{S}} \mathcal{S}'']$ for some point say $\mathcal{S}' \wedge_{\mathcal{S}} \mathcal{S}'' \in \mathsf{P}^1 \setminus \{\mathcal{S}\}$. An element of $T_{\mathcal{S}}\mathsf{P}^1$ is called a *direction* of P^1 at $\mathcal{S}$ and denoted by $\boldsymbol{v}$, and also by $U(\boldsymbol{v}) = U_{\mathcal{S}}(\boldsymbol{v})$ as a subset in $\mathsf{P}^1 \setminus \{\mathcal{S}\}$. If $\boldsymbol{v} \in T_{\mathcal{S}}\mathsf{P}^1$ is represented by an element $\mathcal{S}' \in \mathsf{P}^1 \setminus \{\mathcal{S}\}$, then we also write $\boldsymbol{v}$ as $\overrightarrow{\mathcal{S}\mathcal{S}'}$. A point $\mathcal{S} \in \mathsf{P}^1$ is of type either I or IV if and only if $\#T_{\mathcal{S}}\mathsf{P}^1 = 1$, that is, $\mathcal{S}$ is an end point of P^1 as a tree. On the other hand, a point $\mathcal{S} \in \mathsf{P}^1$ is of type II (resp. type III) if and only if $\#T_{\mathcal{S}}\mathsf{P}^1 > 2$ (resp. $= 2$). A (Berkovich) *strict connected open affinoid* in P^1 is a non-empty subset in P^1 which is the intersection of some finitely many elements of $\{U(\boldsymbol{v}) : \mathcal{S} \in \mathsf{H}^1_{\mathrm{II}}, \boldsymbol{v} \in T_{\mathcal{S}}\mathsf{P}^1\}$. The topology of P^1 has the quasi-open basis $\{U(\boldsymbol{v}) : \mathcal{S} \in \mathsf{P}^1, \boldsymbol{v} \in T_{\mathcal{S}}\mathsf{P}^1\}$, so has the open basis consisting of all Berkovich strict connected open affinoids in P^1. Both $\mathbb{P}^1$ and $\mathsf{H}^1_{\mathrm{II}}$ are dense in P^1, the set $U(\boldsymbol{v})$ is a component of $\mathsf{P}^1 \setminus \{\mathcal{S}\}$ for each $\mathcal{S} \in \mathsf{P}^1$ and each $\boldsymbol{v} \in T_{\mathcal{S}}\mathsf{P}^1$, and for any $\mathcal{S}, \mathcal{S}' \in \mathsf{P}^1$, the interval $[\mathcal{S}, \mathcal{S}']$ is the unique arc in P^1 between $\mathcal{S}$ and $\mathcal{S}'$.

We also denote the *left-half open* interval $[\mathcal{S}, \mathcal{S}'] \setminus \{\mathcal{S}\} \subset \mathsf{P}^1$ by $(\mathcal{S}, \mathcal{S}']$. For every $0 < r \leq 1$, letting $\mathcal{S}(r) \in (0, \mathcal{S}_{\mathrm{can}}]$ be the point induced by the (constant sequence of the) K-closed disk $B(0,r)$, we have

$$\{z\in K:|z|=r\}=\bigcup_{\boldsymbol{v}\in T_{\mathcal{S}(r)}\mathsf{P}^1\setminus\{\overrightarrow{\mathcal{S}(r)0},\overrightarrow{\mathcal{S}(r)\infty}\}}(U(\boldsymbol{v})\cap\mathbb{P}^1). \tag{2.1}$$

The normalized chordal metric $[z,w]_{\mathbb{P}^1}$ on $\mathbb{P}^1$ extends to an upper semicontinuous and separately continuous function $(\mathcal{S},\mathcal{S}')\mapsto[\mathcal{S},\mathcal{S}']_{\mathrm{can}}$ on $\mathsf{P}^1\times\mathsf{P}^1$, which is called the *generalized Hsia kernel* function on P^1 with respect to $\mathcal{S}_{\mathrm{can}}$ ([1, §4.4]); the function $\mathcal{S}\mapsto[\mathcal{S},\mathcal{S}]_{\mathrm{can}}$ is *continuous on any interval* in P^1, and for every $\mathcal{S}\in\mathsf{P}^1$ and every $\boldsymbol{v}\in T_{\mathcal{S}}\mathsf{P}^1$, we have

$$\mathrm{diam}_{\#}\big(U(\boldsymbol{v})\cap\mathbb{P}^1\big)=\begin{cases}[\mathcal{S}_{\mathrm{can}},\mathcal{S}_{\mathrm{can}}]_{\mathrm{can}}=1 & \text{if } \mathcal{S}=\mathcal{S}_{\mathrm{can}},\\ [\mathcal{S}_{\mathrm{can}},\mathcal{S}_{\mathrm{can}}]_{\mathrm{can}}=1 & \text{if } \mathcal{S}\neq\mathcal{S}_{\mathrm{can}} \text{ and } \boldsymbol{v}=\overrightarrow{\mathcal{S}\mathcal{S}_{\mathrm{can}}},\\ [\mathcal{S},\mathcal{S}]_{\mathrm{can}} & \text{if } \mathcal{S}\in(\mathsf{H}^1_{\mathrm{II}}\setminus\{\mathcal{S}_{\mathrm{can}}\})\cup\mathsf{H}^1_{\mathrm{III}} \text{ and } \boldsymbol{v}\neq\overrightarrow{\mathcal{S}\mathcal{S}_{\mathrm{can}}}.\end{cases} \tag{2.2}$$

§ 2.2. Mapping properties of morphisms

Any non-constant morphism f from an open neighborhood of a point $\mathcal{S}\in\mathsf{P}^1$ to P^1 is finite to one *near* $\mathcal{S}$ and induces a *surjection* $f_*=(f_*)_{\mathcal{S}}:T_{\mathcal{S}}\mathsf{P}^1\to T_{f(\mathcal{S})}\mathsf{P}^1$, which is called the *tangent map* of f at $\mathcal{S}$; when $f(\mathcal{S}_{\mathrm{can}})=\mathcal{S}_{\mathrm{can}}$, $(f_*)_{\mathcal{S}_{\mathrm{can}}}$ is regarded as the action on $\mathbb{P}^1(k)$ of the reduction $\tilde{f}\in k(z)$ of f (see, e.g., [7, §2.6, §4.5] for the details).

Let $f:\overline{\mathsf{D}}\setminus\{0\}\to\mathsf{P}^1$ be a non-constant morphism. Then from a general mapping property of a non-constant K-analytic mapping from a disk in $\mathbb{A}^1$ to $\mathbb{P}^1$ (see, e.g., [2, §3]), for every $\mathcal{S}\in(0,\mathcal{S}_{\mathrm{can}}]$ and every $\boldsymbol{v}\in T_{\mathcal{S}}\mathsf{P}^1\setminus\{\overrightarrow{\mathcal{S}0},\overrightarrow{\mathcal{S}\infty}\}$,

$$\text{if } f(U_{\mathcal{S}}(\boldsymbol{v}))\neq\mathsf{P}^1,\ \text{then } f(U_{\mathcal{S}}(\boldsymbol{v}))=U_{f(\mathcal{S})}(f_*\boldsymbol{v}). \tag{2.3}$$

§ 3. Proofs of Theorems 1 and 2

With no loss of generality, we also assume that K is algebraically closed and $|\cdot|$ is non-trivial. We equip $\mathbb{P}^1$ with the normalized chordal metric $[z,w]_{\mathbb{P}^1}$ defined in Section 2, so $\mathrm{diam}_{\#}(\mathbb{P}^1)=1$.

Proof of Theorem 1. Let $f:\overline{\mathsf{D}}\setminus\{0\}\to\mathsf{P}^1$ be a morphism, and suppose that f does not satisfy (1.1). Then by (2.1), for every $\mathcal{S}\in(0,\mathcal{S}_{\mathrm{can}}]$ close enough to 0 and every $\boldsymbol{v}\in T_{\mathcal{S}}\mathsf{P}^1\setminus\{\overrightarrow{\mathcal{S}0},\overrightarrow{\mathcal{S}\infty}\}$, we have $f(U(\boldsymbol{v}))\neq\mathsf{P}^1$, and in turn by (2.2), (2.3), and the continuity of f, there is $\boldsymbol{u}_0\in T_{\mathcal{S}_{\mathrm{can}}}\mathsf{P}^1$ such that $f(\mathcal{S})\subset U(\boldsymbol{u}_0)$ for every $\mathcal{S}\in(0,\mathcal{S}_{\mathrm{can}}]$ close enough to 0. Then, under the assumption that f does not satisfy (1.1), by (2.2) and (2.3), for any $\mathcal{S}\in(0,\mathcal{S}_{\mathrm{can}}]$ close enough to 0, we even have

$$f\big(\mathsf{P}^1\setminus(U(\overrightarrow{\mathcal{S}0})\cup U(\overrightarrow{\mathcal{S}\infty}))\big)\subset U(\boldsymbol{u}_0),$$

and then by (2.1) and a Riemann-type extension theorem (Proposition 1.1), f extends to a morphism from D to a K-analytic space. $\square$

Proof of Theorem 2. We can assume $N = 1$ by an argument similar to that in the final paragraph in [14, Proof of Proposition 7.17] involving not only the existence of a nice lifting of $f|(\mathcal{O}_K \setminus \{0\})$ to a morphism $\mathcal{O}_K \setminus \{0\} \to \mathbb{A}^{N+1}$ through the canonical projection $\pi : \mathbb{A}^{N+1} \setminus \{(0,\dots,0)\} \to \mathbb{P}^N$ (using [6, Theorem 2.7.6]) but also the projection $\mathbb{A}^{N+1} \setminus \{z_0 = 0\} \mapsto [z_0 : z_1] \in \mathbb{P}^1$.

Let Ω be an open subset in P^1. Suppose that $\mathrm{Mor}(\overline{\mathsf{D}} \setminus \{0\}, \Omega)$ is normal and, to the contrary, that there is a morphism $f : \overline{\mathsf{D}} \setminus \{0\} \to \Omega$ having an isolated essential singularity at the origin.

(i). If there is a point $\mathcal{S}_0 \in \mathsf{H}^1_{\mathrm{II}}$ such that $\#(f^{-1}(\mathcal{S}_0) \cap (0, \mathcal{S}_{\mathrm{can}}]) = \infty$, then we can conclude a contradiction by an argument similar to that in the former half of [14, Proof of Theorem 7.17]. For completeness, we include the argument; by the surjectivity of $f_* : T_{\mathcal{S}}\mathsf{P}^1 \to T_{\mathcal{S}_0}\mathsf{P}^1$ for every $\mathcal{S} \in f^{-1}(\mathcal{S}_0) \cap (0, \mathcal{S}_{\mathrm{can}}]$ (and by $\#T_{\mathcal{S}_0}\mathsf{P}^1 = \infty > 2$), there are a sequence $(\mathcal{S}_n)$ in $f^{-1}(\mathcal{S}_0) \cap (0, \mathcal{S}_{\mathrm{can}}]$ tending to 0 as $n \to \infty$ and a direction $\boldsymbol{u}_0 \in T_{\mathcal{S}_0}\mathsf{P}^1$ such that for every $n \in \mathbb{N}$, $\boldsymbol{u}_0 \in f_*(T_{\mathcal{S}_n}\mathsf{P}^1 \setminus \{\overrightarrow{\mathcal{S}_n 0}, \overrightarrow{\mathcal{S}_n \infty}\})$. Then fixing a point $a_0 \in U(\boldsymbol{u}_0) \cap \mathbb{P}^1$, for every $n \in \mathbb{N}$, there is a point $b_n \in \mathbb{P}^1 \cap f^{-1}(a_0)$ such that $\overrightarrow{\mathcal{S}_n b_n} \in T_{\mathcal{S}_n}\mathsf{P}^1 \setminus \{\overrightarrow{\mathcal{S}_n 0}, \overrightarrow{\mathcal{S}_n \infty}\}$. For every $n \in \mathbb{N}$, by (2.1), $\mathcal{S}_n$ is induced by the (constant sequence of the) K-closed disk $B(0, |b_n|)$.

Now setting

$$g_n(z) := f(b_{n!} \cdot z^{n!}) \in \mathrm{Mor}(\overline{\mathsf{D}} \setminus \{0\}, \Omega)$$

for each $n \in \mathbb{N}$, under the assumption that $\mathrm{Mor}(\overline{\mathsf{D}} \setminus \{0\}, \Omega)$ is *normal* in the sense of [4, Introduction], taking a subsequence of (g_n) if necessary, the (*pointwise*) limit $g := \lim_{n\to\infty} g_n$ on $\overline{\mathsf{D}} \setminus \{0\}$ exists and is a *continuous* mapping $\overline{\mathsf{D}} \setminus \{0\} \to \mathsf{P}^1$. Then

$$g(\mathcal{S}_{\mathrm{can}}) = \lim_{n\to\infty} g_n(\mathcal{S}_{\mathrm{can}}) = \lim_{n\to\infty} f(\mathcal{S}_{n!}) = \mathcal{S}_0.$$

On the other hand, we can fix a sequence (ζ_m) in K such that $(\zeta_m)^m = 1$ (so $\zeta_m \in \overline{\mathsf{D}} \setminus \{0\}$) for every $m \in \mathbb{N}$ and that $\lim_{m\to\infty} \zeta_m = \mathcal{S}_{\mathrm{can}}$. Then for any $m \in \mathbb{N}$,

$$g(\zeta_m) = \lim_{n\to\infty} g_n(\zeta_m) = \lim_{n\to\infty} f(b_{n!} \cdot 1^{n!/m}) = \lim_{n\to\infty} f(b_{n!}) = a_0, \quad \text{and in turn}$$
$$g(\mathcal{S}_{\mathrm{can}}) = \lim_{m\to\infty} g(\zeta_m) = \lim_{m\to\infty} a_0 = a_0.$$

Hence we must have $\mathbb{P}^1 \ni a_0 = \mathcal{S}_0 \in \mathsf{H}^1_{\mathrm{II}}$, which is a contradiction.

(ii). If there is a sequence $(\mathcal{S}_n)$ in $(0, \mathcal{S}_{\mathrm{can}}]$ tending to 0 as $n \to \infty$ such that for every $n \in \mathbb{N}$, there is a direction $\boldsymbol{v}_n \in T_{\mathcal{S}_n}\mathsf{P}^1 \setminus \{\overrightarrow{\mathcal{S}_n 0}, \overrightarrow{\mathcal{S}_n \infty}\}$ satisfying $f(U(\boldsymbol{v}_n)) = \mathsf{P}^1$, then taking a subsequence of $(\mathcal{S}_n)$ if necessary, there is a direction $\boldsymbol{u}_0 \in T_{\mathcal{S}_{\mathrm{can}}}\mathsf{P}^1$ such that $f(\mathcal{S}_n) \in \mathsf{P}^1 \setminus U(\boldsymbol{u}_0)$ for any $n \in \mathbb{N}$. Fixing a point $a_0 \in \mathbb{P}^1 \cap U(\boldsymbol{u}_0)$, there is a point $b_n \in \mathbb{P}^1 \cap U(\boldsymbol{v}_n) \cap f^{-1}(a_0)$ for each $n \in \mathbb{N}$.

Now by a (branched) rescaling argument similar to that in the case (i), we must have $U(\boldsymbol{u}_0) \ni a_0 \in \overline{\{f(\mathcal{S}_n) : n \in \mathbb{N}\}} \subset \mathsf{P}^1 \setminus U(\boldsymbol{u}_0)$. This is a contradiction.

(iii). Suppose finally that for every $\mathcal{S} \in \mathsf{H}^1_{\mathrm{II}}$, $\#(f^{-1}(\mathcal{S}) \cap (0, \mathcal{S}_{\mathrm{can}}]) < \infty$ and that for every $\mathcal{S} \in (0, \mathcal{S}_{\mathrm{can}}]$ close enough to 0 and every $\boldsymbol{v} \in T_{\mathcal{S}}\mathsf{P}^1 \setminus \{\overrightarrow{\mathcal{S}0}, \overrightarrow{\mathcal{S}\infty}\}$, $f(U(\boldsymbol{v})) \neq \mathsf{P}^1$.

Then under the former assumption (for $\mathcal{S} = \mathcal{S}_{\mathrm{can}}$), by the continuity of f, there is $\boldsymbol{u}_0 \in T_{\mathcal{S}_{\mathrm{can}}}\mathsf{P}^1$ such that $f(\mathcal{S}) \subset U(\boldsymbol{u}_0)$ for every $\mathcal{S} \in (0, \mathcal{S}_{\mathrm{can}}]$ close enough to 0. Then under the latter assumption, by (2.1), (2.3), and a Riemann-type extension theorem (Proposition 1.1), there must exist a sequence $(\mathcal{S}_n)$ in $(0, \mathcal{S}_{\mathrm{can}}]$ tending to 0 as $n \to \infty$ such that for every $n \in \mathbb{N}$, $f(\mathcal{S}_n) \in U(\boldsymbol{u}_0)$ and there is a direction $\boldsymbol{v}_n \in T_{\mathcal{S}_n}\mathsf{P}^1 \setminus \{\overrightarrow{\mathcal{S}_n 0}, \overrightarrow{\mathcal{S}_n \infty}\}$ satisfying

$$f_*(\boldsymbol{v}_n) = \overrightarrow{f(\mathcal{S}_n)\mathcal{S}_{\mathrm{can}}},$$

and then $f(U(\boldsymbol{v}_n)) \supset \mathsf{P}^1 \setminus U(\boldsymbol{u}_0)$. Fixing a point $a_0 \in \mathbb{P}^1 \setminus U(\boldsymbol{u}_0) = \mathbb{P}^1 \setminus \overline{U(\boldsymbol{u}_0)}$, there is a point $b_n \in \mathbb{P}^1 \cap U(\boldsymbol{v}_n) \cap f^{-1}(a_0)$ for each $n \in \mathbb{N}$.

Now by a (branched) rescaling argument similar to that in the case (i), we have $\mathbb{P}^1 \setminus \overline{U(\boldsymbol{u}_0)} \ni a_0 \in \overline{\{f(\mathcal{S}_n) : n \in \mathbb{N}\}} \subset \overline{U(\boldsymbol{u}_0)}$. This is a contradiction. □

Acknowledgement. The author thanks the referee for a very careful scrutiny and invaluable comments. The author also thank the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

References

[1] Baker, M. and Rumely, R. *Potential theory and dynamics on the Berkovich projective line*, Vol. 159 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI (2010).

[2] Benedetto, R. L. An Ahlfors islands theorem for non-Archimedean meromorphic functions, *Trans. Amer. Math. Soc.*, **360**, 8 (2008), 4099–4124.

[3] Berkovich, V. G. *Spectral theory and analytic geometry over non-Archimedean fields*, Vol. 33 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI (1990).

[4] Favre, C., Kiwi, J. and Trucco, E. A non-Archimedean Montel's theorem, *Compos. Math.*, **148**, 3 (2012), 966–990.

[5] Favre, C. and Rivera-Letelier, J. Théorie ergodique des fractions rationnelles sur un corps ultramétrique, *Proc. Lond. Math. Soc. (3)*, **100**, 1 (2010), 116–154.

[6] Fresnel, J. and Put, van der M. *Rigid analytic geometry and its applications*, Vol. 218 of *Progress in Mathematics*, Birkhäuser Boston, Inc., Boston, MA (2004).

[7] Jonsson, M. Dynamics on Berkovich spaces in low dimensions, Berkovich Spaces and Applications, Springer (2015), 205–366.

[8] Kobayashi, S. *Hyperbolic complex spaces*, Vol. 318 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, Berlin (1998).

[9] Lehto, O. The spherical derivative of meromorphic functions in the neighbourhood of an isolated singularity, *Commentarii Mathematici Helvetici*, **33**, 1 (1959), 196–205.

[10] Lehto, O. and Virtanen, K. I. On the behaviour of meromorphic functions in the neighbourhood of an isolated singularity, *Ann. Acad. Sci. Fenn. Ser. A. I.*, **1957**, 240 (1957), 9.

[11] Nevanlinna, R. *Analytic functions*, Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Band 162, Springer-Verlag, New York (1970).

[12] Okuyama, Y. A Lehto–Virtanen-type theorem and a rescaling principle for an isolated essential singularity of a holomorphic curve in a complex space, *Internat. J. Math.*, **26**, 6 (2015), 1541009 (15 pages).

[13] Okuyama, Y. and Pankka, P. Rescaling principle for isolated essential singularities of quasiregular mappings, *Proc. Amer. Math. Soc.*, **143**, 5 (2015), 2043–2050.

[14] Rodríguez Vázquez, R. Non-archimedean normal families, *arXiv e-prints* (Jul 2016), arXiv:1607.05976.

[15] Tsuji, M. *Potential theory in modern function theory*, Chelsea Publishing Co., New York (1975), Reprinting of the 1959 original.

[16] Put, van der M. Essential singularities of rigid analytic functions, *Nederl. Akad. Wetensch. Indag. Math.*, **43**, 4 (1981), 423–429.