Analyses of Consensus Building Models Based on the Prospect Theory for Equilibrium States

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In this study, we propose a consensus model with nontrivial behaviors. This model uses the prospect theory to describe the human decision-making process in considerable detail, which enables realistic and diverse behaviors. Under the proposed model, the equilibrium states were analyzed based on two settings. In one of the settings, we confirmed markedly diverse behaviors such as first-order and second-order transitions. nontrivial discontinuous transitions were also identified and analyzed in detail on the basis of the Landau theory. In the analysis of the other setting, an intuitive interpretation was made on the basis of the prospect theory.

1. Introduction

In recent years, many studies on social phenomena have been performed from a physical viewpoint, and in particular, opinion formulation has been the subject of many models inspired by statistical physics.[1] For example, one of the most well-known and simple models is based on the Ising model.[2,3] In this model, each agent has two opinions represented as a spin (e.g., right/left, buying/selling, etc.), and their opinions are updated while being influenced by the opinions of surrounding agents. Although the Ising model is simple, it expresses the psychological function of trying to match the opinions of surrounding agents.

The voter model is well known in a similar setting where one chooses between two opinions.[4,5] This model differs from the Ising model in that it probabilistically selects one agent from the surrounding agents and aligns their opinions.

In addition, the majority rule model was proposed.[6,7] These systems have dynamics that unify the opinions within a group of size r. A detailed and realistic behavioral model is the Sznajd model,[8] which incorporates the psychological aspects of human behavior using the social impact theory, making it more realistic than the pre-Sznajd models.

On the other hand, there are few studies on the psychological aspects of humans and even fewer that describe the decision-making process of agents in detail. In this study, we addressed these problems by applying the prospect theory[9,10] while providing more detailed settings than previous studies.

The flow of this paper is as follows: First, in Sect. 2, we give an overview of the prospect theory, which provides the basis for this study. In the third section, we describe the settings of the model and derive the dynamics and equilibrium states using techniques in statistical physics. In Sect. 4, we observe how equilibrium states emerge in a setting where inequality within a group increases. In Sect. 5, we consider equilibrium states in a setting where only the winners are selected among the groups with different opinions and they share the gains. The final section provides a summary and discussion.

2. Prospect Theory

The prospect theory describes decision-making under risk.[9–12] It differs from the expected utility theory in that the overall value of an option is calculated using a value function, $v(x)$, and a probability weighting function, $w(p)$.

The value function $v(x)$ expresses the "value" that a person perceives for the gain x and is generally modeled as:

$$v(x) = \begin{cases} x^{\alpha} & (x \geq 0) \\ -\lambda(-x)^{\alpha} & (x < 0) \end{cases}. \tag{1}$$

By a nonlinear regression procedure, one can obtain estimates of the parameters as $\alpha = 0.88$ and $\lambda = 2.25$.[12] The function $v(x)$ for these parameters is shown in Fig. 1(a). From Fig. 1(a), even if the gain or loss becomes large, their values are not likely to become large. Furthermore, for the same amount of gain and loss, the general tendency to avoid losses is reflected in $\lambda = 2.25$.

The prospect theory also uses a probability weighting function $w(p)$, which captures the idea that people are likely to overreact to small probability events but underreact to large probabilities. On the basis of real-world measurements, the following functional form was assumed in many studies:

$$w(p) = \frac{p^{\gamma}}{(p^{\gamma} + (1-p)^{\gamma})^{1/\gamma}}, \tag{2}$$

where γ is estimated as 0.65 using nonlinear regression.[12] The form of $w(p)$ for this parameter is shown in Fig. 1(b). $w(p)$ becomes large when p is small, and vice versa, which represents the misperceptions of probability.

In the prospect theory, people behave as if they are computing the overall value on the basis of potential outcomes and their respective probabilities, and then they select the option with a higher value. If $(x, p; y, 1-p)$ denotes a prospect of outcome x with the probability p and outcome y with the probability $1-p$, and x is greater than y, then the overall value of the prospect is

$$V(x, p; y, 1-p) = \begin{cases} w(p)v(x) + w(1-p)v(y) & (xy < 0) \\ w(p)v(x) + (1-w(p))v(y) & (xy \geq 0) \end{cases}. \tag{3}$$

Note that the method of computing $V(x, p; y, 1-p)$ differs depending on the sign of xy.

The so-called fourfold pattern of risk attitudes derived from the prospect theory is presented in Table I.[11] Namely, risk-averse behaviors are observed when gains have moderate probabilities or losses have small probabilities, and risk-

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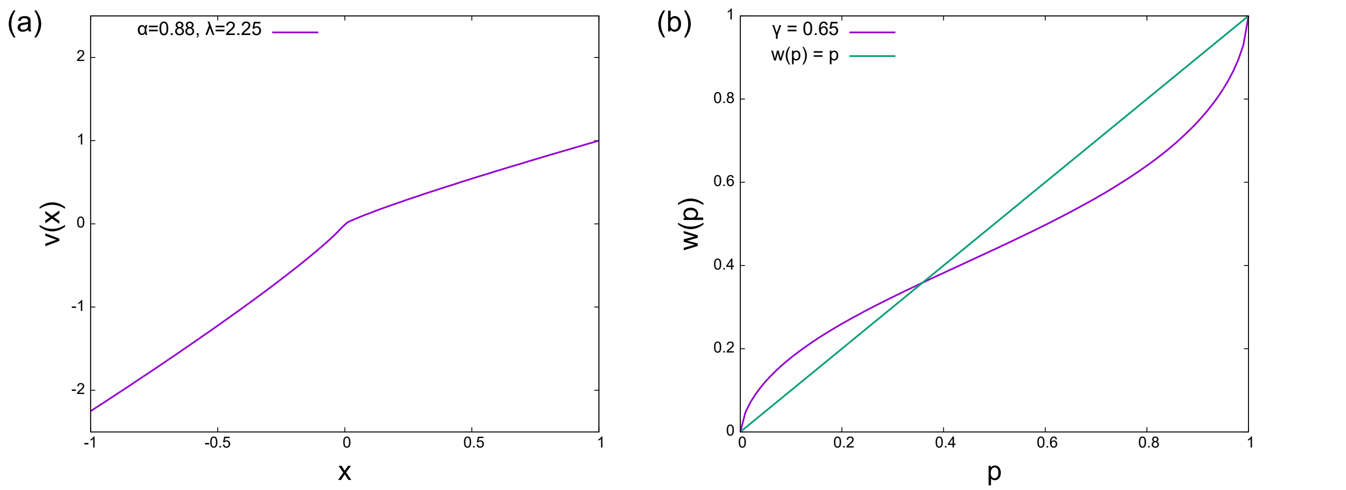

Fig. 1. (Color online) The plot on the left shows the value function $v(x)$ for parameters $\alpha = 0.88$ and $\lambda = 2.25$. Diminishing sensitivity and loss aversion are reflected in the functional form of $v(x)$. The plot on the right shows the probability weighting function $w(p)$ for the parameter $\gamma = 0.65$. $w(p)$ becomes large when p is small, and vice versa, which represents the misperceptions of probability.

seeking behaviors are observed when losses have moderate probabilities or gains have small probabilities. The former is a possible explanation for why many people are likely to buy insurance and the latter explains why people like to buy lottery tickets.

Table I. Fourfold patterns of risk attitudes derived from prospect theory. Risk-averse behaviors are observed when gains have moderate probabilities or losses have small probabilities, and risk-seeking behaviors are observed when losses have moderate probabilities or gains have small probabilities.

	Gains	Losses
High probability	risk-averse	risk-seeking
Low probability	risk-seeking	risk-averse

3. Model and Equilibrium States

In this section, we propose a model based on the prospect theory. We also analyze the equilibrium states using statistical mechanics techniques.

3.1 Model and dynamics

There are N agents with one opinion, i, which can be $+1$ or -1, and the ratio of agents with opinion i is n_i. At each step, one agent is selected to update its state. This agent computes the overall value of having the opinion i, regardless of one's current state (Fig. 2).

Suppose that the overall value of the opinion i depends only on n_i:

$$V_i = V(n_i). \tag{4}$$

In the next section, we describe in more detail the settings for the functional form of $V(n)$. However, we do not consider the functional form of $V(n)$ in this section for the sake of general discussion.

In addition, following the random utility theory,[13,14] it is assumed that the agent cannot specify or measure V_i precisely. This is represented by

$$U_i = V(n_i) + \xi_i, \tag{5}$$

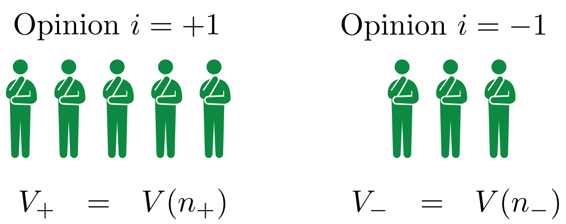

Fig. 2. (Color online) Schematic figure for this model. The ratio of agents with opinion i is represented as n_i. An agent calculates the overall value of having the opinion i and updates its opinion on the basis of the value V_i.

where U_i is the fluctuating overall value for opinion i and ξ_i is an independent random variable that follows a Gaussian distribution with variance γ^2. Here, the agent selects opinion $i = +1$ if $U_+ > U_-$, and $i = -1$ if $U_+ < U_-$. Note that in the prospect theory, individual differences in preferences are attributed to the functional forms of the value function and probability weighting function differing from person to person. Here, this is simplified and expressed as random variables.

When there are only two alternatives, the probability of choosing opinion $+1$ can be calculated analytically (see Appendix for details):

$$P_+ = \Phi(\beta(V(n_+) - V(n_-))), \tag{6}$$

where $\beta = \dfrac{1}{\sqrt{2\gamma^2}}$. Moreover, $\Phi(x)$ is the cumulative distribution function of the normal distribution:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left[-t^2/2\right] dt. \tag{7}$$

This suggests that the probability P_+ of choosing + is determined by $V_+ - V_-$. According to Eq. (6), a larger difference in the values of alternatives causes the agent to more likely select opinion i. Furthermore, β gives the scale of the reciprocal of the magnitude of $V(n_+) - V(n_-)$ and can be regarded as an expression of the clarity of each agent's value judgment.

It can be easily checked whether P_- satisfies

$$P_- = \Phi(\beta(V(n_-) - V(n_+))) = 1 - P_+. \tag{8}$$

Therefore, the dynamics of this model can be derived as:

$$\frac{dn_+}{dt} = (1 - n_+)P_+ - n_+P_-. \tag{9}$$

In the following, we treat the bias $\epsilon = \frac{n_+ - n_-}{2}$ as an order parameter. Note that from $n_+ + n_- = 1$, if ϵ is determined, n_+ and n_- can be obtained. Using Eqs. (6) and (8), we can rewrite the dynamics (9) as

$$\frac{d\epsilon}{dt} = \left(\frac{1}{2} - \epsilon\right)\Phi\left(\beta(V_+ - V_-)\right) - \left(\frac{1}{2} + \epsilon\right)\Phi\left(\beta(V_- - V_+)\right). \tag{10}$$

3.2 Equilibrium states

In the following, the equilibrium states are classified according to the Landau theory.[15)] First, Eq. (10) can be transformed to

$$\frac{d\epsilon}{dt} = -\nabla F(\epsilon), \tag{11}$$

where $F(\epsilon)$ corresponds to the Landau free energy and is obtained by integrating the right-hand side of Eq. (10) with regard to ϵ and reversing the sign:

$$F(\epsilon) = -\int_0^{\epsilon}\left(\frac{1}{2} - \epsilon\right)\Phi\left(\beta(V_+ - V_-)\right)d\epsilon + \int_0^{\epsilon}\left(\frac{1}{2} + \epsilon\right)\Phi\left(\beta(V_- - V_+)\right)d\epsilon.$$

Note that thermal equilibrium is realized by the minimization of $F(\epsilon)$. The global or local minima correspond to the globally or locally stable states, respectively, and the latter are referred to as metastable states.

As we focused on critical phenomena, the value β is close to the critical point β_c and the order parameter ϵ is therefore small. As such, the Landau free-energy expansion for $F(\epsilon)$ reads

$$F(\epsilon) \simeq \frac{1}{2}A\epsilon^2 + \frac{1}{4}B\epsilon^4 + \frac{1}{6}C\epsilon^6, \tag{12}$$

to which we introduced

$$A = 1 - \frac{2\beta}{\sqrt{2\pi}}V'(1/2), \tag{13}$$

$$B = \frac{\beta}{3\sqrt{2\pi}}\left[4\beta^2 V'(1/2)^3 - V'''(1/2)\right]. \tag{14}$$

As for C, if the sign is positive, the value is not critical.

Depending on the sign of the coefficients A and B, it can be shown that the system reaches different equilibrium states when using the Landau theory[15)]. This is summarized in Table II, where we define

$$A_1 = \frac{3B^2}{16C}. \tag{15}$$

In Table II, (G) and (L) mean globally and locally stable states, respectively. The transition is of the first (second) order in the case of $B < 0$ $(B \geq 0)$. In particular, $A = B = 0$ corresponds to the tricritical point.

In addition, the symmetric solution $\epsilon = 0$ is destabilized for $A < 0$. Therefore, we define the critical point as the value of $\beta = \beta_c$ at which $\epsilon = 0$ becomes unstable. This condition is

Table II. Classifications of equilibrium states. In the table, (G) and (L) mean globally and locally stable states, respectively. The transition is of the first (second) order in the case of $B < 0$ $(B \geq 0)$. In particular, $A = B = 0$ corresponds to the tricritical point. In addition, we observe symmetry breaking when $A < 0$.

	$A < 0$	$0 < A < A_1$	$A_1 < A$
$B \geq 0$ (second order)	$\epsilon \neq 0$ (G)	$\epsilon = 0$ (G)	$\epsilon = 0$ (G)
$B < 0$ (first order)	$\epsilon \neq 0$ (G)	$\epsilon \neq 0$ (G)	$\epsilon = 0$ (G)
		$\epsilon = 0$ (L)	$\epsilon \neq 0$ (L)

given by $A = 0$, resulting in

$$\beta_c = \frac{\sqrt{2\pi}}{2V'(1/2)}, \tag{16}$$

using Eq. (13). Furthermore, the sign of B can be calculated as

$$\tilde{B} = \frac{3\sqrt{2\pi}}{\beta_c}B = 2\pi V'(1/2) - V'''(1/2) \tag{17}$$

just above the critical point.

4. Setting I: Expanding the Gap in a Group

In the following, we derive the dynamics on the basis of the prospect theory by applying more detailed settings. Then, from the same analysis as before, we classify the equilibrium state according to the parameters that characterize the system.

4.1 Settings

Again, there are N agents with one opinion, which can be $i = \pm 1$, and the ratio of agents with opinion i is n_i. Suppose that the agent n has the opinion i when a sufficient amount of time has passed. In this case, agent n is assumed to have a gain of $x_0 + an_i$ with probability p and a gain of $x_0 - an_i$ with probability $1 - p$ (Fig. 3). Therefore, this setting is such that if there are more agents who share the same opinion, there will be greater variation in the gain by selecting that opinion. All agents update their respective opinions on the basis of these assumptions. Even when discussion is restricted to $a > 0$, generality is not lost.

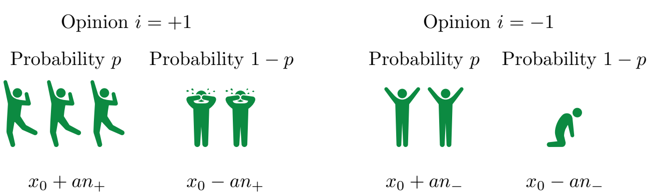

Fig. 3. (Color online) Overview of the expanding gap model. The setting is such that if there are more agents who share the same opinion, then the variation in the gain increases by selecting that opinion.

The most straightforward example of this setting is when there is some kind of "game" going on in each group, such that the inequality in the group increases as the number of people increases. On the other hand, a more realistic setting needs to be considered.

On the basis of the prospect theory, the overall value for the opinion $i = \pm 1$ is given as

$$V_i = \begin{cases} w(p)v(x_0 + an_i) + w(1-p)v(x_0 - an_i) \\ \quad ((x_0 + an_i)(x_0 - an_i) < 0) \\ w(p)v(x_0 + an_i) + (1 - w(p))v(x_0 - an_i) \\ \quad ((x_0 + an_i)(x_0 - an_i) \geq 0) \end{cases} \quad (18)$$

using Eq. (3). By comparing Eq. (18) with Eq. (4), we obtain the function $V(n)$ as

$$V(n) = \tilde{p}v(x_0 + an) + \tilde{q}v(x_0 - an), \quad (19)$$

where $\tilde{p}$ and $\tilde{q}$ are defined as

$$\tilde{p} = w(p), \quad (20)$$

$$\tilde{q} = \begin{cases} w(1-p) & ((x_0 + an)(x_0 - an) < 0) \\ 1 - w(p) & ((x_0 + an)(x_0 - an) \geq 0). \end{cases} \quad (21)$$

4.2 Detailed analysis

In the following,

$$x_\pm = x_0 \pm a/2 \quad (22)$$

is introduced for simplicity of notation. We then have

$$\beta_c = \frac{\sqrt{2\pi}}{2(\tilde{p}v'(x_+) - \tilde{q}v'(x_-))a}, \quad (23)$$

$$\tilde{B} = \frac{3\sqrt{2\pi}}{\beta_c a}B$$
$$= 2\pi(\tilde{p}v'(x_+) - \tilde{q}v'(x_-)) - a^2(\tilde{p}v'''(x_+) - \tilde{q}v'''(x_-)) \quad (24)$$

by substituting $V(n)$ in this model into Eqs. (16) and (17), respectively.

Note that the nonlinearity of $v(x)$ is critical. As the condition $\beta > 0$ implies $\tilde{p}v'(x_+) - \tilde{q}v'(x_-) > 0$, if $v(x)$ were linear (i.e., $v'''(x) = 0$), then $\tilde{B}$ cannot be negative. Thus, the nonlinearity of $v(x)$ introduced by the prospect theory leads to the first-order transition.

Therefore, by determining the sign of $\tilde{B}$ for each parameter, we can classify the equilibrium states. There are three parameters p, x_0, and a that determine this system, but the signs of β_c and $\tilde{B}$ depend only on p and x_0/a from Eqs. (23) and (24). The resulting classifications are shown in Fig. 4 for x_0/a and p. The existence of phase transitions, especially both first- and second-order transitions, can be confirmed. To interpret this result, we first note that the aggregation of agents' opinions means that many agents select alternatives with high gain variability and are therefore risk-seeking.

One major feature is that phase transitions do not occur when p is small. This can be interpreted as when p is small, the benefit of aggregating opinions becomes small, i.e., risk-averse. Another important point is that first-order phase transitions occur near $x_0/a \simeq -0.5$, because when the opinions are equal (i.e., $n_\pm \simeq 1/2$), the gain obtained with probability $1-p$ is $x_0 - an_\pm \simeq 0$. When the gain is close to 0, the "value" obtained from the nature of the value function $v(x)$ increases rapidly and can be interpreted as sudden cohesion to one side of the opinion (i.e., first-order transitions). Furthermore, when the gain is positive ($x_0/a > 0$), the area where the phase transition occurs is small, which can be interpreted as indicating a risk-averse attitude. On the other hand, when the gain is negative ($x_0/a < 0$), the region where the phase transition occurs is large, reflecting a risk-seeking attitude.

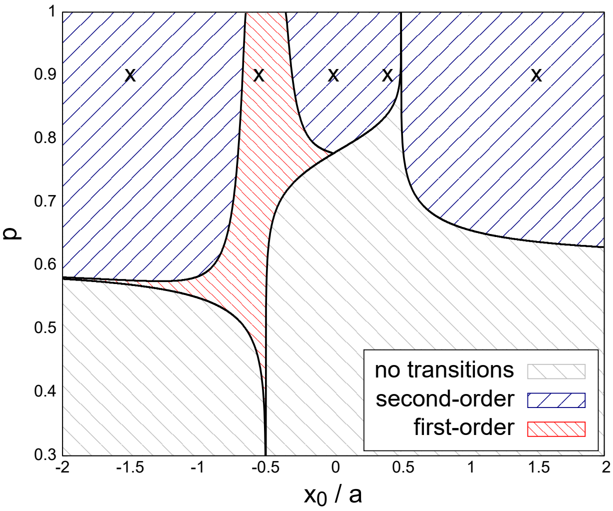

Fig. 4. (Color online) Resulting classifications of phase transitions for setting I. The existence of phase transitions, especially both first- and second-order transitions, can be confirmed. The parameters used in the simulations described below are indicated by × in the figure.

To confirm the classification of equilibrium states shown in Fig. 4, the values of ϵ at equilibrium for the values of β were obtained by both numerical simulation and analyses, as shown in Fig. 5. In the numerical simulation, we simulated $N = 1000$ agents and the results for 20 samples are shown with error bars. In the numerical analysis, the stable solution (solid line) and unstable solution (dotted line) calculated on the basis of the Landau free energy $F(\epsilon)$ are shown. Here, the parameter $p = 0.9$ was fixed, whereas x_0/a was varied. The corresponding parameters are shown as × in Fig. 4.

In Fig. 5, (a), (b), and (d) indicate second-order phase transitions, and the first-order phase transition is shown in Fig. 5(c). These behaviors are expected from Fig. 4.

In addition, the case with $x_0/a = 0.40$ is shown in Fig. 6(a). Based on Fig. 4, it is expected to be a second-order transition, but the numerical simulations demonstrated first-order behavior. For a more detailed analysis, the Landau free energy was calculated numerically for several values of β near the phase transition point shown in Fig. 6(b). The globally stable states are shown as circles and the locally stable states are shown as triangles, and a first-order-like bifurcation before the second-order transition, as predicted from Fig. 4, is observed. Thus, before the second-order transition predicted from Fig. 4 occurs, there is a first-order transition-like bifurcation, which becomes a global stability point; therefore, the asymmetric solution that appears owing to the second-order transition does not appear in the numerical simulation. This behavior is not predictable from Fig. 4, but it is reasonable considering that Fig. 4 was obtained from the analysis near the critical point.

5. Setting II: Winners-Take-All Game

In this section, we will deal with a game in a more realistic setting than in the previous section.

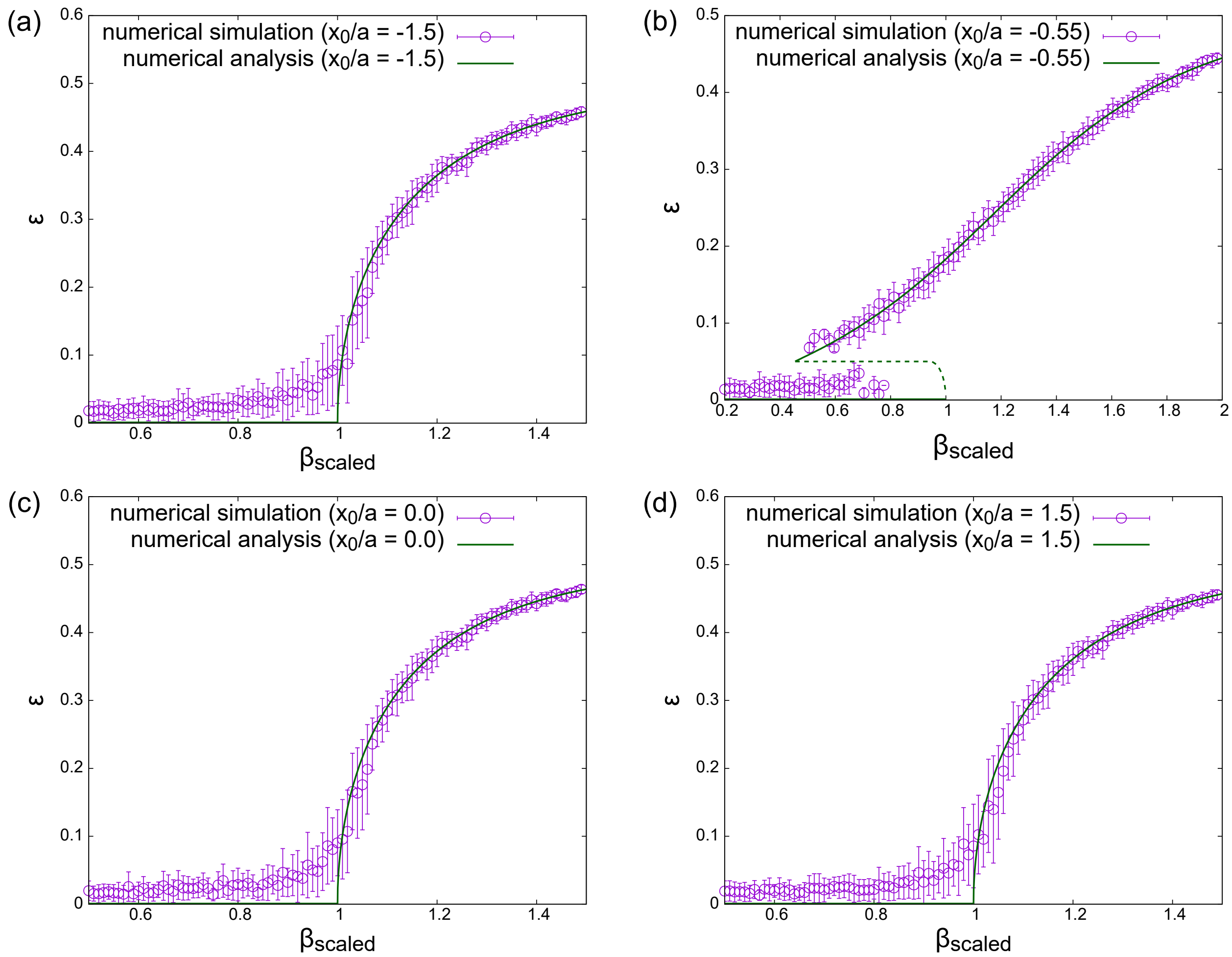

Fig. 5. (Color online) Dependence of equilibrium state on β for certain parameters indicated by $\times$ in Fig. 4. Here, β_{scaled} is defined as β/β_c. The (meta)stable state (solid line) and unstable solution (dotted line) obtained numerically from the Landau free energy Eq. (12) are consistent with the numerical simulation (dots). In each case, $p = 0.9$ was used. (a) $x_0/a = -1.5$. (b) $x_0/a = -0.55$ shows the first-order transition. The meta-stable state was also confirmed. (c) $x_0/a = 0.0$. (d) $x_0/a = 1.5$.

5.1 Settings

All agents must have either opinion $i = \pm 1$ and a certain bet is made within each opinion group. The entry fee is c in each case, and in each bet, only N_0 randomly selected people are winners, splitting the stakes among the group (Fig. 7). Here, we discuss only the case where $N_0 < N/2$. The game is expressed in prospect form as follows: An agent with opinion i in the final state will gain $c\left(\frac{N}{N_0}n_i - 1\right)$ with probability $p(n_i)$ and gain $-c$ with probability $1 - p(n_i)$, where $p(n) = \min(1, N_0/(Nn))$.

Note that this game can be applied to the case where $c < 0$. In this case, everyone receives $-c$ as a reward, but N_0 agents are randomly selected from the agents with opinion i and have to pay $-N_0 c n_i$. This can be considered as a situation in which each agent chooses the optimal community to be a free rider. The aggregation of agents' opinions means that there are many free riders in a community.

On the basis of the prospect theory, the overall value for the opinion i is given as

$$V_i = w(p(n_i))v(cn_i N/N_0 - c) + w(1 - p(n_i))v(-c). \quad (25)$$

We note that the condition $N_0 < N/2$ leads to $Nn_i/N_0 - 1 > 0$ for $n_i \simeq 1/2$; thus, division in cases for the evaluation of overall values is not necessary. By comparing Eq. (25) with Eq. (4), we obtain $V(n)$ as

$$V(n) = w(p(n))v(cnN/N_0 - c) + w(1 - p(n))v(-c). \quad (26)$$

5.2 Detailed analysis

The classification of the equilibrium states using the same procedure as the analysis described in Setting I is shown in Fig. 8. The results when $w(p) = p$ and $w(p) = p^\gamma/(p^\gamma + (1 - p)^\gamma)^{1/\gamma}$ are shown in Figs. 8(a) and 8(b), respectively.

Again, we note that the aggregation of agents' opinions means that many agents select alternatives with high variability of gain and are therefore risk-seeking.

As shown in Fig. 8(a), first- and second-order phase transitions occur in the region where $c < 0$, which suggests that the system is risk-seeking when $c < 0$. This represents the typical risk-averse behavior toward gains and risk-seeking behavior toward losses.

On the other hand, as shown in Fig. 8(b), second-order phase transitions occur in the region where $c > 0$, revealing a reversal of risk preferences. This is consistent with the reversal of preference for low-probability outcomes presented in Table I.

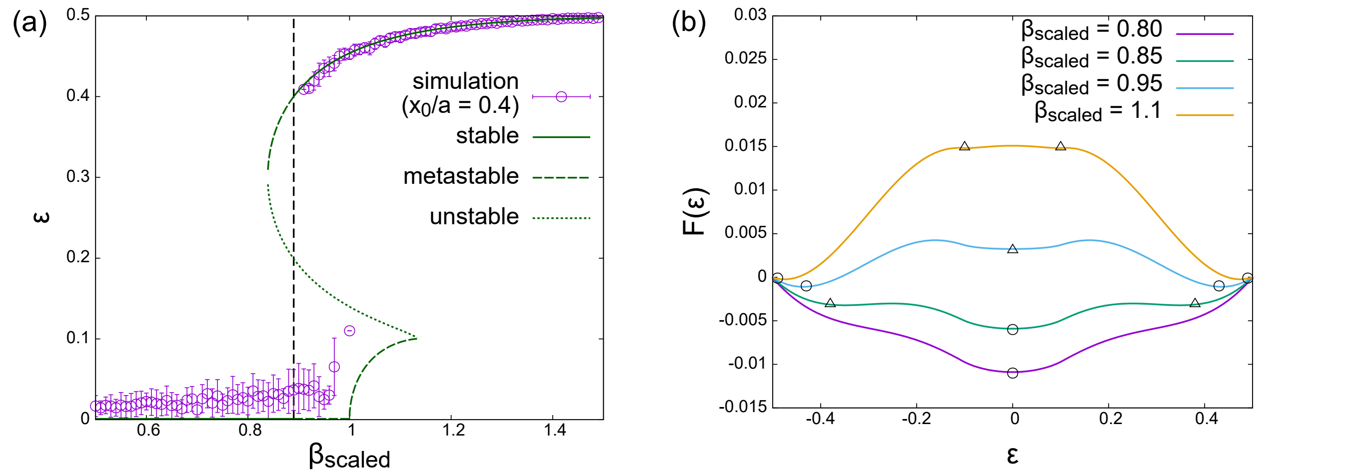

Fig. 6. (Color online) Behaviors of the system at $p = 0.9$ and $x_0/a = 0.40$. (a) Dependence of the equilibrium state on β. The steady states obtained from the Landau free energy are classified into global stable points (solid lines), meta-stable states (dashed lines), and unstable states (dotted lines). Here, $\beta_{\mathrm{scaled}} = 0.89$ is also shown, where $\epsilon = 0$ becomes the meta-stable state. Of note, this transition may be of the first order, although it was expected to be of the second order. (b) Landau free energies at several values of β near the critical point. Globally stable states are indicated by circles and locally stable states are indicated by triangles. A first-order transition-like bifurcation occurs before the second-order transition predicted from Fig. 4.

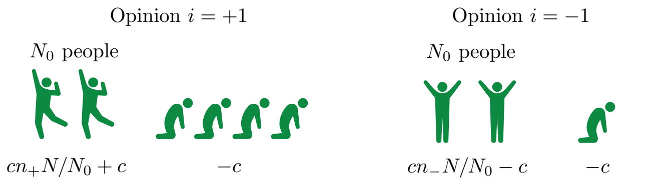

Fig. 7. (Color online) Overview of winners-take-all game. The entry fee is c in each case, and in each bet, only N_0 randomly selected people are winners, splitting the stakes among the group.

6. Conclusion

In this study, we constructed an opinion formulation model based on the prospect theory. The prospect theory not only provides a more realistic model, but is also highly important for the emergence of multiple equilibrium states. For example, in Setting I, the nonlinearity of the value function $v(x)$ plays an essential role in the first-order transitions, and in Setting II, the existence of the probability weighting function $w(p)$ plays an essential role in the aggregation at $c > 0$. Therefore, we achieved our goal of constructing an opinion formation model that incorporates the psychological aspects of people to obtain multiple equilibrium states.

On the other hand, some problems remain in terms of correspondence with reality. In this study, we introduced a probabilistically fluctuating utility, U_i, as in the expected utility theory, and assumed that decision-making depends on it. In the prospect theory, the variability of human decision-making is due to differences in the shapes of individual value functions and probability weighting functions, and it was necessary to make such modifications in this study. In addition, both Settings I and II dealt with a certain type of simplified game, which needs to be examined in order to consider the correspondence with reality.

In addition, it is possible to expand the number of choices to more than three. In such cases, we can expect nontrivial behavior (e.g., cohesion in one opinion or a decrease in the number of people in one opinion),[16] which will lead to more diverse equilibrium states.

We are grateful to Ryosuke Yoneda and Hidetaka Manabe for helpful discussions .

Appendix: Derivation of the selection probability P_+ of $i = +1$ in a consensus model based on the prospect theory

Let $\mathcal{N}(x \mid \mu, \sigma^2)$ be the Gaussian distribution with a mean μ and variance σ^2, and $\Phi(x)$ be the cumulative distribution function of the Gaussian distribution with a mean 0 and variance 1.

First, we note the following integrals[17] for the normal distribution and its cumulative distribution function:

$$\int_{-\infty}^{\infty} \mathcal{N}(a \mid \mu, \sigma^2)\Phi(\lambda a)da = \Phi\left(\frac{\mu}{(\lambda^{-2} + \sigma^2)^{1/2}}\right). \quad \text{(A·1)}$$

Next, consider the situation where the gains of the alternatives $i = \pm 1$ are given by $U_{\mathrm{i}} = V_{\mathrm{i}} + \xi_{\mathrm{i}}$. Here, we assume that ξ_{i} is an independent random number that follows a Gaussian distribution $\mathcal{N}(\xi_{\mathrm{i}} \mid 0, \gamma^2)$ with a mean 0 and variance γ^2. In this case, the selection probability P_+ of $i = +1$ is given as the probability that $U_+ > U_-$.

Using the formula (A·1), we have

$$\begin{aligned} P_+ &= \mathrm{Prob}(V_+ - V_- + \xi_+ > \xi_-) \\ &= \int_{-\infty}^{\infty} d\xi_+ \mathcal{N}(\xi_+ \mid 0, \gamma^2) \\ &\quad \times \int_{-\infty}^{\infty} d\xi_- \mathbb{I}(V_+ - V_- + \xi_- > \xi_+)\mathcal{N}(\xi_- \mid 0, \gamma^2) \\ &= \int_{-\infty}^{\infty} d\xi_+ \mathcal{N}(\xi_+ \mid 0, \gamma^2)\Phi((V_+ - V_- - \xi_+)/\gamma) \\ &= \int_{-\infty}^{\infty} d\tilde{\xi}_+ \mathcal{N}(\tilde{\xi}_+ \mid V_+ - V_-, \gamma^2)\Phi(\tilde{\xi}_+/\gamma) \\ &= \Phi\left(\frac{V_+ - V_-}{\sqrt{2\gamma^2}}\right), \end{aligned} \quad \text{(A·2)}$$

where $\tilde{\xi}_+ = \xi_+ + V_+ - V_-$ is introduced.

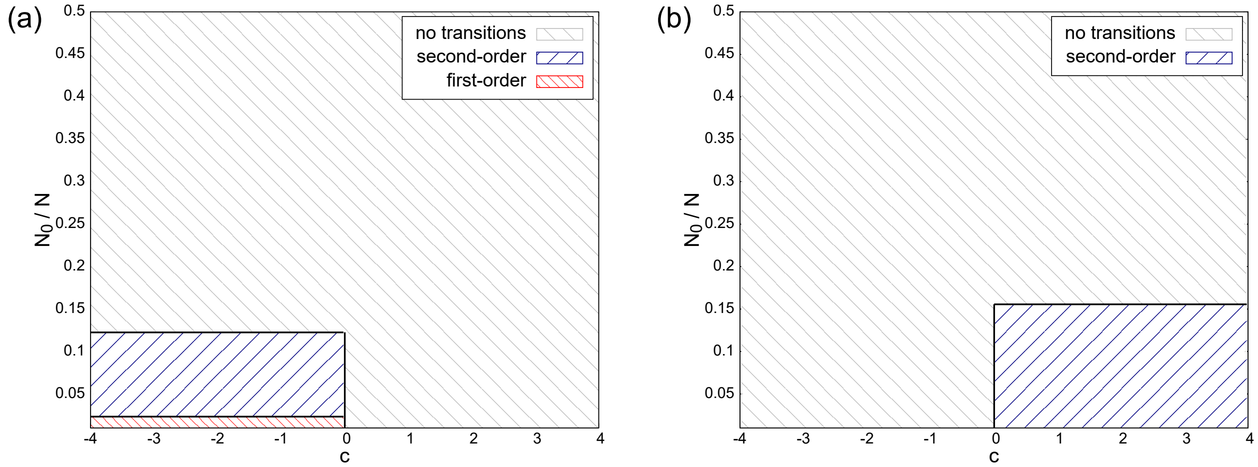

Fig. 8. (Color online) Resulting classifications of phase transitions for setting II. (a) $w(p) = p$. Risk-seeking attitudes toward loss lead to a phase transition or aggregation of opinions at $c < 0$. (b) $w(p) = p^{\gamma}/(p^{\gamma} + (1 - p)^{\gamma})^{1/\gamma}$. This is consistent with the reversal of preference for low probability outcomes presented in Table I.

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