ON A LOCAL INVERSION OF THE X-RAY TRANSFORM FROM ONE SIDED DATA

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ABSTRACT. We explain how the theory of A-analytic maps of A. Bukhgeim can apply to a local CT inversion problem, in which the data is restricted to lines leaning on a given arc.

1. INTRODUCTION

By the time the commercial CT became a widespread diagnostic method in medicine, it was also apparent that X-ray radiation is harmful to human body. In mitigation, the engineering and mathematics communities have proposed various methods to lower the radiation dosage, in particular by inverting the Radon transform from a restricted set of lines. It is well known, that discretization of the set of directions leads to non-unique image reconstructions, see [11]. Moreover, in two dimensions, the inversion of the classical Radon transform is non-local, and, thus, the usage of only those lines that pass through the region of interest may not be enough to uniquely invert it. Several works identify specific subsets of lines which still provide unique reconstruction in the region of interest. Among the mathematics works, which use, roughly, half the data set, we refer to [4, 23, 5] or, in the constant attenuation case to [17, 16, 13, 20, 22]; see also references below.

In this brief note, we are concerned with the inversion question in which the (fan beam) data is collected from "one side". More precisely, let $\Omega \subset \mathbb{R}^2$ be a convex domain and Λ be an arc of its boundary $\varGamma$, see Figure 1 (left) below. The chord L joining the endpoints of the arc Λ partitions the domain in two subdomains $\Omega^{\pm}$, where Ω^{+} denotes the domain enclosed by $\Lambda \cup L$. For a function f compactly supported in Ω, we explain that unique determination of $f|_{\Omega^+}$ from its attenuated X-ray transform over lines leaning on Λ is theoretically possible. Note that, if f happens to also be supported in Ω^-, then its X-ray data is incomplete: while the measurements are affected by the possible nonzero values of $f|_{\Omega^-}$, an entire cone of directions through points in Ω^- are missing in the data.

The unique determination of $f|_{\Omega^+}$ does follow from the support theorem in [7]; also in the attenuated case provided the attenuation is analytic. However. those arguments have yet to yield a method of reconstruction. In here we use the theory of A-analytic maps originally developed by A. Bukhgeim in [8] to address the inversion of the attenuated X-ray transform from complete data set; see [3, 24] for the application to the attenuated case and [25, 26] for extensions to higher order tensors. For different approaches to the inversion of the attenuated X-ray transform from complete data we refer to the original work in [18, 19], and further developments in [15, 6, 4, 12, 14].

The unique determination result here follows from a Carleman type formula for A-analytic maps as in [1, 2]. The novelty of this work is in the explicit Carleman weight-operator, see

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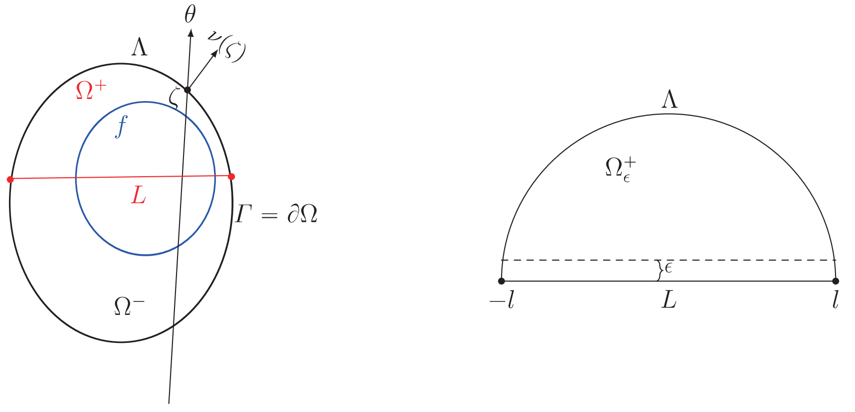

FIGURE 1. (left) Geometric setup: $\partial\Omega^+ = \Lambda \cup L$. (right) Domain Ω_ϵ^+.

equation (6) below, specifically tailored for the convex hull of the arc Λ. The arguments here have been recently refined by the authors to yield a reconstruction method in [10] via the explicit Bukhgeim-Cauchy operator in [9].

The X-ray transform of f is given by,

$$Xf(z,\theta) := \int_{-\infty}^{\infty} f(z+s\theta)ds, \quad (z,\theta) \in \Omega \times \mathbf{S}^1. \tag{1}$$

The reconstruction of f from its ray data is approached through the known equivalence between the X-ray transform and the boundary value problems for the transport equation: Let $\Gamma_\pm := \{(\zeta,\theta) \in \Gamma \times \mathbf{S}^1 : \pm\nu(\zeta)\cdot\theta > 0\}$ denote the outgoing $(+)$, respectively incoming $(-)$ submanifolds of the unit tangent bundle of Γ, with $\nu(\zeta)$ being the outer normal at $\zeta \in \Gamma$ and θ is a direction in the unit sphere $\mathbf{S}^1$. If $u(z,\theta)$ is the unique solution to

$$\theta \cdot \nabla u(z,\theta) = f(z) \quad (z,\theta) \in \Omega \times \mathbf{S}^1, \tag{2a}$$

$$u|_{\Gamma_-} = 0, \tag{2b}$$

then its trace on Γ_+ satisfies

$$u|_{\Gamma_+}(\zeta,\theta) = Xf(\zeta,\theta), \quad (\zeta,\theta) \in \Gamma_+. \tag{3}$$

In our problem here the data Xf is only available on

$$\Lambda_\pm := \{(\zeta,\theta) \in \Lambda \times \mathbf{S}^1 : \pm\nu(\zeta)\cdot\theta > 0\}.$$

Upon a rotation and translation of the domain Ω, we assume without loss of generality that the arc Λ lies in the upper half plane with the endpoints on the real axis lying symmetrically about the origin. In particular, $\Omega \cap \{\mathbb{I}\mathrm{m}\, z = 0\} = L = (-l, l)$, for some $l > 0$. For $\epsilon > 0$, define

$$\Omega_\epsilon^+ = \{z \in \Omega : \mathbb{I}\mathrm{m}\, z > \epsilon\}; \tag{4}$$

see Figure 1 on the right.

2. A Carleman type formula for $\mathcal{L}$-analytic maps in Ω^+

In this section we breifly recall some known properties of A-analytic functions, on which our reconstruction method is based, and present an explicit Carleman weight operator tailored for Ω^+_ϵ. For $z = x + \mathrm{i}y$, let $\overline{\partial} = (\partial_x + \mathrm{i}\partial_y)/2$, and $\partial = (\partial_x - \mathrm{i}\partial_y)/2$ be the Cauchy-Riemann operators.

A sequence valued map $\Omega \ni z \mapsto \mathbf{u}(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), ...\rangle$ in $C(\overline{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty)$ is called *$\mathcal{L}$-analytic*, if

$$\left[\overline{\partial} + \mathcal{L}\partial\right] \mathbf{u}(z) = 0, \quad z \in \Omega, \tag{5}$$

where $\mathcal{L}$ is the left shift operator, $\mathcal{L}\langle u_0, u_{-1}, u_{-2}, \cdots\rangle = \langle u_{-1}, u_{-2}, \cdots\rangle$, and l_∞ is the space of bounded sequences. Note that we use the sequences of non-positive indexes to conform with the notation in Bukhgeim's original work [8].

Unique determination of f follows via a Carleman type formula as in [1], provided a suitable quenching function is known. In here we made explicit such a function tailored for the subdomain Ω^+. More precisely, for $\lambda > 0$, we consider the Carleman weight operator function

$$\Phi_\lambda(z) = e^{-\mathrm{i}\lambda z} e^{\mathrm{i}\lambda \overline{z}\mathcal{L}}, \quad z \in \overline{\Omega^+}, \tag{6}$$

and its inverse $\Phi_\lambda^{-1}(z) = \Phi_\lambda(-z)$. By direct computation, one can check that Φ_λ satisfies the operator valued equation

$$\overline{\partial}\Phi_\lambda(z) + \mathcal{L}\partial\Phi_\lambda(z) = e^{-\mathrm{i}\lambda z} e^{\mathrm{i}\lambda \overline{z}\mathcal{L}} \mathrm{i}\lambda\mathcal{L} + \mathcal{L}(-\mathrm{i}\lambda) e^{-\mathrm{i}\lambda z} e^{\mathrm{i}\lambda \overline{z}\mathcal{L}} = 0.$$

Consequently, if $\mathbf{u}(z)$ is $\mathcal{L}$-analytic in Ω^+, then $\Phi_\lambda(z)\mathbf{u}(z)$ is also $\mathcal{L}$-analytic in Ω^+, so its values can be determined from the boundary $\partial\Omega^+ = \Lambda \cup L$ by

$$\Phi_\lambda(z)\mathbf{u}(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Lambda\cup L} (d\zeta - \mathcal{L}d\overline{\zeta}) G(\zeta - z) \Phi_\lambda(\zeta) \mathbf{u}(\zeta),$$

where $G(z) = (z - \mathcal{L}\overline{z})^{-1}$ is the Green kernel for the differential operator in (5); see [8]. By using the commutating properties $[\Phi_\lambda^{-1}(z), \Phi_\lambda(\zeta)] = 0$, and $[\Phi_\lambda^{-1}(z), \mathcal{L}] = 0$, for $z \in \Omega^+$ and $\zeta \in \Lambda \cup L$, we obtain

$$\mathbf{u}(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Lambda\cup L} (d\zeta - \mathcal{L}d\overline{\zeta}) G(\zeta - z) \Phi_\lambda(\zeta - z) \mathbf{u}(\zeta) \tag{7}$$

We consider for $s \in (0, 1)$, the following space

$$l^{2,s}_\infty(\overline{\Omega}) := \left\{ \mathbf{u} = \langle u_{-1}, u_{-2}, ...\rangle : \sup_{\xi\in\overline{\Omega}} \sum_{j=1}^{\infty} s^{-2j} |u_{-j}(\xi)|^2 < \infty \right\}.$$

The left shift operator $\mathcal{L} : l^{2,s}_\infty \to l^{2,s}_\infty$ is bounded, and the operator norm $\|\mathcal{L}\| = s$.

Theorem 2.1. *For $\epsilon > 0$, let $\Omega^+_\epsilon \subset \mathbb{R}^2$ be the subdomain in* (4)*, d be diameter of Ω^+, and let $s < \dfrac{\epsilon}{d}$. If $\mathbf{u}$ is $\mathcal{L}$-analytic in Ω^+ with $\mathbf{u}|_L \in l^{2,s}_\infty(L)$, then it is uniquely determined by its trace on Λ by*

$$\mathbf{u}(z) = \lim_{\lambda\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\Lambda} (d\zeta - \mathcal{L}d\overline{\zeta}) G(\zeta - z) \Phi_\lambda(\zeta - z) \mathbf{u}(\zeta), \quad z \in \Omega^+_\epsilon, \tag{8}$$

where Φ_λ is the Carleman weight operator in (6).

Proof. For $\lambda > 0$, we consider the Carleman weight operator function Φ_λ in (6). We argue that the integral over the segment L in (7) vanishes in the limit with $\lambda \to \infty$,

$$\lim_{\lambda\to\infty} \int_L (d\zeta - \mathcal{L}d\overline{\zeta}) G(\zeta - z)\Phi_\lambda(\zeta - z)\mathbf{u}(\zeta) = 0. \tag{9}$$

For $z = x + iy \in \Omega^+_\epsilon$ and $\zeta \in L$, we have $\epsilon \leqslant |\overline{z} - \overline{\zeta}| \leqslant d$ and $\left|e^{i\lambda(z-\zeta)}\right| = e^{-\lambda y} \leqslant e^{-\lambda\epsilon}$. If $\|\cdot\|$ denotes the operator norm in $l^{2,s}$, we obtain for any $z \in \Omega^\epsilon$ that

$$\sup_{\zeta\in L} \|\Phi_\lambda(\zeta - z)\| = \|e^{\mathrm{i}\lambda(z-\zeta)}e^{-\mathrm{i}\lambda(\overline{z}-\overline{\zeta})\mathcal{L}}\| \leqslant e^{-\lambda(\epsilon - sd)}.$$

Since $s < \dfrac{\epsilon}{d}$, by letting $\lambda \to \infty$, we conclude (9) $\square$

The source f is recovered in Ω^+_ϵ, by

$$f(z) = 2\,\mathbb{R}\mathrm{e}\{\partial u_{-1}(z)\},$$

where u_{-1} is the first component in (8). The reconstruction to Ω^+ can be completed by a layer stripping argument.

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