The monotonicity method for the inverse crack scattering problem

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1 Introduction

Let $\Gamma \subset \mathbb{R}^2$ be a smooth non-intersecting open arc (crack), and we assume that Γ can be extended to an arbitrary smooth, simply connected, closed curve $\partial\Omega$ enclosing a bounded domain Ω in $\mathbb{R}^2$. Let $k > 0$ be the wave number, and let $\theta \in \mathbb{S}^1$ be incident direction. We consider the following direct scattering problem: For $\theta \in \mathbb{S}^1$ determine u^s such that

$$\Delta u^s + k^2 u^s = 0 \text{ in } \mathbb{R}^2 \setminus \Gamma, \tag{1.1}$$

$$u^s = -\mathrm{e}^{ik\theta\cdot x} \text{ on } \Gamma \tag{1.2}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{1.3}$$

where $r = |x|$, and (1.3) is the *Sommerfeld radiation condition*. It is well known that there exists a unique solution u^s and it has the following asymptotic behaviour:

$$u^s(x) = \frac{\mathrm{e}^{ikr}}{\sqrt{r}}\Big\{u^\infty(\hat{x},\theta) + O\big(1/r\big)\Big\},\ r \to \infty,\quad \hat{x} := \frac{x}{|x|}. \tag{1.4}$$

The function u^∞ is called the *far field pattern* of u^s. With the far field pattern u^∞, we define the *far field operator* $F : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ by

$$Fg(\hat{x}) := \int_{\mathbb{S}^1} u^\infty(\hat{x},\theta) g(\theta) ds(\theta),\ \hat{x} \in \mathbb{S}^1. \tag{1.5}$$

The inverse scattering problem we consider is to reconstruct the unknown arc Γ from the far field pattern $u^\infty(\hat{x},\theta)$ for all $\hat{x} \in \mathbb{S}^1$, all $\hat{x} \in \mathbb{S}^1$ with one $k > 0$. In other words, given the far field operator F, reconstruct Γ.

In order to solve such a problem, we use the monotonicity method. The feature of this method is to understand the inclusion relation of an unknown target and artificial object by comparing the data operator with some operator corresponding to an artificial one. For recent developments of the monotonicity method, we refer to [2, 3, 4]. The following theorems are our main results for solving the inverse crack scattering problem.

Theorem 1.1 (Theorem 1.1 in [1]). *Let $\sigma \subset \mathbb{R}^2$ be a smooth non-intersecting open arc. Then,*

$$\sigma \subset \Gamma \qquad \Longleftrightarrow \qquad H_\sigma^* H_\sigma \leq_{\mathrm{fin}} -\mathrm{Re}F, \tag{1.6}$$

where the Herglotz operator $H_\sigma : L^2(\mathbb{S}^1) \to L^2(\sigma)$ is given by

$$H_\sigma g(x) := \int_{\mathbb{S}^1} \mathrm{e}^{ik\theta\cdot x} g(\theta) ds(\theta),\ x \in \sigma, \tag{1.7}$$

and the inequality on the right-hand side in (1.6) denotes that $-\mathrm{Re}F - H_\sigma^* H_\sigma$ has only finitely many negative eigenvalues, and the real part of an operator F is self-adjoint operators given by $\mathrm{Re}F := \dfrac{1}{2}(F + F^*)$.

Theorem 1.2 (Theorem 1.2 in [1]). *Let $B \subset \mathbb{R}^2$ be a bounded open set. Then,*

$$\Gamma \subset B \quad \Longleftrightarrow \quad -\mathrm{Re}F \leq_{\mathrm{fin}} H^*_{\partial B} H_{\partial B}, \tag{1.8}$$

where $H_{\partial B} : L^2(\mathbb{S}^1) \to L^2(\partial B)$ is given by

$$H_{\partial B} g(x) := \int_{\mathbb{S}^1} \mathrm{e}^{ik\theta \cdot x} g(\theta) ds(\theta), \ x \in \partial B. \tag{1.9}$$

2 Proof of Theorem

We will only prove Theorem 1.1 because Theorem 1.2 is proved by the same argument. We denote by

$$H^{1/2}(\Gamma) := \{u\big|_{\Gamma} \colon u \in H^{1/2}(\partial\Omega)\}$$

$$\tilde{H}^{1/2}(\Gamma) := \{u\big|_{\Gamma} \colon u \in H^{1/2}(\partial\Omega), supp(u) \subset \overline{\Gamma}\}$$

$$H^{-1/2}(\Gamma) := \left(\tilde{H}^{1/2}(\Gamma)\right)'$$

$$\tilde{H}^{-1/2}(\Gamma) := \left(H^{1/2}(\Gamma)\right)'$$

We have the following inclusion relation:

$$\tilde{H}^{1/2}(\Gamma) \subset H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma) \subset H^{-1/2}(\Gamma), \tag{2.1}$$

We denote by the Herglotz operator $\hat{H}_\Gamma : L^2(\mathbb{S}^1) \to H^{1/2}(\Gamma)$

$$\hat{H}_\Gamma g = v_g\big|_{\Gamma}, \tag{2.2}$$

We remark that $\hat{H}_\Gamma : L^2(\mathbb{S}^1) \to H^{1/2}(\Gamma)$ has just a different range from $H_\Gamma : L^2(\mathbb{S}^1) \to L^2(\Gamma)$. We denote by the single layer operator $S : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$

$$S\varphi(x) := \int_{\Gamma} \varphi(y)\Phi(x,y) ds(y), \ x \in \Gamma, \tag{2.3}$$

where $\Phi(x,y)$ is the Green's function for Helmholtz equation in $\mathbb{R}^2$,

$$\Phi(x,y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \ x \neq y. \tag{2.4}$$

Lemma 2.1 (Kirsch and Ritter 2000, [5]). *The far field operator F has the following factorization:*

$$F = -\hat{H}^*_\Gamma S^{-1} \hat{H}_\Gamma. \tag{2.5}$$

Furthermore, S^{-1} is of the form

$$S^{-1} = C + K, \tag{2.6}$$

wher K is some compact operator, and C is some self-adjoint and coercive operator, i.e., there exists $c_0 > 0$ such that

$$\langle \varphi, C\varphi \rangle \geq c_0 \left\|\varphi\right\|^2 \ \ for \ all \ \varphi. \tag{2.7}$$

($\Longrightarrow$) Assume that $\sigma \subset \Gamma$. Let $R : L^2(\Gamma) \to L^2(\sigma)$ be the restriction operator, i.e., $Rf := f|_\sigma$. Then,

$$H_\sigma = RH_\Gamma. \tag{2.8}$$

Let $J : H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ be a compact embedding map. Then,

$$H_\Gamma = J\hat{H}_\Gamma, \tag{2.9}$$

which leads to

$$H_\sigma = RJ\hat{H}_\Gamma. \tag{2.10}$$

Then,

$$\begin{aligned} -\mathrm{Re}F - H_\sigma^* H_\sigma &= \hat{H}_\Gamma^*\big[\mathrm{Re}S^{-1} - J^*R^*RJ\big]\hat{H}_\Gamma \\ &= \hat{H}_\Gamma^*\big[C + \mathrm{Re}K - J^*R^*RJ\big]\hat{H}_\Gamma \\ &=: \hat{H}_\Gamma^*\big[C + \tilde{K}\big]\hat{H}_\Gamma, \end{aligned} \tag{2.11}$$

where $\tilde{K}$ is a self-adjoint and compact operator. Let $\{(\lambda_j, \psi_j) | j \in \mathbb{N}\}$ be complete eigensystem of $\tilde{K}$. Let $V := \mathrm{span}\{\psi_j | \lambda_j < -c_0\}$, where a constant number $c_0 > 0$ appears in Lemma 2.1, and it is finite dimensional. Then, for all $v \in V^\perp = \mathrm{span}\{\psi_j | \lambda_j \geq -c_0\}$

$$\langle (C + \tilde{K})v, v\rangle \geq 0, \tag{2.12}$$

which implies that for all $g \in \big[\hat{H}_\Gamma^*(V)\big]^\perp$ ($\Longleftrightarrow \hat{H}_\Gamma g \in V^\perp$)

$$\langle (-\mathrm{Re}F - H_\sigma^* H_\sigma)g, g\rangle = \langle (C + K)\hat{H}_\Gamma g, \hat{H}_\Gamma g\rangle \geq 0, \tag{2.13}$$

and $\dim[\hat{H}_\Gamma^*(V)] \leq \dim(V) < \infty$. By Corollary 3.3 of [3], we conclude that

$$H_\sigma^* H_\sigma \leq_{\mathrm{fin}} -\mathrm{Re}F. \tag{2.14}$$

($\Longleftarrow$) Let $\sigma \not\subset \Gamma$. We assume on the contrary $H_\sigma^* H_\sigma \leq_{\mathrm{fin}} -\mathrm{Re}F$, i.e, by Corollary 3.3 of [3] there exists a finite dimensional subspace V s.t. for all $g \in V^\perp$

$$\langle (-\mathrm{Re}F - H_\sigma^* H_\sigma)g, g\rangle \geq 0. \tag{2.15}$$

We choose a small open arc σ_0 s.t. $\sigma_0 \subset \sigma$ and $\sigma_0 \cap \Gamma = \emptyset$. Then, we have for all $g \in V^\perp$

$$\begin{aligned} \|H_{\sigma_0} g\|^2_{L^2(\sigma_0)} &\leq \|H_\sigma g\|^2_{L^2(\sigma)} \\ &\leq \langle (-\mathrm{Re}F)g, g\rangle \\ &= \langle (\mathrm{Re}S^{-1})\hat{H}_\Gamma g, \hat{H}_\Gamma g\rangle \\ &\leq \big\|\mathrm{Re}S^{-1}\big\| \, \|H_\Gamma g\|^2_{L^2(\Gamma)}. \end{aligned} \tag{2.16}$$

Lemma 2.2 (Harrach et al. 2019, [3])**.** *Let X, Y, and Z be Hilbert spaces, and let $A : X \to Y$ and $B : X \to Z$ be bounded linear operators, and let $V \subset X$ be a finite dimensional subspace. Then,*

$$\exists C > 0 : \ \|Ax\|^2 \leq C\,\|Bx\|^2 \ \textit{ for all } x \in V^\perp \qquad \Longleftrightarrow \qquad \mathrm{Ran}(A^*) \subseteq \mathrm{Ran}(B^*) + V, \tag{2.17}$$

By this lemma, we have

$$\mathrm{Ran}(H_{\sigma_0}^*) \subset \mathrm{Ran}(H_\Gamma^*) + V. \tag{2.18}$$

On the other hand,

Lemma 2.3 (Harrach et al. 2019, [3])**.** *Let X, Y, $V \subset Z$ be subspaces of a vector space Z. If*

$$X \cap Y = \{0\}, \quad \textit{and} \quad X \subseteq Y + V, \tag{2.19}$$

then, $\dim(X) \leq \dim(V)$.

Lemma 2.4 (Furuya et al. 2020, [1])**.** *(a)* $\dim(\mathrm{Ran}(H_{\sigma_0}^*)) = \infty$

(b) $\mathrm{Ran}(H_{\sigma_0}^*) \cap \mathrm{Ran}(H_\Gamma^*) = \{0\}$.

As $X = \mathrm{Ran}(H_{\sigma_0}^*)$, $Y = \mathrm{Ran}(H_\Gamma^*)$, and $V = V$, we apply contraposition of Lemma 2.3. We remark that $\infty = \dim(\mathrm{Ran}(H_{\sigma_0}^*)) \nleq \dim(V) < \infty$. Then, $\mathrm{Ran}(H_{\sigma_0}^*) \not\subset \mathrm{Ran}(H_\Gamma^*) + V$, which contradicts (2.18). Therefore, we conclude that $H_\sigma^* H_\sigma \nleq_{\mathrm{fin}} -\mathrm{Re}F$. □

3 Numerical examples

Based on Theorem 1.1, we give numerical examples. The indicator function in our examples is given by

$$I(\sigma) := \#\left\{\text{negative eigenvalues of } - \mathrm{Re}F - H_\sigma^* H_\sigma\right\}. \tag{3.1}$$

The idea to reconstruct Γ is to plot the value of $I(\sigma)$ for many of small σ in the sampling region. Then, we expect from Theorem 1.1 that the value of the function $I(\sigma)$ is low if σ is close to Γ.

Here, σ is chosen in two ways; One is the vertical line segment $\sigma_{i,j}^{ver} := z_{i,j} + \{0\} \times [-\frac{R}{2M}, \frac{R}{2M}]$ where $z_{i,j} := (\frac{Ri}{M}, \frac{Rj}{M})$ $(i, j = -M, -M+1, ..., M)$ denote the center of $\sigma_{i,j}^{ver}$, and $\frac{R}{M}$ is the length of $\sigma_{i,j}^{ver}$, and $R > 0$ is length of sampling square region $[-R, R]^2$, and $M \in \mathbb{N}$ is large to take a small segment. The other is horizontal one $\sigma_{i,j}^{hor} := z_{i,j} + [-\frac{R}{2M}, \frac{R}{2M}] \times \{0\}$.

The far field operator F is approximated by the matrix

$$F \approx \frac{2\pi}{N}\left(u^\infty(\hat{x}_l, \theta_m)\right)_{1 \le l,m \le N} \in \mathbb{C}^{N \times N}, \tag{3.2}$$

where $\hat{x}_l = \left(\cos(\frac{2\pi l}{N}), \sin(\frac{2\pi l}{N})\right)$ and $\theta_m = \left(\cos(\frac{2\pi m}{N}), \sin(\frac{2\pi m}{N})\right)$. The operator $H_\sigma^* H_\sigma$ is approximated by

$$H_\sigma^* H_\sigma \approx \frac{2\pi}{N}\left(\int_\sigma e^{iky\cdot(\theta_m - \hat{x}_l)} dy\right)_{1 \le l,m \le N} \in \mathbb{C}^{N \times N}. \tag{3.3}$$

In our examples, we fix $R = 1.5$, $M = 100$, $N = 20$, and wavenumber $k = 1$, and consider the true shape of Γ as a blue curve in Figure 1. Figures 2 and 3 are given by plotting the values of the vertical and horizontal indicator functions

$$I_{ver}(z_{i,j}) := I(\sigma_{i,j}^{ver}), \qquad I_{hor}(z_{i,j}) := I(\sigma_{i,j}^{hor}), \tag{3.4}$$

for each $i, j = -100, 99, ..., 100$, respectively.

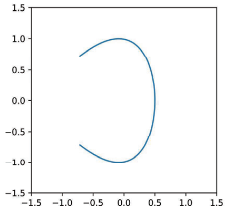

Figure 1: true shape of Γ

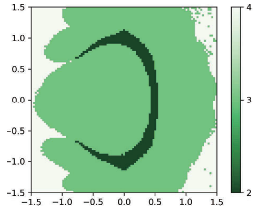

Figure 2: vertical

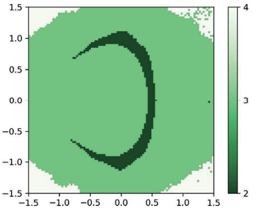

Figure 3: horizontal

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