Reconstruction of the defect by the enclosure method for inverse problems of the magnetic Schrödinger operator

Ryusei Yamashita

Tokyo Metropolitan University

This study is based on the paper [5]. We show a reconstruction formula of the convex hull of the defect D from the Dirichlet to Neumann map associated with the magnetic Schrödinger operator by using the enclosure method proposed by Ikehata [2], assuming certain higher regularity for the potentials of the magnetic Schrödinger operator, under the Dirichlet condition or the Robin condition on the boundary ∂D in the two and three dimensional case.

Let $\Omega \subset R^n (n = 2, 3)$ be a bounded domain where the boundary $\partial\Omega$ is C^2 and let D be an open set satisfying $\overline{\mathrm{D}} \subset \Omega$ and $\Omega \setminus \overline{\mathrm{D}}$ is connected. The defect D consists of the union of disjoint bounded domains $\{D_j\}_{j=1}^n$, where the boundary of D is Lipschitz continuous. First, we define the DN map for the magnetic Schrödinger equation with no defect D in Ω. Here, let $D_A^2 u := \sum_{j=1}^n D_{A,j}(D_{A,j}u)$, where $D_{A,j} := \frac{1}{i}\partial_j + A_j$ and $A = (A_1, A_2, \cdots, A_n)$.

Definition 1. Suppose $q \in L^\infty(\Omega), q \geq 0, A \in C^1(\overline{\Omega}, R^n)$. For a given $f \in H^{1/2}(\partial\Omega)$, we say $u \in H^1(\Omega)$ is a weak solution to the following boundary value problem for the magnetic Schrödinger equation

$$\begin{cases} D_A^2 u + qu = 0 \text{ in } \Omega, \\ u = f \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

if $u = f$ on $\partial\Omega$ and u satisfies

$$\int_\Omega (D_A u) \cdot \overline{D_A \varphi} + qu\overline{\varphi}\, dx = 0$$

for any $\varphi \in H^1(\Omega)$ such that $\varphi|_{\partial\Omega} = 0$. Here, $\overline{\varphi}$ is the complex conjugate of φ.

The DN map $\Lambda_{q,A}$ is defined as follows.

Definition 2. (Weak formulation of DN map)
The DN map $\Lambda_{q,A} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is defined as follows by the duality:

$$\langle \Lambda_{q,A} f\ ,\ \overline{g} \rangle = \int_\Omega (D_A u) \cdot \overline{D_A v} + qu\overline{v}\, dx, \quad f, g \in H^{1/2}(\partial\Omega),$$

where $u \in H^1(\Omega)$ is the weak solution of (1.1) and $v \in H^1(\Omega)$ is any function satisfying $v|_{\partial\Omega} = g$.

We define the weak solution of the magnetic Schrödinger equation with a defect D in Ω under the Robin boundary condition on ∂D.

Definition 3. (Robin case)
Suppose $q \in L^\infty(\Omega \setminus \overline{D}), q \geq 0, \lambda \in C^1(\partial D), \lambda \geq 0$ and $A \in C^1(\overline{\Omega \setminus \overline{D}}, R^n)$. Let ν is the outward unit normal vector to $\Omega \setminus \overline{D}$. For a given $f \in H^{1/2}(\partial\Omega)$, we say $u \in H^1(\Omega \setminus \overline{D})$ is a weak solution to the following value problem for the magnetic Schrödinger equation

$$\begin{cases} D_A^2 u + qu = 0 \text{ in } \Omega \setminus \overline{D}, \\ \nu \cdot (\nabla + iA)u + \lambda u = 0 \text{ on } \partial D, \\ u = f \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

if $u = f$ on $\partial\Omega$ and u satisfies

$$\int_{\Omega\setminus\overline{D}} (D_A u) \cdot \overline{D_A \varphi} + qu\overline{\varphi}\, dx + \int_{\partial D} \lambda u \overline{\varphi}\, dS = 0$$

for any $\varphi \in H^1(\Omega \setminus \overline{D})$ such that $\varphi|_{\partial\Omega} = 0$.

The DN map $\Lambda_{q,A,D}^{(R)}$ is defined as follows.

Definition 4. (DN map of the Robin case)
The DN map $\Lambda^{(R)}_{q,A,D} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ is defined as follows by the duality:

$$\langle \Lambda^{(R)}_{q,A,D} f \ , \ \overline{g} \rangle = \int_{\partial D} \lambda u \overline{v} \, dS + \int_{\Omega \setminus \overline{D}} (D_A u) \cdot \overline{D_A v} + q u \overline{v} \, dx, \quad f, g \in H^{1/2}(\partial\Omega),$$

where $u \in H^1(\Omega \setminus \overline{D})$ is the weak solution of (1.3) and $\varphi \in H^1(\Omega \setminus \overline{D})$ is any function $\varphi|_{\partial\Omega} = g$.
In the special case $\lambda = 0$, we denote $\Lambda^{(N)}_{q,A,D}$ instead of $\Lambda^{(R)}_{q,A,D}$.

Remark 1. The weak solution of the magnetic Schrödinger equation with a defect D in Ω under the Dirichlet boundary condition on ∂D and the DN map $\Lambda^{(D)}_{q,A,D}$ can be defined in a similar way.

Next, we introduce an indicator function that plays an important role in the enclosure method. We denote by S^{n-1} the set of n-dimensional unit vectors ($n = 2, 3$). For a given $\omega \in S^{n-1}$, we can take an orthogonal unit vector $\omega^\perp \in S^{n-1}$, namely $\omega \cdot \omega^\perp = 0$. Then we can construct a solution $v_\tau(x; \omega) := e^{\tau x \cdot (\omega + i\omega^\perp)}(1 + r_\tau(x; \omega))$ of $D_A^2 v + qv = 0$, where $r_\tau(x; \omega)$ is chosen suitably associated with the parameter $\tau \in R$. This solution is called the complex geometrical optics solutions.

Definition 5. (Indicator function)
Let $t, \tau \in \mathbb{R}$. Then, the indicator function $I_\omega(\tau; t)$ is defined as follows.

$$I^{(R)}_\omega(\tau; t) := \langle (\Lambda_{q,A} - \Lambda^{(R)}_{q,A,D})(e^{-\tau t} v_\tau(x; \omega)), \overline{e^{-\tau t} v_\tau(x; \omega)} \rangle$$

Here, $\overline{v_\tau}$ is the complex conjugate of v_τ. In the special case $\lambda = 0$, we denote $\Lambda^{(N)}_{q,A,D}$ instead of $\Lambda^{(R)}_{q,A,D}$. Also, $I^{(D)}_\omega(\tau; t)$ can be defined by $\Lambda^{(D)}_{q,A,D}$. We define the support function $h_D(\omega)$ as follows :

$$h_D(\omega) = \sup_{x \in D} x \cdot \omega, \ \omega \in S^{n-1}.$$

Then it is well-known that the convex hull conv(D) of D is obtained as follows.

$$\text{conv}(D) := \cap_{\omega \in S^{n-1}} \{x \in R^n | x \cdot \omega < h_D(\omega)\}.$$

Since the indicator function $I_\omega(\tau; t)$ is determined from the DN map, if the support function $h_D(\omega)$ is obtained from the indicator function $I_\omega(\tau; t)$, the convex hull conv(D) of inclusion D can be reconstructed from the observation data on boundary $\partial\Omega$. Now, we give the formula of the reconstruction of the support function from the indicator function under a certain smallness condition for the vector potential A.

Theorem 1. *Suppose ∂D is Lipschitz continuous. Let $n = 2, 3, q \in H^2(\Omega), q \geq 0, A \in H^3(\Omega)$ and $C(\Omega)\|A\|_{H^2(\Omega)} \leq \frac{1}{2}$. Then, we have*

$$\lim_{\tau \to \infty} \frac{\log |I^{(D)}_w(\tau; 0)|}{2\tau} = h_D(w), \lim_{\tau \to \infty} \frac{\log |I^{(N)}_w(\tau; 0)|}{2\tau} = h_D(w),$$

for any $\omega \in S^{n-1}$. Here, the constant $C(\Omega)$ depends only on Ω.

For a given $\omega \in S^{n-1}$, we furthermore assume the following condition $(D)_\omega$ for the Robin case.

$(D)_\omega$**:** Suppose ∂D is C^2 and the set $T(\omega) := \{x \in \overline{D} \mid h_D(\omega) - x \cdot \omega = 0\}$ consists of only one point $x_0 \in \partial D$. Furthermore, we assume that in the neighborhood of x_0 the boundary ∂D can be expressed as $y = f(s), |s| < \epsilon, s \in R^{n-1}$, and there exists $K_0, K_1 > 0, m_\omega \geq 2$ such that

$$K_0 |s|^{m_w} \leq f(s) \leq K_1 |s|^{m_w} \quad (|s| < \epsilon).$$

Theorem 2. (*Robin case*) *Suppose $\lambda \neq 0, \lambda \geq 0$ and $\lambda \in C^1(\partial D)$. Let $n = 2, 3, q \in H^2(\Omega), q \geq 0, A \in H^3(\Omega)$ and $C(\Omega)\|A\|_{H^2(\Omega)} \leq \frac{1}{2}$. We assume that the condition $(D)_\omega$ holds as $2 \leq m_w < 3$ for some $\omega \in S^{n-1}$. Then, we have*

$$\lim_{\tau \to \infty} \frac{\log |I^{(R)}_\omega(\tau; 0)|}{2\tau} = h_D(\omega).$$

See [5] for the proof of Theorem 1. We present the basic estimates for the DN maps in the Robin case.

Proposition 1. *Let $\lambda \neq 0, \lambda \geq 0$ and $\lambda \in C^1(\partial D)$. Let L be a constant satisfying $\|\lambda\|_{L^\infty(\partial D)} \leq L$. Assume ∂D is C^2. Take any $y_0 \in \partial D$, for a given $f \in H^{\frac{1}{2}}(\partial\Omega)$, $v \in H^1(\Omega)$ is a weak solution of* (1.1). *Let $q = \frac{1}{2}$ when $n = 3$ and $q = 1 - \epsilon$ for any $0 < \epsilon < 1$ when $n = 2$. Then, there exist positive constants $C_1 = C_1(\Omega, D, \epsilon), C_2 = C_2(\Omega, L, \epsilon)$ such that*

$$\begin{aligned}
&\int_D |D_A v|^2 dx - C_2\{\int_D |v|^2\, dx + (\int_{\partial D} |y - y_0|^q |\frac{\partial v}{\partial \nu}|\, dS)^2 + \int_{\partial D} |v|^2\, dS\} \\
&\leq \langle (\Lambda_{q,A} - \Lambda^{(R)}_{q,A,D}) f\ ,\ \overline{f} \rangle \\
&\leq C_1(\|D_A v\|^2_{L^2(D)} + \|v\|^2_{L^2(D)}) + C_2\{(\int_{\partial D} (|y - y_0|^q |\frac{\partial v}{\partial \nu}|\, dS)^2 + \int_{\partial D} |v|^2\, dS\}.
\end{aligned}$$

To prove Proposition 1, we prepare the following two lemmas.

Lemma 1. *Let $v \in H^1(\Omega)$ and $u \in H^1(\Omega \setminus \overline{D})$ are weak solutions of* (1.1) *and* (1.3), *respectively. We have for $w := u - v$,*

$$\begin{aligned}
&\langle (\Lambda_{q,A} - \Lambda^{(R)}_{q,A,D}) f\ ,\ \overline{f} \rangle \\
&= \int_{\Omega \setminus \overline{D}} |D_A w|^2 + q|w|^2\, dx + \int_D |D_A v|^2 + q|v|^2\, dx - (\int_{\partial D} \lambda u \overline{v} - \lambda |u|^2 + \lambda \overline{u} v\, dS).
\end{aligned}$$

We need the following estimate for the Robin case. We follow the argument in [3], where the proof is given for the three-dimensional case.

Lemma 2. *Assume ∂D is C^2. Let $L \geq 0$ be a constant satisfying $\|\lambda\|_{L^\infty(\partial D)} \leq L$. Take any $y_0 \in \partial D$. For a given $f \in H^{\frac{1}{2}}(\partial\Omega)$, $v \in H^1(\Omega)$ and $u \in H^1(\Omega \setminus \overline{D})$ are weak solutions of* (1.1) *and* (1.3), *respectively. Let $q = \frac{1}{2}$ when $n = 3$ and $q = 1 - \epsilon$ for any $0 < \epsilon < 1$ when $n = 2$. Then, there exists a positive constant C such that*

$$\begin{aligned}
&\int_{\partial D} |u - v|^2\, dS \leq \\
&C \|D_A w\|_{L^2(\Omega \setminus \overline{D})} \Big(\int_{\partial D} |y - y_0|^q |\frac{\partial v}{\partial \nu}|\, dS + \| |A|^2 + q \|_{L^\infty(D)} \int_D |v|\, dx + L \int_{\partial D} |v|\, dS \Big).
\end{aligned}$$

Remark 2. To show Lemma 2, we need to assume that λ is a real-valued function for the case $A \neq 0$. If $A = 0$, we can allow λ to be a complex-valued function (see Ikehata [3]).

By Lemma 1 and 2, we obtain Proposition 1. To prove the asymptotic formula for the indicator function under the Robin condition on ∂D, we need the following basic lemmas.

Lemma 3. *Let $v_\tau = v_\tau(x; \omega) = e^{\tau x \cdot (\omega + i\omega^\perp)}(1 + r_\tau(x; \omega))$ be the complex geometrical optics solution as $\zeta = \tau(\omega - i\omega^\perp)$, where $\tau > 0$ and $\omega, \omega^\perp \in S^{n-1}$ satisfying $\omega \cdot \omega^\perp = 0$. Assume $\|A\|_{H^2(\Omega)}$ is sufficiently small. Then, there exists a constant C such that*

$$\frac{1}{4}\tau^2 \int_D e^{2\tau x \cdot \omega}\, dx \leq \int_D |D_A v_\tau|^2 dx \leq C\tau^2 \int_D e^{2\tau x \cdot \omega} dx$$

for sufficient large τ and

$$\int_D |v_\tau|^2 dx \leq C \int_D e^{2\tau x \cdot \omega}\, dx.$$

Lemma 4. (cf. Ikehata [2, Proposition 2.3])
Let ∂D is Lipschitz continuous. There exists $C_\omega > 0, \tau_\omega > 0$ such that

$$\tau^2 \int_D e^{-2\tau(h_D(\omega) - x \cdot \omega)}\, dx \geq C_\omega \tau^{1 - p_\omega} \quad (\tau \geq \tau_\omega)$$

with

$$p_\omega = \begin{cases} 2 & (n = 3) \\ 1 & (n = 2), \end{cases}$$

for $\omega \in S^{n-1}$. Especially, when we assume furthermore the condition $(D)_\omega$ and the graph $y = f(s)$ representing ∂D, satisfies $f(s) \leq g(s) = L|s|^{m_\omega}$ near $x_0 \in T(\omega)$. We have following estimate:

$$\tau^2 \int_D e^{-2\tau(h_D(\omega) - x \cdot \omega)}\, dx \geq \begin{cases} C_w \tau^{1 - \frac{2}{m_\omega}} & (n = 3), \\ C_w \tau^{1 - \frac{1}{m_\omega}} & (n = 2) \end{cases}$$

for any $\tau \geq \tau_w$.

Lemma 5. (cf. Ikehata [1, Lemma 4.2])
Assume $(D)_\omega$ *for* $\omega \in S^{n-1}$ *and* $x_0 \in T(\omega)$ *which appeared in the assumption* $(D)_\omega$.

(1) *Let* $n = 3$. *Then, there exist constants* τ_ω *and* K *such that*

$$\left(\tau \int_{\partial D} |x - x_0|^{\frac{1}{2}} e^{\tau(x\cdot\omega - h_D(\omega)} \, dS\right)^2 \leq K\tau^{2-\frac{5}{m_\omega}} \quad (\tau \geq \tau_w),$$

and

$$\int_{\partial D} e^{\tau(x\cdot\omega - h_D(\omega)} \, dS \leq K\tau^{-\frac{2}{m_\omega}}.$$

(2) *Let* $n = 2$. *Then, for any* $0 < \epsilon < 1$, *there exist* τ_ω *and* K *such that*

$$\left(\tau \int_{\partial D} |x - x_0|^{1-\epsilon} e^{\tau(x\cdot\omega - h_D(\omega)} \, dS\right)^2 \leq K\tau^{2-\frac{4-2\epsilon}{m_\omega}} \quad (\tau \geq \tau_\omega).$$

Proof of Theorem 2. By the definition of $I_\omega^{(R)}(\tau; t)$ and Proposition 1, we have

$$I_3(\tau) \leq I_\omega^{(R)}(\tau, 0)e^{-2\tau h_D(\omega)} = I_\omega^{(R)}(\tau; h_D(\omega)) \leq I_4(\tau),$$

where

$$\begin{aligned} I_3(\tau) &= \int_D |D_A e^{-\tau(h_D(\omega))} v_\tau|^2 \, dx - C_2(L)\{\int_D |e^{-\tau(h_D(\omega))} v_\tau|^2 \, dx \\ &+ (\int_{\partial D} |x - x_0|^q |D_A e^{-\tau(h_D(\omega))} v_\tau| \, dS)^2 + \int_{\partial D} |e^{-\tau(h_D(\omega))} v_\tau|^2 \, dS\}, \\ I_4(\tau) &= C_1(D)(\int_D |D_A e^{-\tau(h_D(\omega))} v_\tau|^2 \, dx + \int_D |e^{-\tau(h_D(\omega))} v_\tau|^2 \, dx) \\ &+ C_2(L)\{(\int_{\partial D} |x - x_0|^q |D_A e^{-\tau(h_D(\omega))} v_\tau| \, dS)^2 + \int_{\partial D} |e^{-\tau(h_D(\omega))} v_\tau|^2 \, dS\}. \end{aligned}$$

Since $x \cdot \omega - h_D(\omega) \leq 0$ $(x \in D)$, it follows

$$I_4(\tau) \leq C\tau^2.$$

Lemma 4 implies for large $\tau \geq \tau_\omega$

$$\begin{aligned} & C\tau^2 \int_D e^{2\tau(x\cdot\omega - h_D(\omega))} \, dx - C' \int_D e^{2\tau(x\cdot\omega - h_D(\omega))} \, dx \\ \geq \; & \frac{C}{2}\tau^2 \int_D e^{2\tau(x\cdot\omega - h_D(\omega))} \, dx \geq C \begin{cases} \tau^{1-\frac{2}{m_\omega}} & (n = 3) \\ \tau^{1-\frac{1}{m_\omega}} & (n = 2). \end{cases} \end{aligned}$$

On the other hand, Lemma 5 implies for large $\tau \geq \tau_\omega$

$$\begin{aligned} \int_{\partial D} \left| e^{\tau(x\cdot(\omega + i\omega^\perp) - h_D(\omega))}(1 + r) \right|^2 dS &\leq C \int_{\partial D} e^{2\tau(x\cdot\omega - h_D(\omega))} \, dS \\ &\leq \begin{cases} C\tau^{-\frac{2}{m_\omega}} & (n = 3) \\ C & (n = 2). \end{cases} \end{aligned}$$

Furthermore, since there exists a constant C such that $|r(x)| \leq C, |\nabla r(x)| \leq C\tau$ $(x \in D)$, we can estimate as follows:

$$\begin{aligned} & \left(\int_{\partial D} |x - x_0|^q \left| \frac{\partial}{\partial \nu} \left(e^{\tau(x\cdot(\omega + i\omega^\perp)) - \tau h_D(\omega)}(1 + r(x)) \right) \right| dS \right)^2 \\ \leq \; & \left(\int_{\partial D} |x - x_0|^q \left(|\tau(\omega + i\omega^\perp) \cdot \nu(1 + r(x))| + |\nabla r(x)| \right) e^{\tau(x\cdot\omega - h_D(\omega))} \, dS \right)^2 \\ \leq \; & C\left(\tau \int_{\partial D} |x - x_0|^q e^{\tau(x\cdot\omega - h_D(\omega))} \, dS\right)^2 \leq \begin{cases} CK\tau^{2-\frac{5}{m_\omega}} & (n = 3) \\ CK\tau^{2-\frac{4-2\epsilon}{m_\omega}} & (n = 2). \end{cases} \end{aligned}$$

Combining these estimates, we have for $\tau \geq \tau_\omega$

$$I_3(\tau) \geq \begin{cases} \frac{C}{2}\tau^{1-\frac{2}{m_\omega}} - C\tau^{-\frac{2}{m_\omega}} - CK\tau^{2-\frac{5}{m_\omega}} & (n=3) \\ \frac{C}{2}\tau^{1-\frac{1}{m_\omega}} - C - CK\tau^{2-\frac{4-2\epsilon}{m_\omega}} & (n=2). \end{cases}$$

Note that $1 - \frac{2}{m_\omega} > 2 - \frac{5}{m_\omega}$ for $n = 3$ and $1 - \frac{1}{m_\omega} > 2 - \frac{4-2\epsilon}{m_\omega}$ for $n = 2$, since $2 \leq m_\omega < 3$. Here, we take $0 < \epsilon < 1$ sufficiently small such that $3 - 2\epsilon > m_\omega$. So, for τ_ω large enough, there exists a positive constant C such that

$$I_3(\tau) \geq \begin{cases} C\tau^{1-\frac{2}{m_\omega}} & (n=3) \\ C\tau^{1-\frac{1}{m_\omega}} & (n=2) \end{cases}$$

for $\tau \geq \tau_\omega$. Thus, it follows

$$C\tau^\alpha \leq e^{-2\tau h_D(\omega)} I_\omega^{(R)}(\tau, 0) \leq C\tau^2 \quad (\tau \geq \tau_\omega),$$

where

$$\alpha := \begin{cases} 1 - \frac{2}{m_\omega} & (n=3) \\ 1 - \frac{1}{m_\omega} & (n=2). \end{cases}$$

Then, we have

$$\log C + \alpha \log \tau \leq -2\tau h_D(\omega) + \log |I_\omega^{(R)}(\tau, 0)| \leq \log C + 2\log \tau \quad (\tau \geq \tau_\omega).$$

Now, we can conclude

$$\lim_{\tau\to\infty} \frac{\log |I_\omega^{(R)}(\tau, 0)|}{2\tau} = h_D(\omega).$$

□

References

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Department of Mathematical Sciences
Tokyo Metropolitan University,
Minami-Ohsawa 1-1, Hachioji, Tokyo, 192-0397 Japan.
yamashita@uitec.ac.jp