# On an inverse Robin eigenvalue problem appearing in thin coating problems

Toshiaki Yachimura
Institute for the Advanced Study of Human Biology (WPI-ASHBi),
Kyoto University Institute for Advanced Study

## 1 Introduction and problem setting

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with boundary $\partial\Omega$ of class $C^2$, and $\gamma$, $\Gamma_D$ be disjoint nonempty closed subsets of the boundary $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \gamma$. Let $h \in C^0(\gamma)$ and $h > 0$. In this paper, we consider the following Robin eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ hu + \partial_\nu u = 0 & \text{on } \gamma, \end{cases} \tag{1.1}$$

where $\nu$ is the outward unit normal vector of $\partial\Omega$. We only consider the principal eigenvalue and eigenfunction of (1.1), and assume that the principal eigenfunction is positive and it is normalized by

$$\int_\Omega |u|^2 \, dx = 1.$$

Our aim in this paper is to study an inverse problem of the Robin eigenvalue problem. In particular, we consider the recovery of an unknown Robin coefficient $h$ defined in the inaccessible part $\gamma$ of the boundary $\partial\Omega$, given the principal eigenvalue $\lambda(h)$ and the Neumann data $\partial_\nu u(h)|_{\Gamma_D}$ on the accessible part $\Gamma_D$.

The inverse problem appears in thin coating problems [10, 20, 24]. Physically speaking, it is closely related to the problem that we determine a thin insulator coating for a heat conductor by measurements of the first eigenvalue and the Neumann data of an accessible part of the boundary.

This inverse eigenvalue problem is considered by [3, 2]. They dealt with the Robin inverse eigenvalue problem when the support of the Robin coefficient is sufficiently small and gave a non-iterative algorithm for detecting the Robin coefficient from the measurements of an eigenvalue and a Neumann data of the accessible part of the boundary.

We remark that the setting of the inverse problem is similar to the detection problem of internal corrosion. Let us consider the following problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ hu + \partial_\nu u = 0 & \text{on } \gamma. \end{cases} \tag{1.2}$$

The detection problem of internal corrosion is to recover of an unknown Robin coefficient $h$ defined in the inaccessible part $\gamma$ of the boundary $\partial\Omega$, given the Neumann data $\partial_\nu u(h)|_{\Gamma_D}$ on the accessible part $\Gamma_D$. Physically speaking, Neumann data $\partial_\nu u(h)|_{\Gamma_D}$ is the current and the Robin coefficient $h$ is the corrosion. There are many results for uniqueness, stability, and reconstruction algorithm for the inverse problem of (1.2). For the details about the inverse problem, see [13, 7, 1, 8, 6, 9, 5, 15, 4, 12, 22] and the references therein.

In this paper, we prove the uniqueness of the inverse problem and establish the identification by using a Neumann tracking type functional. Moreover, we show numerical results by using the gradient descent method. This paper is based on [21] and [25].

## 2 Uniqueness of the inverse eigenvalue problem

In this section, we prove uniqueness of the inverse problem.

**Theorem 2.1.** *[21, Theorem 2] Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with $C^2$ boundary and $\gamma$, $\Gamma_D$ be disjoint nonempty closed subsets of the boundary $\partial\Omega$ such that $\partial\Omega = \Gamma_D \cup \gamma$. Let $(\lambda(h_j), u_j)$ be a solution of the Robin eigenvalue problems (1.1), corresponding to the Robin coefficients $h_j$, with $h_j \in C^0(\gamma)$ and $h_j > 0$ for $j = 1, 2$. If $(\lambda(h_1), \partial_\nu u_1|_{\Gamma_D}) = (\lambda(h_2), \partial_\nu u_2|_{\Gamma_D})$, then we have $h_1 = h_2$.*

*Proof.* Put $w := u_1 - u_2$. From the assumption $(\lambda(h_1), \partial_\nu u_1|_{\Gamma_D}) = (\lambda(h_2), \partial_\nu u_2|_{\Gamma_D})$, we have

$$\begin{cases} -\Delta w = \lambda w & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_D, \\ \partial_\nu w = 0 & \text{on } \Gamma_D, \\ h_1 w + \partial_\nu w + (h_1 - h_2) u_2 = 0 & \text{on } \gamma. \end{cases}$$

By using Holmgren's unique continuation theorem (see [14]), we obtain $w \equiv 0$ in $\Omega$. Hence $(h_1 - h_2)u_2 = 0$ on $\gamma$. Let us assume that there exists a point $x_0 \in \gamma$ such that $h_1(x_0) \neq h_2(x_0)$. Then by continuity of $h_1$ and $h_2$, there exists an open subset $U \subset \gamma$ such that $h_1 \neq h_2$ in $U$. Thus we have $u_2 = 0$ on $U$. Due to the boundary condition for $u_2$, we also obtain $\partial_\nu u_2 = 0$ on $U$. Hence,

$$\begin{cases} -\Delta u_2 = \lambda u_2 & \text{in } \Omega, \\ u_2 = 0 & \text{on } U, \\ \partial_\nu u_2 = 0 & \text{on } U. \end{cases}$$

Applying Holmgren's theorem again, we obtain $u_2 \equiv 0$ in $\Omega$. However, this is in contradiction with $u_2 \not\equiv 0$. Hence we have $h_1 = h_2$. $\square$

# 3 Neumann tracking type functional and its properties

Let us introduce a Neumann tracking type functional in order to solve the inverse problem numerically. Let $\mathscr{A}$ be the admissible set of Robin coefficients, defined by

$$\mathscr{A} = \left\{ h \in C^1(\gamma) \,:\, h > 0 \right\}.$$

We consider a Neumann tracking type functional $\mathcal{F}$ over the admissible set $\mathscr{A}$ defined by:

$$\mathcal{F}(h) = \frac{1}{2}\int_{\Gamma_D} (\partial_\nu u(h) - g)^2 \, ds + \frac{1}{2}\left|\lambda(h) - \lambda\right|^2, \tag{3.3}$$

where $(\lambda, g)$ are the given spectral data. By Theorem 2.1, we can easily show that the functional (3.3) has a unique minimizer in $\mathscr{A}$ which is the solution of the inverse problem.

**Proposition 3.1.** *[21, Proposition 11] There exists a unique function $h \in \mathscr{A}$ of the functional* (3.3) *such that*

$$0 = \mathcal{F}(h) \leq \mathcal{F}(\psi) \quad \forall \psi \in \mathscr{A}.$$

*Moreover $h$ is the solution of the inverse problem.*

*Proof.* Let $h$ be the solution of the inverse problem. Then we obtain $\lambda(h) = \lambda$ and $\partial_\nu u|_{\Gamma_D} = g$. Thus $h$ is a minimum for $\mathcal{F}$ with $\mathcal{F}(h) = 0$. On the other hand, we assume that $\mathcal{F}(h) = 0$. Then we can easily see that $h$ is the solution of the inverse problem.

Also let $\tilde{h}$ be another minimum for $\mathcal{F}$. Then $\lambda(h) = \lambda(\tilde{h})$ and $\partial_\nu u|_{\Gamma_D}(h) = \partial_\nu u|_{\Gamma_D}(\tilde{h})$. Thus by Theorem 2.1 we obtain $h = \tilde{h}$. $\square$

By Proposition 3.1, we may consider the minimization problem for $\mathcal{F}$ to solve the inverse problem numerically. In order to compute the Fréchet derivative of the functional $\mathcal{F}$ with respect to $h$, we need to prove the Fréchet differentiability of the solution $u(h)$ in (1.1) with respect to the Robin coefficient $h$. By a perturbation argument of the Fourier expansions with respect to the Robin eigenvalue problem (1.1) and the elliptic regularity theory for mixed boundary problem (see [17, Theorem 7.36.6, p.621]), we can prove the following eigenvalue estimate and Fréchet differentiability for $u(h)$ in $H^2$ (see [21, Theorem 4 and Corollary 9]):

$$\frac{\left\| u(h+\xi) - u(h) - u'(h)[\xi] \right\|_{H^2(\Omega)}}{\|\xi\|_{C^1(\gamma)}} \to 0 \text{ as } \|\xi\|_{C^1(\gamma)} \to 0, \tag{3.4}$$

$$\frac{\left|\lambda(h+\xi) - \lambda(h) - \lambda'\right|}{\|\xi\|^{3/2}_{C^1(\gamma)}} \to 0 \text{ as } \|\xi\|_{C^1(\gamma)} \to 0, \tag{3.5}$$

where $u'(h)[\xi]$ is the solution of the following sensitivity problem:

$$\begin{cases} -\Delta u' - \lambda(h)u' = \lambda' u(h) & \text{in } \Omega, \\ u' = 0 & \text{on } \Gamma_D, \\ hu' + \partial_\nu u' = -\xi u(h) & \text{on } \gamma, \\ \displaystyle\int_\Omega u(h)u'\,dx = 0. \end{cases}$$

Here, $\lambda'$ is given by $\lambda' = \displaystyle\int_\gamma \xi u(h)^2\,ds$. By (3.4) and (3.5), we can easily compute the Fréchet derivative of the functional $\mathcal{F}$ with respect to $h$.

**Theorem 3.2.** *[21, Theorem 12] The Fréchet derivative of the functional $\mathcal{F}$ at the point $h \in \mathscr{A}$ in the direction $\xi$ is*

$$\mathcal{F}'(h)[\xi] = \int_\gamma \left\{ u(h)\varphi + (\lambda(h) - \lambda)u(h)^2 \right\} \xi\,ds, \tag{3.6}$$

*where $\varphi$ is the solution of the following problem:*

$$\begin{cases} -\Delta\varphi = \lambda(h)\varphi & \text{in } \Omega, \\ \varphi = \partial_\nu u(h) - g & \text{on } \Gamma_D, \\ h\varphi + \partial_\nu\varphi = 0 & \text{on } \gamma, \\ \displaystyle\int_\Omega u(h)\varphi\,dx = 0. \end{cases} \tag{3.7}$$

*Proof.* By (3.4) and (3.5), we obtain

$$\begin{aligned} &\mathcal{F}(h+\xi) - \mathcal{F}(h) \\ &= \frac{1}{2}\left\|\partial_\nu u(h+\xi) - g\right\|^2_{L^2(\Gamma_D)} + \frac{1}{2}\left|\lambda(h+\xi) - \lambda\right|^2 - \frac{1}{2}\left\|\partial_\nu u(h) - g\right\|^2_{L^2(\Gamma_D)} - \frac{1}{2}\left|\lambda(h) - \lambda\right|^2 \\ &= \frac{1}{2}\left\|\partial_\nu u(h) + \partial_\nu u'(h)[\xi] + o(\|\xi\|_{C^1(\gamma)}) - g\right\|^2_{L^2(\Gamma_D)} \\ &\qquad + \frac{1}{2}\left|\lambda(h) + \lambda' + o(\|\xi\|^{3/2}_{C^1(\gamma)}) - \lambda\right|^2 - \frac{1}{2}\left\|\partial_\nu u(h) - g\right\|^2_{L^2(\Gamma_D)} - \frac{1}{2}\left|\lambda(h) - \lambda\right|^2 \\ &= \int_{\Gamma_D} (\partial_\nu u(h) - g)\partial_\nu u'(h)[\xi]\,ds + \lambda'(\lambda(h) - \lambda) + o(\|\xi\|_{C^1(\gamma)}). \end{aligned}$$

Let us focus on the first term. By the Green's second identity we obtain

$$\begin{aligned} 0 &= \int_\Omega \big((-\Delta\varphi - \lambda(h)\varphi)u' - (-\Delta u' - \lambda(h)u' - \lambda' u(h))\varphi\big)\,dx \\ &= \int_\Omega (\varphi\Delta u' - u'\Delta\varphi)\,dx. \end{aligned}$$

By the divergence theorem we have

$$\begin{aligned} 0 &= \int_{\Gamma_D} \varphi \partial_\nu u' \, ds + \int_\gamma (\varphi \partial_\nu u' - u' \partial_\nu \varphi) \, ds \\ &= \int_{\Gamma_D} \varphi \partial_\nu u' \, ds + \int_\gamma \big(\varphi(-hu' - \xi u(h)) + u' h \varphi\big) \, ds \\ &= \int_{\Gamma_D} (\partial_\nu u(h) - g) \partial_\nu u' \, ds - \int_\gamma \xi u(h) \varphi \, ds. \end{aligned}$$

Thus we obtain

$$\int_{\Gamma_D} (\partial_\nu u(h) - g) \partial_\nu u' \, ds = \int_\gamma \xi u(h) \varphi \, ds.$$

Therefore, since $\lambda' = \int_\gamma \xi u(h)^2 \, ds$, we have that the Fréchet derivative $\mathcal{F}'$ of the functional $\mathcal{F}$ is given by

$$\begin{aligned} \mathcal{F}'(h)[\xi] &= \int_\gamma \xi u(h) \varphi \, ds + \lambda'(\lambda(h) - \lambda) \\ &= \int_\gamma \left\{ u(h)\varphi + (\lambda(h) - \lambda) u(h)^2 \right\} \xi \, ds. \end{aligned}$$ □

By (3.6), the gradient descent direction of the functional $\mathcal{F}$ is

$$\xi = -\left( u(h)\varphi + (\lambda(h) - \lambda) u(h)^2 \right).$$

We use a gradient descent type algorithm to solve the minimization problem for the functional $\mathcal{F}$.

# 4 Reconstruction algorithm and numerical tests

Let $tol$ be a fixed tolerance level and $\tau_k > 0$ the step sizes at each iteration $k$. In all numerical experiments below, we keep $\tau_k$ fixed.

**Algorithm 1** Reconstruction algorithm.

Inputs: spectral data $(\lambda, g)$ and initial guess $h_0$. Set $k = 0$ and iterate:

1: Compute the principal eigenfunction $u_k$ and eigenvalue $\lambda_k$ with Robin coefficient $h_k$, solving Problem (1.1).

2: Compute the solution $\varphi_k$ of the problem (3.7).

3: Compute the descent direction $\delta_k$ with the formula

$$\delta_k = -\left( u_k \varphi_k + (\lambda_k - \lambda) u_k^2 \right). \tag{4.8}$$

4: Define $h_{k+1} = h_k + \tau_k \delta_k$.

5: If $\|\delta_k\|_{C^1(\gamma)} > tol$, set $k = k + 1$ and repeat.

In what follows, we consider the annular region $\Omega = B(0,2) \setminus \overline{B(0,1)}$, with $\gamma = \partial B(0,1)$ and $\Gamma_D = \partial B(0,2)$. Also, the initial guess is $h \equiv 1$ on $\gamma$. We consider the reconstruction of the Robin coefficient

$$h(x,y) = 1 + \frac{xy}{2} - \frac{x^2 y}{5}$$

for $(x,y) \in \gamma = \partial B(0,1)$, the interior part of the boundary of the annular region $\Omega$. We present reconstruction from noiseless data (Figure 1), and noisy data: 2% (Figure 2). All the computations are done using FreeFem++ [11].

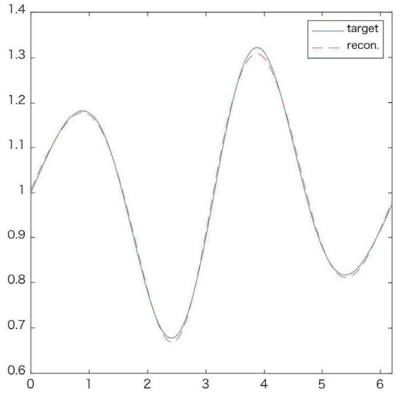


Figure 1: Reconstruction from noiseless data. Iteration = 1000.

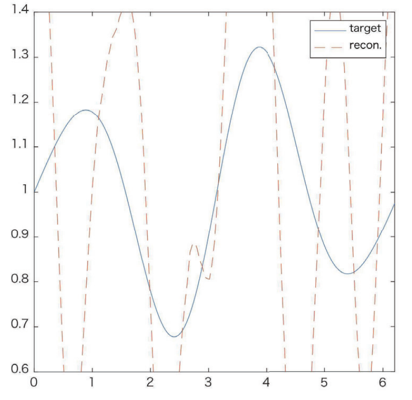


Figure 2: Reconstruction from 2% noisy data Iteration = 1000.

We can see that the algorithm performs well in case of no noise, while the quality of the reconstruction is bad for noise levels of 2%. This is because of the ill-posedness of the inverse problem.

There are so many techniques to treat such an ill-posedness of inverse problems. One of the major techniques is Tikhonov regularization, see [16, 18, 14] and references therein. It is a technique that prevents the ill-posedness of inverse problems by adding a penalty of $L^2$ in an objective functional. In our situation, we can consider a Tikhonov regularization functional $\mathcal{F}_{\mathrm{reg}}$ for the functional $\mathcal{F}$ defined by

$$\mathcal{F}_{\mathrm{reg}}(h) = \frac{1}{2}\int_{\Gamma_D} (\partial_\nu u(h) - g)^2\, ds + \frac{1}{2}\left|\lambda(h) - \lambda\right|^2 + \frac{\eta}{2}\int_\gamma h^2\, ds, \tag{4.9}$$

where $\eta > 0$ is a regularization parameter.

This technique is certainly useful, but we have to choose suitable regularization parameter $\eta > 0$ in (4.9). To avoid this difficulty, we use the early stopping method which is widely used in the field of machine learning, see [23, 19] and references therein. This method is a kind of regularization method that stops updating when the value of a functional no longer decreases. The merit of this method is that we can use the functional $\mathcal{F}$ directly, and we do not need to choose a suitable regularization parameter $\eta > 0$.

We show reconstruction from 2% noisy data when iteration number is 1000 (Figure 3), and 2% noisy data when iteration number is 40 (Figure 4). We can see that the early stopping method works well and prevent ill-posedness of the inverse problem of (1.1).

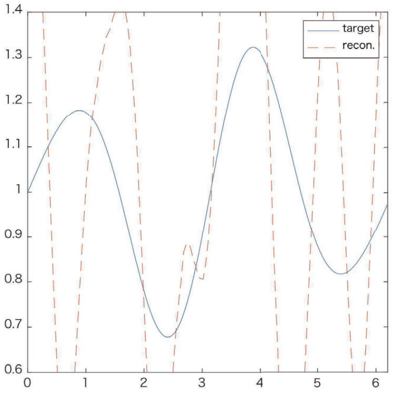


Figure 3: Reconstruction from 2% noisy data Iteration = 1000.

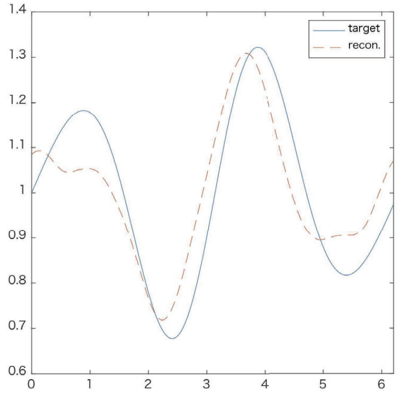


Figure 4: Reconstruction from 2% noisy data Iteration = 40.

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