Non-relativistic limit of the semi-relativistic Pauli-Fierz model*†

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Abstract

The non-relativistic limit of the semi-relativistic Pauli-Fierz Hamiltonian

$$\sqrt{c^2(-i\nabla\otimes\mathbb{1}-A(x))^2+m^2c^4}-mc^2+V\otimes\mathbb{1}+\mathbb{1}\otimes H_{\rm rad}$$

is considered. Here c denots the speed of light, m the mass of a charged particle, A a quantized radiation field, V an external potential and $H_{\rm rad}$ the free field Hamiltonian. By the limit $c\to\infty$ in the sense of strong semigroup, we derive the Pauli-Fierz Hamiltonian in non-relativistic quantum electrodynamics:

$$\frac{1}{2m}(-i\nabla\otimes\mathbb{1}-A(x))^2+V\otimes\mathbb{1}+\mathbb{1}\otimes H_{\rm rad}.$$

1 Non-relativistic limit of subordinator

In 1905 Albert Einstein discovered that a particle with momentum $p\in\mathbb{R}^3$ and mass m has the kinetic energy $\sqrt{c^2|p|^2+m^2c^4}$. Since we have

$$\sqrt{c^2|p|^2+m^2c^4}-mc^2=\frac{1}{2m}|p|^2+\mathcal{O}(\frac{|p|^4}{m^3c^2}),$$

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†This article is dedicated to late Syun'etsu Hiroshima.

intuitively we have

$$\exp\Big(-t\big(\sqrt{c^2(-\Delta)+m^2c^4}-mc^2\big)\Big)\to\exp\Big(-t\frac{1}{2m}(-\Delta)\Big)$$

as $c\to\infty$. This intuition becomes substantial by means of the so-called non-relativistic limit discussed in this article. Define

$$H_c=\sqrt{-c^2\Delta+m^2c^4}-mc^2+V.$$

By using the Feynman-Kac formula we can show that $H_c\to H_\infty$ as $c\to\infty$ in a specific sense, and the limit operator is the Schrödinger operator

$$H_\infty=-\frac{1}{2m}\Delta+V.$$

We call this non-relativistic limit.

Now we introduce a Feynman-Kac formula of e^{-tH_c}. Let $(B_t)_{t\geq 0}$ be 3-dimensional Brownian motion on the Wiener space $(\mathscr{X},\mathcal{B},\mathcal{W}^x)$, where $\mathscr{X}=C([0,\infty);\mathbb{R}^3)$ is the set of $\mathbb{R}^3$-valued continuous paths on $[0,\infty)$, $\mathcal{W}^x$ denotes the Wiener measure such that $\mathcal{W}^x(B_0=x)=1$. It is established that

$$(f,e^{-tH_\infty}g)=\int_{\mathbb{R}^3}\mathbb{E}^x\big[f(x)g(B_{t/m})e^{-\int_0^t V(B_{s/m})ds}\big]dx. \tag{1.1}$$

Here $\mathbb{E}^x[\dots]=\int_{\mathscr{X}}\dots d\mathcal{W}^x$ denotes the expectation with respect to $\mathcal{W}^x$.

Next we consider a Feynman-Kac formula of e^{-tH_c}. To do that we need a subordinator in addition to Brownian motion. We recall that $(T_t)_{t\geq 0}$ is a subordinator if and only if it is a one-dimensional Lévy process and $[0,\infty)\ni t\mapsto T_t\in\mathbb{R}$ is almost surely nondecreasing. For every $c>0$ consider the subordinator $(T_t^c)_{t\geq 0}$ on a probability space $(\mathcal{S},\mathcal{F},P)$ with parameter c such that

$$\mathbb{E}_P[e^{-uT_t^c}]=e^{-t(\sqrt{2c^2u+m^2c^4}-mc^2)},$$

where $u\in\mathbb{R}$ and $\mathbb{E}_P[\dots]=\int_{\mathcal{S}}\dots dP$. By using the distribution

$$\rho_t^c(s)=\frac{ct}{\sqrt{2\pi}}e^{mc^2t}s^{-3/2}\exp\left(-\frac{1}{2}\left(\frac{c^2t^2}{s}+m^2c^2s\right)\right)1_{[0,\infty)}(s)$$

of T_t^c we have

$$\mathbb{E}_P[e^{-uT_t^c}]=\int_{\mathbb{R}}e^{-us}\rho_t^c(s)ds.$$

Substituting $-\frac{1}{2}\Delta$ into u formally above, we have

$$\mathbb{E}_P[e^{T_t^c \frac{1}{2}\Delta}] = e^{-t(\sqrt{-c^2\Delta u + m^2c^4} - mc^2)},$$

Adding an external potential V we have the related Feynman-Kac formulae:

$$(f, e^{-tH_c}g) = \int_{\mathbb{R}^3} \mathbb{E}^x[f(x)[\mathbb{E}_P[g(B_{T_t^c})e^{-\int_0^t V(B_{T_s})ds}]]dx. \tag{1.2}$$

We refer [5] for the detail of (1.2).

Proposition 1.1 ([7, Section 4.6]) *Let f be a bounded continuous function on $\mathbb{R}$. Then*

$$\lim_{c\to\infty} \mathbb{E}_P[f(T_t^c)] = f(t/m).$$

It can be allowed to say that $\rho_t^c(s) \to \delta(s - t/m)$ as $c \to \infty$ by Proposition 1.1. We derive the non-relativistic limit of e^{-tH_c}.

Corollary 1.2 *Let V be a bounded continuous function. Then*

$$s - \lim_{c\to\infty} e^{-tH_c} = e^{-tH_\infty}.$$

Proof: We suppose that V is nonnegative without loss of generality. It is enough to show the weak limit

$$\lim_{c\to\infty} (f, e^{-tH_c}g) = (f, e^{-tH_\infty}g). \tag{1.3}$$

Since $H_c \geq 0$ for every $c > 0$, $\|e^{-tH_c}\| \leq 1$ uniformly with respect to $c > 0$. It is also sufficient to show (1.3) for arbitrary $f, g \in \mathscr{S}(\mathbb{R})$ by a simple limiting argument. Note that by Proposition 1.1 it can be seen that

$$(f, e^{-t(\sqrt{-\Delta+m^2c^4}-mc^2+V)}g) = \int_{\mathbb{R}^3} \mathbb{E}^{x,0}[\bar{f}(x)g(B_{T_t^c})e^{-\int_0^t V(B_{T_s^c})ds}]dx$$

$$\to \int_{\mathbb{R}^3} \mathbb{E}^x[\bar{f}(x)g(B_{t/m})e^{-\int_0^t V(B_{s/m})ds}]dx = (f, e^{-t(-\frac{1}{2m}\Delta+V)}g)$$

as $c \to \infty$. □

2 Non-relativistic limit of RPF model

We consider a system of quantum matters minimally coupled to a quantized radiation field. This model describes an interaction between non-relativistic spinless n-electrons and photons. Let

$$\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathcal{F}$$

be the total Hilbert space describing the joint electron-photon state vectors. $L^2(\mathbb{R}^3)$ describes the state space of a single electron moving in $\mathbb{R}^3$ and $\mathcal{F}$ that of photons. Here $\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^3 \times \{1,2\}))$ is the boson Fock space over Hilbert space $L^2(\mathbb{R}^3 \times \{1,2\})$ of the set of L^2-functions on $\mathbb{R}^3 \times \{1,2\}$. The elements of the set $\{1,2\}$ account for the fact that a photon is a transversal wave perpendicular to the direction of its propagation, which has two components. $\mathcal{H}$ can be decomposed into infinite direct sum:

$$\mathcal{H} = \oplus_{n=0}^{\infty} \mathcal{H}^{(n)},$$

where $\mathcal{H}^{(n)} = L^2(\mathbb{R}^3) \otimes \mathcal{F}^{(n)}$. The Fock vacuum in $\mathcal{F}$ is denoted by Ω as usual. We introduce the free field Hamiltonian on $\mathcal{F}$. Let $\omega = \omega(k) = |k|$. $\omega(k)$ describes the energy of a single photon with momentum k. The free field Hamiltonian $H_{\rm rad}$ on $\mathcal{F}$ is given in terms of the second quantization

$$H_{\rm rad} = {\rm d}\Gamma(\omega).$$

Here ω denotes the multiplication in $L^2(\mathbb{R}^3 \times \{1,2\})$ by $(\omega f)(k,j) = \omega(k) f(k,j)$ for $(k,j) \in \mathbb{R}^3 \times \{1,2\}$.

On the other hand the charged matter, electron, is governed by Schrödinger operator of the form

$$H_{\rm p} = -\frac{1}{2m}\Delta + V$$

in $L^2(\mathbb{R}^3)$. Here m denotes the mass of electron. To introduce the minimal coupling we define quantized radiation fields. Let $a(f)$ and $a^\dagger(f)$ be the annihilation operator and the creation operator on $\mathcal{F}$ smeared by $f \in L^2(\mathbb{R}^3 \times \{1,2\})$, respectively. Let us identify $L^2(\mathbb{R}^3 \times \{1,2\})$ with $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ by

$$L^2(\mathbb{R}^3 \times \{1,2\}) \ni f(\cdot,1) \cong f(\cdot,1) \oplus 0 \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3),$$
$$L^2(\mathbb{R}^3 \times \{1,2\}) \ni f(\cdot,2) \cong 0 \oplus f(\cdot,2) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3).$$

We set $a^\sharp(f\oplus 0)=a^\sharp(f,1)$ and $a^\sharp(0\oplus f)=a^\sharp(f,2)$. Hence we obtain canonical commutation relations:

$$[a(f,j),a^\dagger(g,j')]=\delta_{jj'}(\bar{f},g),\quad [a(f,j),a(g,j')]=0=[a^\dagger(f,j),a^\dagger(g,j')].$$

We define the quantized radiation field with a cutoff function $\hat{\varphi}$. Put

$$\wp_\mu(x,j)=\frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}}e_\mu(k,j)Fe^{-ikx},\quad \tilde{\wp}_\mu(x,j)=\frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}}e_\mu(k,j)Fe^{ikx}$$

for each $x\in\mathbb{R}^3$, $j=1,2$ and $\mu=1,2,3$. Here cutoff function $\hat{\varphi}$ is the Fourier transform of the charge distribution $\varphi\in\mathscr{S}'(\mathbb{R}^3)$. Although physically it should be $\hat{\varphi}=1/(2\pi)^{3/2}$, we have to introduce cutoff function $\hat{\varphi}$ to ensure that $\wp_\mu(x,j)\in L^2(\mathbb{R}^3_k)$ for each x. The vectors $e(k,1)$ and $e(k,2)$ are called polarization vectors, that is, $(e(k,1),e(k,2),k/|k|)$ forms a right-hand system at each $k\in\mathbb{R}^3$;

$$e(k,i)\cdot e(k,j)=\delta_{ij},\quad e(k,j)\cdot k=0,\quad e(k,1)\times e(k,2)=\frac{k}{|k|}.$$

The quantized radiation field with cutoff function $\hat{\varphi}$ is defined by

$$A_\mu(x)=\frac{1}{\sqrt{2}}\sum_{j=1,2}\Big(a^\dagger(\wp_\mu(x,j),j)+a(\tilde{\wp}_\mu(x,j),j)\Big),\quad \mu=1,2,3.$$

Unless otherwise stated we suppose the following assumptions.

Assumption 2.1 (Cutoff functions) $\varphi\in\mathscr{S}'(\mathbb{R}^3)$ *satisfies that (1)* $\hat{\varphi}\in L^1_{\rm loc}(\mathbb{R}^3)$*, (2)* $\hat{\varphi}(-k)=\overline{\hat{\varphi}(k)}$*, (3)* $\sqrt{\omega}\hat{\varphi},\hat{\varphi}/\sqrt{\omega},\hat{\varphi}/\omega\in L^2(\mathbb{R}^3)$*.*

In the case of $\hat{\varphi}/\sqrt{\omega}\in L^2(\mathbb{R}^3)$ and $\overline{\hat{\varphi}(k)}=\hat{\varphi}(-k)$, $A_\mu(x)$ is symmetric, and moreover essentially selfadjoint on the finite particle subspace $\mathcal{F}_{\rm fin}$ of $\mathcal{F}$. We denote the closure of $A_\mu(x)\lceil_{\mathcal{F}_{\rm fin}}$ by the same symbol. Write

$$A_\mu=\int_{\mathbb{R}^3}^\oplus A_\mu(x)dx,\quad A=(A_1,A_2,A_3).$$

A_μ is a selfadjoint operator on

$$D(A_\mu)=\left\{F\in\mathcal{H}\,\middle|\,F(x)\in D(A_\mu(x))\text{ a.e. and }\int_{\mathbb{R}^3}\|A_\mu(x)F(x)\|^2_{\mathcal{F}}dx<\infty\right\}$$

and acts as $(A_\mu F)(x) = A_\mu(x)F(x)$ for $F \in D(A_\mu)$ for a.e. $x \in \mathbb{R}^3$. Since $k \cdot e(k,j) = 0$, the polarization vectors introduced above are chosen in the way that $\sum_{\mu=1}^3 \nabla_\mu \wp_j^\mu(x) = 0$, implying the Coulomb gauge condition

$$\sum_{\mu=1}^3 \nabla_\mu A_\mu = 0.$$

This in turn yields $\sum_{\mu=1}^3 [\nabla_\mu, A_\mu] = 0$. Let us define the Pauli-Fierz Hamiltonian. The interaction is obtained by minimal coupling:

$$-i\nabla_\mu \otimes \mathbb{1} \mapsto -i\nabla_\mu \otimes \mathbb{1} - A_\mu$$

to $H_\mathrm{p} \otimes \mathbb{1} + \mathbb{1} \otimes H_\mathrm{rad}$.

Definition 2.2 (The Pauli-Fierz Hamiltonian) *The Pauli-Fierz Hamiltonian of one electron with mass m is defined by*

$$H_\mathrm{PF} = \frac{1}{2m}(-i\nabla \otimes \mathbb{1} - A)^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes H_\mathrm{rad}.$$

In what follows we omit the tensor notation $\otimes$ for the sake of simplicity. Thus

$$H_\mathrm{PF} = \frac{1}{2m}(-i\nabla - A)^2 + V + H_\mathrm{rad}.$$

We introduce classes of external potentials. We say $V \in C_\mathrm{kato}$ if and only if $D(\Delta) \subset D(V)$ and there exist $0 \le a < 1$ and $0 \le b$ such that $\|Vf\| \le a\| -(1/2)\Delta f\| + b\|f\|$ for $f \in D(\Delta)$. H_PH with $V \in C_\mathrm{kato}$ is self-adjoint on $D(-\Delta) \cap D(H_\mathrm{rad})$.

Definition 2.3 (Semi-relativistic Pauli-Fierz Hamiltonian) *H_RPF is defined by*

$$H_\mathrm{RPF} = \sqrt{c^2(-i\nabla - A)^2 + m^2c^4} - mc^2 + V + H_\mathrm{rad}.$$

The functional integration and the self-adjointness of H_RPF is shown in [1, 2, 4]. We introduce classes of external potentials which is a counterpart of C_kato. We say $V \in C_\mathrm{rkato}$ if and only if $D(\sqrt{-\Delta}) \subset D(V)$ and there exist $0 \le a < 1$ and $0 \le b$ such that $\|Vf\| \le a\|\sqrt{-\Delta} f\| + b\|f\|$ for $f \in D(\Delta)$. H_RPH with $V \in C_\mathrm{rkato}$ is self-adjoint on $D(\sqrt{-\Delta}) \cap D(H_\mathrm{rad})$. In the previous section we could see that

$$\sqrt{-\Delta + m^2c^4} - mc^2 + V \to -\frac{1}{2m}\Delta + V$$

as $c \to \infty$ strongly in the sense of semigroup. In a similar way to this we shall show the non-relativistic limit of the semi-relativistic Pauli-Fierz Hamiltonian. Using $(T_t^c)_{t\geq 0}$ we can see that

$$(F, e^{-tH_{\rm RPF}}G) = \int_{\mathbb{R}^3} \mathbb{E}_{\mathcal{W}^x\otimes P}\big[e^{-\int_0^t V(B_{T_s^c})ds}(\mathrm{J}_0 F(x), e^{-i\hat{A}_{\rm E}(K_t^{\rm rel}(c))}\mathrm{J}_t G(B_{T_t^c}))\big]dx, \tag{2.1}$$

and the functional integral representation of $e^{-tH_{\rm PF}}$ with mass m is given by

$$(F, e^{-tH_{\rm PF}}G) = \int_{\mathbb{R}^3} \mathbb{E}_{\mathcal{W}^x}\left[e^{-\int_0^t V(B_{s/m})ds}(\mathrm{J}_0 F(x), e^{-i\hat{A}_{\rm E}(K_t)}\mathrm{J}_t G(B_{t/m}))\right]dx. \tag{2.2}$$

Let $\mathrm{j_t} : \mathrm{L}^2(\mathbb{R}^3) \to \mathrm{L}^2(\mathbb{R}^4)$ be such that $\mathrm{j_t^* j_s} = \mathrm{e}^{-|\mathrm{s}-\mathrm{t}|\omega(-\mathrm{i}\nabla)}$. Let

$$\mathrm{I}_m^c = \bigoplus_{\mu=1}^3 \sum_{j=1}^{2^n} \int_{T_{t_{j-1}}^c}^{T_{t_j}^c} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\circ dB_s^\mu,$$

$$\mathrm{I}_m = \bigoplus_{\mu=1}^3 \sum_{j=1}^{2^n} \int_{t_{j-1}/m}^{t_j/m} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)\circ dB_s^\mu.$$

Then $K_t^{\rm rel}(c)$ and K_t are defined by the limits: $\mathrm{I}_m^c \to K_t^{\rm rel}(c)$ and $\mathrm{I}_m \to K_t$ as $m\to\infty$ strongly in $L^2(\mathscr{X}\times\mathcal{S})\otimes(\oplus^3 L^2(\mathbb{R}^4))$. The functional integral representation is due to [3] for $e^{-tH_{\rm PF}}$ and [4] for $e^{-tH_{\rm RPF}}$. Using (2.1) and (2.2) we show that $e^{-tH_{\rm RPF}} \to e^{-tH_{\rm RPF}}$ as $c\to\infty$ strongly. In what follows we set $\mathbb{E}^{x,0} = \mathbb{E}_{\mathcal{W}^x\otimes P}$ and $\mathbb{E}^x = \mathbb{E}_{\mathcal{W}^x}$.

Lemma 2.4 *It follows that*

$$\lim_{c\to\infty} K_t^{\rm rel}(c) = K_t$$

strongly in $L^2(\mathscr{X}\times\mathcal{S})\otimes(\oplus^3 L^2(\mathbb{R}^4))$.

Proof: We have

$$\|K_t^{\rm rel}(c) - K_t\| \leq \|K_t^{\rm rel}(c) - \mathrm{I}_n^c\| + \|\mathrm{I}_n^c - \mathrm{I}_n\| + \|\mathrm{I}_n - K_t\|.$$

We have

$$\mathbb{E}^x[\|\mathrm{I}_n^c - \mathrm{I}_k^c\|^2] \le 3T_t^c\|\hat{\varphi}/\sqrt{\omega}\|^2\Big(\sum_{j=n+1}^{k} 2^{-j/2}\Big)^2.$$

Here $\|\cdot\|$ denotes the norm on $\oplus^3 L^2(\mathbb{R}^4)$. From this we have

$$\mathbb{E}^{x,0}[\|\mathrm{I}_n^c - K_t^{\mathrm{rel}}(c)\|^2] \le 3\mathbb{E}^0[T_t^c]\|\hat{\varphi}/\sqrt{\omega}\|^2\Big(\sum_{j=n+1}^{\infty} 2^{-j/2}\Big)^2.$$

Since $\mathbb{E}^0[T_t^c] = t/m$ which is independent of $c > 0$, we obtain that

$$\mathbb{E}^{x,0}[\|\mathrm{I}_n^c - K_t^{\mathrm{rel}}(c)\|^2] \le 3\frac{t}{m}\|\hat{\varphi}/\sqrt{\omega}\|^2\Big(\sum_{j=n+1}^{\infty} 2^{-j/2}\Big)^2$$

and we conclude that

$$\mathbb{E}^{x,0}[\|\mathrm{I}_n^c - K_t^{\mathrm{rel}}(c)\|^2] \to 0 \tag{2.3}$$

as $n \to \infty$ uniformly in c. Let $\varepsilon > 0$ be arbitrary. There exists n_0 such that for all $n > n_0$ $\mathbb{E}^{x,0}[\|K_t^{\mathrm{rel}}(c) - \mathrm{I}_n^c\|^2] < \varepsilon^2$ and $\mathbb{E}^{x,0}[\|\mathrm{I}_n - K_t\|^2] < \varepsilon^2$ uniformly in c. Now we estimate $\|\mathrm{I}_n^c - \mathrm{I}_n\|$. We have

$$\mathrm{I}_n^c - \mathrm{I}_n = \bigoplus_{\mu=1}^{3}\sum_{j=1}^{2^n}\Big(\int_{T_{t_{j-1}}^c}^{T_{t_j}^c} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu - \int_{t_{j-1}/m}^{t_j/m} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu\Big).$$

We note that $s \to \int_a^s \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu$ and $s \to \int_s^b \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu$ are almost surely continuous. Hene

$$(S,T) \to \mathbb{E}^x\Big[\Big(\int_S^T \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu, \int_{t_{j-1}/m}^{t_j/m} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu\Big)\Big]$$

is continuous. This implies that for every j,

$$\begin{aligned}
&\mathbb{E}^{x,0}\Big[\Big(\int_{T_{t_{j-1}}^c}^{T_{t_j}^c} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu, \int_{t_{j-1}/m}^{t_j/m} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu\Big)\Big] \\
&\to \mathbb{E}^x\Big[\Big(\int_{t_{j-1}/m}^{t_j/m} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu, \int_{t_{j-1}/m}^{t_j/m} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu\Big)\Big] \\
&= \frac{(t_j - t_{j-1})}{m}\|\hat{\varphi}/\sqrt{\omega}\|^2
\end{aligned} \tag{2.4}$$

as $c \to \infty$. Hence

$$\begin{aligned}
&\mathbb{E}^{x,0}[\|\mathrm{I}_n^c - \mathrm{I}_n\|^2] \\
&= 3\sum_{j=1}^{2^n} \mathbb{E}^{x,0}\left[\left\| \int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu - \int_{t_{j-1}/m}^{t_j/m} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu \right\|^2\right].
\end{aligned}$$

Since we have

$$\begin{aligned}
&\mathbb{E}^{x,0}\left[\left\| \int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu - \int_{t_{j-1}/m}^{t_j/m} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu \right\|^2\right] \\
&= \mathbb{E}^{x,0}\left[\left\| \int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu \right\|^2\right] + \mathbb{E}^{x,0}\left[\left\| \int_{t_{j-1}/m}^{t_j/m} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu \right\|^2\right] \\
&\quad - 2\mathbb{E}^{x,0}\left[\left(\int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu, \int_{t_{j-1}/m}^{t_j/m} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu \right)\right] \\
&= \frac{1}{m}\|\hat{\varphi}/\sqrt{\omega}\|^2(\mathbb{E}^0[T^c_{t_j} - T^c_{t_{j-1}}] + t_j - t_{j-1}) \\
&\quad - 2\mathbb{E}^{x,0}\left[\left(\int_{T^c_{t_{j-1}}}^{T^c_{t_j}} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu, \int_{t_{j-1}/m}^{t_j/m} \mathrm{j}_{t_{j-1}}\tilde{\varphi}(\cdot - B_s)dB_s^\mu \right)\right].
\end{aligned}$$

Note that $\mathbb{E}^0[T^c_{t_j} - T^c_{t_{j-1}}] = t_j - t_{j-1}$ and (2.4). We can see that

$$\mathbb{E}^{x,0}[\|\mathrm{I}_n^c - \mathrm{I}_n\|^2] \to 0$$

as $c \to \infty$. We have

$$\lim_{c\to\infty}(\mathbb{E}^{x,0}[\|K_t^{\mathrm{rel}}(c) - K_t\|]^2)^{1/2} \le 2\varepsilon + \lim_{c\to\infty}(\mathbb{E}^{x,0}[\|\mathrm{I}_n^c - \mathrm{I}_n\|])^{1/2} = 2\varepsilon.$$

Thus the lemma is proven. □

The main result of this article is the next theorem.

Theorem 2.5 (Non-relativistic limit) *Suppose that V is bounded and continuous. Then for every $t \ge 0$ it follows that*

$$\mathrm{s}-\lim_{c\to\infty} e^{-tH_{\mathrm{RPF}}} = e^{-tH_{\mathrm{PF}}}.$$

Proof: Suppose that $F, G \in C_0^\infty(\mathbb{R}^3)\otimes\mathcal{F}_{\mathrm{rad}}$. From Lemma 2.4 and

$$(F, e^{-tH_{\mathrm{RPF}}}G) = \int_{\mathbb{R}^3} \mathbb{E}^{x,0}\left[e^{-\int_0^t V(B_{T_s^c})ds}(\mathrm{J}_0F(x), e^{-i\hat{A}_{\mathrm{E}}(K_t^{\mathrm{rel}}(c))}\mathrm{J}_tG(B_{T_t^c}))\right]dx$$

it follows that

$$\lim_{c\to\infty}(F, e^{-tH_{\rm RPF}}G) = \int_{\mathbb{R}^3} \mathbb{E}^x \left[e^{-\int_0^t V(B_{s/m})ds} (\mathrm{J}_0 F(x), e^{-i\hat{A}_{\rm E}(K_t)} \mathrm{J}_t G(B_{t/m})) \right] dx$$
$$= (F, e^{-tH_{\rm PF}}G).$$

Since $H_{\rm RPF} \geq \inf_{x\in\mathbb{R}^3} V(x) = g \geq -\infty$, $e^{-H_{\rm RPF}} \leq e^{-tg}$. Let $F, G \in \mathcal{H}_{\rm PF}$. There exists $F_n, G_n \in C_0^\infty(\mathbb{R}^3) \otimes \mathcal{F}_{\rm rad}$ such that $F_n \to F$ and $G_n \to G$ strongly as $n \to \infty$. By the uniform bound $e^{-tH_{\rm RPF}} \leq e^{-tg}$, we can show $\lim_{c\to\infty}(F, e^{-tH_{\rm RPF}}G) = (F, e^{-tH_{\rm PF}}G)$. Finally since the weak convergence of $e^{-tH_{\rm RPF}}$ implies the strong convergence, the theorem follows. □

Remark 2.6 Theorem 2.5 has been already published in [6, Theorem 3.137]. Although this article was planed to be published in 2019, it delayed however by 2 years and then [6] has been published before the publication of this article. Hence this is not the reprint of [6].

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