Singular Bogoliubov Transformations

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1 Introduction

From a representation theoretical point of view, quantum mechanics (resp. quantum field theory) may be regarded as representations of canonical commutation relations (CCR) and/or canonical anti-commutation relations (CAR) with finite (resp. infinite) degrees of freedom. Models in quantum mechanics and quantum field theory are constructed based on Hilbert space representations of CCR and/or CAR. There exist basically two categories for representations of CCR and/or CAR respectively, i.e., reducible and irreducible, and each of them is divided into two classes: equivalent and inequivalent. Quantum theories based on equivalent representations of CCR and/or CAR are physically equivalent, being *different only in the framework of the physical picture*. On the other hand, quantum theories based on inequivalent irreducible representations are *essentially different from each other, describing non-comparable physical situations*.

We have learned from studies on models in quantum mechanics and quantum field theory that inequivalent representations of CCR or CAR are associated with "characteristic" quantum phenomena such as the Aharonov–Bohm effect [1, 2, 3], the Bose–Einstein condensation [4, 11, 14] and infrared or ultraviolet renormalizations in some models in quantum field theory [8].[1] This structure is very interesting and the following philosophical point of view is suggested:

> *The Universe uses inequivalent representations of CCR and/or CAR to create "characteristic" quantum phenomena in which macroscopic quantities (external magnetic fields, masses, charges, particle densities etc.) appear as labels indexing families of mutually inequivalent representations of CCR and/or CAR.*

[1]For a comprehensive description of inequivalent representations of CCR and CAR in correspondence to "characteristic" quantum phenomena, see [10].

From this point of view, it is important to find inequivalent representations of CCR and CAR respectively as many as possible and to make clear their physical correspondences.

Complementarily to the contents of the preceding paragraph, we want to add a remark which should be kept in mind: *Roles of equivalent representations and inequivalent irreducible representations of CCR and/or CAR are different.* Although equivalent representations are physically equivalent to each other as mentioned above, they may be *mathematically important.* For example, there may be mathematical problems which are not so easy to solve in a representation, but relatively or very easy to solve in other representations equivalent to the former.[2] From this point of view, it is important also to find equivalent representations of CCR and CAR respectively as many as possible.

As is well known, there is a method, called a Bogoliubov transformation, which generates a new representation of CCR (resp. CAR) from a given representation of CCR (resp. CAR). A necessary and sufficient condition for a Bogoliubov transformation to generate a representation equivalent to a standard representation, called a Fock representation, has been established (see, e.g., [17, 20, 21, 22, 23]). It seems, however, that inequivalent representations generated by Bogoliubov transformations have not been noted very much. In fact, there exist physically interesting examples of inequivalent representations; see, e.g., [6] (resp. [7]) in which a family of mutually inequivalent irreducible representations of CCR (resp. CAR) is constructed and boson masses (resp. fermion masses) appear as the labels of the family.

A standard Bogoliubov transformation is defined from a pair (T, S) of everywhere defined bounded linear operators on a one-particle Hilbert space in a Fock space. It would be natural to ask: what happens if T or S is unbounded? We call a Bogoliubov transformation with T or S unbounded a *singular Bogoliubov transformation.* Fundamental properties of singular Bogoliubov transformations have been studied in [9]. In the present paper, we report some basic results in [9].

[2] A typical example to which such a case applies is a one-dimensional quantum harmonic oscillator. The spectrum of the Hamiltonian is easily found in the Born–Heisenberg–Jordan representation of the CCR with one degree of freedom rather than in the Schrödinger one with the same degree, which is equivalent to the former. In quantum field theory, the Q-space representation (e.g., [5, 15, 19, 24]), which is equivalent to the Fock representation of the CCR over a Hilbert space, is very useful.

2 Representations of the CCR over an inner product space

In this section we review elementary aspects of representations of CCR.

Let $\mathscr{F}$ be a complex Hilbert space and $\mathscr{D}$ be a dense subspace of $\mathscr{F}$. Let $\mathscr{V}$ be a complex inner product space with inner product $\langle\ ,\ \rangle_{\mathscr{V}}$ and norm $\|\cdot\|_{\mathscr{V}}$.[3]

Definition 2.1 Suppose that, for each $f \in \mathscr{V}$, a densely defined closed linear operator $C(f)$ on $\mathscr{F}$ is given. Then the triple $(\mathscr{F}, \mathscr{D}, \{C(f), C(f)^*|f \in \mathscr{V}\})$ is called a representation of the CCR over $\mathscr{V}$ if the following (i)–(iii) hold:

(i) (invariance of $\mathscr{D}$) For all $f \in \mathscr{V}$,

$$\mathscr{D} \subset D(C(f)) \cap D(C(f)^*), \quad C(f)\mathscr{D} \subset \mathscr{D}, \quad C(f)^*\mathscr{D} \subset \mathscr{D}.$$

(ii) (anti-linearity in test vectors) For all $f, g \in \mathscr{V}$ and $\alpha, \beta \in \mathbb{C}$,

$$C(\alpha f + \beta g) = \alpha^* C(f) + \beta^* C(g)$$

on $\mathscr{D}$, where, for $z \in \mathbb{C}$, z^* denotes the complex conjugate of z.

(iii) (the CCR over $\mathscr{V}$) For all $f, g \in \mathscr{V}$,

$$[C(f), C(g)^*] = \langle f, g\rangle_{\mathscr{V}}, \quad [C(f), C(g)] = 0 \quad \text{on } \mathscr{D}.$$

Definition 2.2 Two representations $(\mathscr{F}, \mathscr{D}, \{C(f), C(f)^*|f \in \mathscr{V}\})$ and $(\mathscr{F}', \mathscr{D}', \{C'(f), C'(f)^*|f \in \mathscr{V}\})$ of the CCR over $\mathscr{V}$ are said to be equivalent if there exists a unitary operator $U : \mathscr{F} \to \mathscr{F}'$ such that, for all $f \in \mathscr{V}$,

$$UC(f)U^{-1} = C'(f). \tag{1}$$

Remark 2.3 Equation (1) implies that $UC(f)^*U^{-1} = C'(f)^*$, $f \in \mathscr{V}$.

Definition 2.4 Let $\mathfrak{A}$ be a set of (not necessarily bounded) linear operators on a Hilbert space $\mathscr{X}$.

(i) The set $\mathfrak{A}$ is said to be reducible if there is a non-trivial closed subspace $\mathscr{M}$ of $\mathscr{X}$ (i.e., $\mathscr{M} \neq \{0\}$, $\mathscr{X}$) such that every $A \in \mathfrak{A}$ is reduced by $\mathscr{M}$ (i.e., $P_{\mathscr{M}}A \subset AP_{\mathscr{M}}$, where $P_{\mathscr{M}}$ is the orthogonal projection onto $\mathscr{M}$).

[3] We sometimes omit the subscript $\mathscr{V}$ in $\langle\ ,\ \rangle_{\mathscr{V}}$ and norm $\|\cdot\|_{\mathscr{V}}$.

(ii) The set $\mathfrak{A}$ is said to be irreducible if it is not reducible.

Definition 2.5 A representation $(\mathscr{F}, \mathscr{D}, \{C(f), C(f)^*|f \in \mathscr{V}\})$ of the CCR over $\mathscr{V}$ is said to be reducible (resp. irreducible) if the set $\{C(f), C(f)^*|f \in \mathscr{V}\}$ is reducible (resp. irreducible).

For a Hilbert space $\mathscr{X}$, we denote by $\mathfrak{B}(\mathscr{X})$ the space of everywhere defined bounded linear operators on $\mathscr{X}$.

Definition 2.6 For a set $\mathfrak{A}$ of linear operators on a Hilbert space $\mathscr{X}$,

$$\mathfrak{A}' := \{T \in \mathfrak{B}(\mathscr{X})|TA \subset AT, \forall A \in \mathfrak{A}\}$$

is called the strong commutant of $\mathfrak{A}$.

The following fact is well known (see, e.g., [8, Proposition 5.9]):

Lemma 2.7 *Let $\mathfrak{A}$ be a set of linear operators on $\mathscr{X}$.*

(i) *If $\mathfrak{A}' = \mathbb{C}I := \{\alpha I|\alpha \in \mathbb{C}\}$ (I denotes identity), then $\mathfrak{A}$ is irreducible.*

(ii) *If $\mathfrak{A}$ is an irreducible set of densely defined closed linear operators on $\mathscr{X}$ and $*$-invariant (i.e., $A \in \mathfrak{A} \Longrightarrow A^* \in \mathfrak{A}$), then $\mathfrak{A}' = \mathbb{C}I$.*

3 Fock representation of CCR

Let $\mathscr{H}$ be a complex Hilbert space and denote by $\otimes_{\mathrm{s}}^n \mathscr{H}$ the n-fold symmetric tensor product Hilbert space of $\mathscr{H}$. We set $\otimes_{\mathrm{s}}^0 \mathscr{H} := \mathbb{C}$. The boson Fock space over $\mathscr{H}$ is defined by

$$\begin{aligned}\mathscr{F}_{\mathrm{b}}(\mathscr{H}) :&= \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n \mathscr{H} \\ &= \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} | \Psi^{(n)} \in \otimes_{\mathrm{s}}^n \mathscr{H},\, n \geq 0,\, \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty \right\}.\end{aligned}$$

The subspace

$$\mathscr{F}_0(\mathscr{H}) := \{\Psi \in \mathscr{F}_{\mathrm{b}}(\mathscr{H})|\exists n_0 \in \mathbb{N} \text{ such that } \Psi^{(n)} = 0, \forall n \geq n_0\}$$

is called the finite particle subspace of $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$. It follows that $\mathscr{F}_0(\mathscr{H})$ is dense in $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$.

For each $f \in \mathscr{H}$, a densely defined closed linear operator $A(f)$ on $\mathscr{F}_{\mathrm{b}}(\mathscr{H})$, called the annihilation operator with test vector $f \in \mathscr{H}$, is defined in such a way that the adjoint $A(f)^*$ of $A(f)$, the creation operator with test vector f, is of the form:

$$D(A(f)^*) = \left\{ \Psi \in \mathscr{F}_{\mathrm{b}}(\mathscr{H}) | \sum_{n=1}^{\infty} \| \sqrt{n} S_n (f \otimes \Psi^{(n-1)}) \|^2 < \infty \right\},$$
$$(A(f)^*\Psi)^{(0)} = 0,$$
$$(A(f)^*\Psi)^{(n)} = \sqrt{n} S_n (f \otimes \Psi^{(n-1)}), \quad n \geq 1,$$

where S_n denotes the symmetrization operator on the n-fold tensor product Hilbert space $\otimes^n \mathscr{H}$ of $\mathscr{H}$. The following proposition is well known (or easy to show):

Proposition 3.1

(i) For all $f \in \mathscr{H}$, $\mathscr{F}_0(\mathscr{H}) \subset D(A(f)) \cap D(A(f)^*)$ and $A(f)$ and $A(f)^*$ leave $\mathscr{F}_0(\mathscr{H})$ invariant.

(ii) $\{A(f), A(f)^* | f \in \mathscr{H}\}$ satisfies the CCR over $\mathscr{H}$:

$$[A(f), A(g)^*] = \langle f, g \rangle_{\mathscr{H}}, \quad [A(f), A(g)] = 0 \quad (f, g \in \mathscr{H})$$

on $\mathscr{F}_0(\mathscr{H})$.

This proposition shows that, for any subspace $\mathscr{D}$ of $\mathscr{H}$,

$$\pi_{\mathrm{F}}(\mathscr{D}) := (\mathscr{F}_{\mathrm{b}}(\mathscr{H}), \mathscr{F}_0(\mathscr{H}), \{A(f), A(f)^* | f \in \mathscr{D}\})$$

is a representation of the CCR over $\mathscr{D}$. It is called the Fock representation of the CCR over $\mathscr{D}$. Concerning irreducibility of $\pi_{\mathrm{F}}(\mathscr{D})$, we have:

Proposition 3.2 *Let $\mathscr{D}$ be a dense subspace of $\mathscr{H}$. Then $\pi_{\mathrm{F}}(\mathscr{D})$ is irreducible.*

Proof. See [8, Theorem 5.14]. ∎

4 Standard Bogoliubov transformations

The main topic of the present paper is a singular Bogoliubov transformation. But, for comparison, we first review a standard Bogoliubov transformation [8, 17, 20, 21, 22, 23].

Let J be a conjugation on $\mathscr{H}$ and suppose that there exist linear operators $T, S \in \mathfrak{B}(\mathscr{H})$ satisfying

$$T^*T - S^*S = I, \quad T^*JS = S^*JT. \tag{2}$$

Let $\mathscr{D}$ be a dense subspace of $\mathscr{H}$. Then it is easy to see that $A(Tf)+A(JSf)^*$ is a densely defined closable operator. Hence one can define

$$\widehat{A}(f) := \overline{A(Tf) + A(JSf)^*}, \quad f \in \mathscr{D},$$

where, for a closable operator C, we denote its closure by $\overline{C}$. It is easy to prove the following lemma:

Lemma 4.1 The triple

$$\pi_{\widehat{A}}(\mathscr{D}) := (\mathscr{F}_{\mathrm{b}}(\mathscr{H}), \mathscr{F}_0(\mathscr{H}), \{\widehat{A}(f), \widehat{A}(f)^* | f \in \mathscr{D}\})$$

is a representation of the CCR over $\mathscr{D}$.

The correspondence: $(A(\cdot), A(\cdot)^*) \to (\widehat{A}(\cdot), \widehat{A}(\cdot)^*)$ is called a Bogoliubov transformation, which preserves the CCR over $\mathscr{D}$.

The following proposition is essentially known:

Proposition 4.2 *Assume that $\mathscr{H}$ is separable and that S is not Hilbert-Schmidt. Then $\pi_{\widehat{A}}(\mathscr{D})$ is inequivalent to any direct sum representation of $\pi_{\mathrm{F}}(\mathscr{D})$.*

The operators T and S may satisfy additional conditions:

$$TT^* - JSS^*J = I, \quad JST^* = TS^*J. \tag{3}$$

The following theorem is well known.

Theorem 4.3 *Assume that $\mathscr{H}$ is separable and suppose that (2) and (3) hold. Then $\pi_{\widehat{A}}(\mathscr{D})$ is equivalent to $\pi_{\mathrm{F}}(\mathscr{D})$ if and only if S is Hilbert-Schmidt.*

5 Singular Bogoliubov transformations

In what follows, we present only results. For proofs of them, see [9].

5.1 Definitions

Let T and S be densely defined (not necessarily bounded) linear operators on $\mathscr{H}$ such that there exists a dense subspace $\mathscr{D} \subset D(T) \cap D(S)$ and the following equations hold:

$$\langle Tf, Tg \rangle - \langle Sf, Sg \rangle = \langle f, g \rangle \,, \tag{4}$$

$$\langle Tf, JSg \rangle = \langle Sf, JTg \rangle \,, \quad f, g \in \mathscr{D}, \tag{5}$$

where J is a conjugation on $\mathscr{H}$.

For each $f \in \mathscr{D}$, one can define

$$B(f) := \overline{A(Tf) + A(JSf)^*}. \tag{6}$$

Conditions (4) and (5) imply the following proposition:

Proposition 5.1

$$\pi_B(\mathscr{D}) := (\mathscr{F}_{\mathrm{b}}(\mathscr{H}), \mathscr{F}_0(\mathscr{H}), \{B(f), B(f)^* | f \in \mathscr{D}\}) \tag{7}$$

is a representation of the CCR over $\mathscr{D}$*.*

We call the correspondence T_B: $(A(\cdot), A(\cdot)^*) \mapsto (B(\cdot), B(\cdot)^*)$ a singular Bogoliubov transformation if S or T is unbounded (then both T and S are unbounded).

Remark 5.2 *Suppose that T or S is bounded. Then both T and S are bounded and*

$$\overline{T}^* \overline{T} - \overline{S}^* \overline{S} = I, \quad \overline{T}^* J \overline{S} = \overline{S}^* J \overline{T}.$$

Hence, in this case, (4) and (5) are equivalent to (2) with (T, S) replaced by $(\overline{T}, \overline{S})$ and T_B becomes a standard Bogoliubov transformation.

5.2 Inequivalence to any direct sum representation of the Fock representation $\pi_{\mathrm{F}}(\mathscr{D})$

To discuss if $\pi_B(\mathscr{D})$ is inequivalent to $\pi_{\mathrm{F}}(\mathscr{D})$, we first present a general fact:

Lemma 5.3 *Let $\mathscr{F}$ be a Hilbert space and $\mathscr{D}$ be a dense subspace of $\mathscr{H}$. Let*

$$\pi_C(\mathscr{D}) := (\mathscr{F}, \mathscr{D}, \{C(f), C(f)^* | f \in \mathscr{D}\})$$

be a representation of the CCR over $\mathscr{D}$. Suppose that $\pi_C(\mathscr{D})$ is equivalent to a direct sum representation $\oplus_{n=1}^N \pi_{\mathrm{F}}(\mathscr{D})$ of $\pi_{\mathrm{F}}(\mathscr{D})$ with $N < \infty$ or $N = \infty$. Then there exists a non-zero vector $\Omega \in \cap_{f \in \mathscr{D}} D(C(f))$ such that $C(f)\Omega = 0$, $f \in \mathscr{D}$.

Remark 5.4 *A non-zero vector* $\Omega \in \cap_{f\in\mathscr{D}} D(C(f))$ *such that* $C(f)\Omega = 0$, $f \in \mathscr{D}$ *is called a vacuum vector for* $\pi_C(\mathscr{D})$.

Lemma 5.5 *(absence of vacuum vectors for* $\pi_B(\mathscr{D})$*) Assume that* $\mathscr{H}$ *is separable. Suppose that* T *is unbounded and* $T\mathscr{D}$ *is dense in* $\mathscr{H}$. *Then there exist no non-zero vectors* $\Omega \in \cap_{f\in\mathscr{D}} D(B(f))$ *such that* $B(f)\Omega = 0$, $f \in \mathscr{D}$.

Theorem 5.6 *Assume that* $\mathscr{H}$ *is separable. Suppose that* T *is unbounded and* $T\mathscr{D}$ *is dense in* $\mathscr{H}$. *Then* $\pi_B(\mathscr{D})$ *is inequivalent to any direct sum representation of the Fock representation* $\pi_{\mathrm{F}}(\mathscr{D})$. *In particular, if* $\pi_B(\mathscr{D})$ *is irreducible, then* $\pi_B(\mathscr{D})$ *is inequivalent to* $\pi_{\mathrm{F}}(\mathscr{D})$.

5.3 Irreducibility

Assumption (I) There exists a dense subspace $\mathscr{D}_1$ of $\mathscr{H}$ such that the following (i) and (ii) hold:

(i) $\mathscr{D}_1 \subset D(TT^*) \cap D(JSS^*J) \cap D(ST^*) \cap D(JTS^*J)$.

(ii) $T^*\mathscr{D}_1 \subset \mathscr{D}$, $(S^*J)\mathscr{D}_1 \subset \mathscr{D}$ and

$$TT^* - JSS^*J = I, \quad ST^*J = JTS^* \quad \text{on } \mathscr{D}_1.$$

Lemma 5.7 *Suppose that Assumption (I) holds. Then, for all* $f \in \mathscr{D}_1$,

$$A(f) = B(T^*f) - B(S^*Jf)^* \quad \text{on } \mathscr{F}_0(\mathscr{H}). \tag{8}$$

Theorem 5.8 *Suppose in addition to (4) and (5) that Assumption (I) holds. Then* $\pi_B(\mathscr{D})$ *is irreducible.*

The subset

$$\mathscr{H}_J := \{f \in \mathscr{H} | Jf = f\}$$

becomes a real Hilbert space and each $f \in \mathscr{H}$ is uniquely written as

$$f = f_1 + if_2$$

with $f_1, f_2 \in \mathscr{H}_J$.

Another criterion for the irreducibility of $\pi_B(\mathscr{D})$ is given as follows:

Theorem 5.9 *Suppose in addition to (4) and (5) that*

$$JS \subset SJ, \quad JT \subset TJ.$$

Let $\mathscr{D}_J := \mathscr{D} \cap \mathscr{H}_J$, $R := (1+i)T + (1-i)S$ *and suppose that* $R\mathscr{D}_J$ *is dense in* $\mathscr{H}$. *Then* $\pi_B(\mathscr{D})$ *is irreducible.*

6 A general class of singular Bogoliubov transformations

Let K and L be injective (not necessarily bounded) symmetric operators on a Hilbert space $\mathscr{H}$ such that

$$\mathscr{D}_{K,L} := D(K^{-1}L) \cap D(KL^{-1}) \tag{9}$$

is dense in $\mathscr{H}$ and, for a conjugation J on $\mathscr{H}$,

$$JK \subset KJ, \quad JL \subset LJ. \tag{10}$$

Then one can define densely defined linear operators

$$T_\pm := \frac{1}{2}(K^{-1}L \pm KL^{-1}) \tag{11}$$

with $D(T_\pm) = \mathscr{D}_{K,L}$. It follows from (10) that

$$JT_\pm \subset T_\pm J.$$

Lemma 6.1 *For all $f, g \in \mathscr{D}_{K,L}$,*

$$\langle T_+ f, T_+ g\rangle - \langle T_- f, T_- g\rangle = \langle f, g\rangle\,, \tag{12}$$

$$\langle T_+ f, JT_- g\rangle = \langle T_- f, JT_+ g\rangle\,. \tag{13}$$

Let $\mathscr{D}$ be a dense subspace of $\mathscr{H}$ such that $\mathscr{D} \subset \mathscr{D}_{K,L}$. Then one can define a densely defined closed linear operator

$$B_{K,L}(f) := \overline{A(T_+ f) + A(JT_- f)^*}, \quad f \in \mathscr{D}. \tag{14}$$

Proposition 6.2

$$\pi_{K,L}(\mathscr{D}) := (\mathscr{F}_{\mathrm{b}}(\mathscr{H}), \mathscr{F}_0(\mathscr{H}), \{B_{K,L}(f), B_{K,L}(f)^* | f \in \mathscr{D}\})n \tag{15}$$

is a representation of the CCR over $\mathscr{D}$.

If T_+ or T_- is unbounded, then the correspondence: $(A(\cdot), A(\cdot)^*) \mapsto (B_{K,L}(\cdot),\, B_{K,L}(\cdot)^*)$ is a singular Bogoliubov transformation.

A simple application of Lemma 5.5 and Theorem 5.6 yields the following theorem:

Theorem 6.3 *Let $\mathscr{H}$ be separable. Suppose that T_+ is unbounded and $T_+\mathscr{D}$ is dense. Then:*

(i) There exists no non-zero vector $\Omega \in \cap_{f \in \mathscr{D}} D(B_{K,L}(f))$ *such that* $B_{K,L}(f)\Omega = 0$, $f \in \mathscr{D}$.

(ii) The representation $\pi_{K,L}(\mathscr{D})$ *is inequivalent to any direct sum representation of the Fock representation* $\pi_{\mathrm{F}}(\mathscr{D})$*. In particular, if* $\pi_{K,L}(\mathscr{D})$ *is irreducible, then* $\pi_{K,L}(\mathscr{D})$ *is inequivalent to* $\pi_{\mathrm{F}}(\mathscr{D})$.

With regard to irreducibility of $\pi_{K,L}(\mathscr{D})$, we have the following result.

Theorem 6.4 *Suppose that* $(K^{-1}L+iKL^{-1})(\mathscr{D}\cap\mathscr{H}_J)$ *is dense in* $\mathscr{H}$ *. Then* $\pi_{K,L}(\mathscr{D})$ *is irreducible.*

Proof. We need only to apply Theorem 5.9 to the case where $T = T_+$, $S = T_-$. In this case $R = K^{-1}L + iKL^{-1}$. ∎

Example 6.5 An example of singular Bogoliubov transformations in the form $\pi_{K,L}(\mathscr{D})$ is associated with the Casimir effect [12, 16, 18, 25] in the context of a quantum scalar field. See [8] for mathematical details. A singular Bogoliubov transformation appears also in a quantum scalar field theory [6].

References

[1] Aharonov, Y., Bohm, D.: Significance of electromagnetic potentials in the quantum theory. Phys. Rev. **115**, 485–491 (1959)

[2] Arai, A.: Momentum operators with gauge potentials, local quantization of magnetic flux, and representation of canonical commutation relations. J. Math. Phys. **33**, 3374–3378 (1992)

[3] Arai, A.: Representation-theoretic aspects of two-dimensional quantum systems in singular vector potentials: canonical commutation relations, quantum algebras, and reduction to lattice quantum systems. J. Math. Phys. **39**, 2476–2498 (1998)

[4] Arai, A.: Mathematical Principles of Quantum Statistical Mechanics. In Japanese. Kyoritsu-shuppan, Tokyo, 2008

[5] Arai, A.: Functional Integral Methods in Quantum Mathematical Physics. In Japanese. Kyoritsu-shuppan, Tokyo, 2010

[6] Arai, A.: A family of inequivalent Weyl representations of canonical commutation relations with applications to quantum field theory. Rev. Math. Phys. **28**, 1650007 (2016)

[7] Arai, A.: Inequivalence of quantum Dirac fields of different masses and the underlying general structures involved. In: Functional Analysis and Operator Theory for Quantum Physics, 31–53, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2017

[8] Arai, A.: Analysis on Fock Spaces and Mathematical Theory of Quantum Fields. World Scientific, Singapore, 2018

[9] Arai, A.: Singular Bogoliubov transformations and inequivalent representations of canonical commutation relations. Rev. Math. Phys. **31**, 1950026 (2019)

[10] Arai, A.: Inequivalent Representations of Canonical Commutation and Anti-Commutation Relations. Springer, to be published

[11] Araki, H., Woods, E. J.: Representations of the canonical commutation relations describing a nonrelativistic infinite free Bose gas. J. Math. Phys. **4**, 637–662 (1963)

[12] Casimir, H. B. G.: On the attraction between two perfectly conducting plates. Proc. Koninklijke Nederlandse Akademie van Wetenschappen **51**, 793–795 (1948)

[13] Dappiaggi, C., Nosari, G., Pinamonti, N.: The Casimir effect from the point of view of algebraic quantum field theory. Math. Phys. Anal. Geom.**19**, 12 (2016)

[14] Ezawa, H.: Quantum mechanics of a many-boson system and the representation of canonical variables. J. Math. Phys. **6**, 380–404 (1965)

[15] Ezawa, H., Arai, A.: Quantum Field Theory and Statistical Mechanics. In Japanese. Nippon-Hyoron-sha, Tokyo, 1988

[16] Ezawa, H., Nakamura, K., Watanabe, K.: The Casimir force from Lorentz's. Frontiers in Quantum Physics (Lim, S. C., Abd-Shukor, R., Kwek, K. H. eds.), Springer, Singapore, 160–169 (1998)

[17] Hiroshima, F., Sasaki, I., Spohn, H., Suzuki, A.: Enhanced Binding in Quantum Field Theory. COE Lecture Note Vol. 38, Institute of Mathematics for Industry, Kyushu University, 2012

[18] Lamoreaux, S. K.: Demonstration of the Casimir force in the 0.6 to 6 μm range. Phys. Rev. Lett. **78**, 5–8 (1998); Erratum Phys. Rev. Lett. **81**, 5475–5476 (1998)

[19] Lörenczi, J., Hiroshima, F., Betz, V.: Feynman–Kac-Type Theorems and Gibbs Measures on Path Space. De Gruyter, Berlin, 2011

[20] Reed, M., Simon, B.: Methods of Modern Mathematical Physics III: Scattering Theory. Academic Press, New York, 1979

[21] Ruijsenaars, S. N. M.: On Bogoliubov transformations for systems of relativistic charged particles. J. Math. Phys. **18**, 517–526 (1977)

[22] Ruijsenaars, S. N. M.: On Bogoliubov transformations. II. The general case. Ann. of Phys. **116**, 105–134 (1978)

[23] Shale, D.: Linear symmetries of free boson fields. Trans. Amer. Math. Soc. **103**, 149–167 (1962)

[24] Simon, B.: The $P(\phi)_2$ Euclidean Quantum Field Theory. Princeton University Press, Princeton, 1974

[25] Sparnaay, M. J.: Measurements of attractive forces between flat plates. Physica **24**, 751–764 (1958).