Towards new uncertainty relations

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I Introduction

Since Heisenberg's paper [4], uncertainty relation is a central topic of quantum measurement theory. It is now known that Heisenberg's error-disturbance relation, one of his uncertainty relations,

$$\varepsilon(Q)\eta(P) \geq \frac{\hbar}{2} \tag{1}$$

is violated in general, where Q and P, respectively, are the position and the momentum of a nonrelativistic single-particle system. The *error-free* linear measurement [13] is constructed in the 1980s, and universally valid uncertainty relations [14, 15, 16, 17, 2, 18] are shown. An error called the noise-operator based quantum root-mean-square (q-rms) error is used in previous investigations. In the paper, we define another q-rms error for discrete observables, and give universally valid uncertainty relations for such a q-rms error.

In Section II, we introduce preliminaries on algebraic quantum theory and quantum measurement theory. Completely positive (CP) instrument and measuring process are defined. In Section III, we define a q-rms error for discrete observables, and give universally valid uncertainty relations.

II Algebraic Quantum Theory and Measurement

Here we assume the following axiomatic system.

Axiom 1. *A physical system in an experimental situation is described by a W^*-probability space $(\boldsymbol{B}(\mathcal{H}), \rho)$, a pair of a W^*-algebra and a normal state on it.*

In this talk, we assume that all Hilbert spaces $\mathcal{H}$ are separable. We do not distinguish normal states ρ on $\boldsymbol{B}(\mathcal{H})$ and density operators $\tilde{\rho}$ via the isomorphism $\tilde{\cdot} : \boldsymbol{B}(\mathcal{H})_* \to \boldsymbol{T}(\mathcal{H})$ such that

$$\rho(X) = \mathrm{Tr}[\tilde{\rho}X], \quad X \in \boldsymbol{B}(\mathcal{H}). \tag{2}$$

We also assume the Born statistical formula.

Axiom 2. *When an observable A of $\boldsymbol{B}(\mathcal{H})$ is precisely measured in a normal state ρ, the probability $\Pr\{A \in \Delta \| \rho\}$ that the spectrum of A belonging to Δ emerge is given by*

$$\Pr\{A \in \Delta \| \rho\} = \mathrm{Tr}[\rho E^A(\Delta)]. \tag{3}$$

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Let $A = \sum aE^A(a)$ be a discrete observable of $\boldsymbol{B}(\mathcal{H})$.

Postulate 1 (von Neumann-Lüders projection postulate)**.** *When a state ρ is prepared, the state after the measurement of A is given by*

$$\frac{\sum_{a\in\Delta} E^A(a)\rho E^A(a)}{\mathrm{Tr}[\rho E^A(\Delta)]} = \frac{\left(\sum_{a\in\mathbb{R}} E^A(a)\rho E^A(a)\right)\cdot E^A(\Delta)}{\mathrm{Tr}[\rho E^A(\Delta)]}.$$

Under the repeatability hypothesis, a standard assumption in the 1930s, von Neumann proved that this postulate is derived only for nondegenerate discrete observables in his famous book [7]. He also constructed a model of quantum measurement called a von Neumann model. Lüders [5] generalized the projection postulate for the degenerate case.

Postulate 2 (Repeatable hypothesis)**.** *If an observable A is measured twice in succesion in the object system, then we get the same value each time.*

This postulate is valid only for discrete observables. The linear map $\mathcal{E}_A : \boldsymbol{B}(\mathcal{H}) \to \{A\}' := \{E^A(\Delta) \mid \Delta \in \mathcal{B}(\mathbb{R})\}'$ defined by

$$\mathcal{E}_A(X) = \sum_{a\in\mathbb{R}} E^A(\{a\})XE^A(\{a\}), \quad X \in \boldsymbol{B}(\mathcal{H}) \tag{4}$$

is a conditional expectation [22, 6]. It is, however, known that such conditional expectations do not exist for the continuous case [1]. Furthermore, there exist measurements of observables which cannot be described by (the predual of) conditional expectations.

Following those investigations, Davies and Lewis [3] abandoned the repreatability hypothesis (Postulate 2) and introduced the notion of instrument which describes general state changes caused by the measurement. Ozawa [12] introduced the notions of completely positive (CP) instrument and of measuring process and established the one-to-one correspondence between them in quantum mechanical systems (type I factors). The author and Ozawa introduced the normal extension property (NEP) for CP instruments and generalized the above one-to-one correspondence between CP instruments with the NEP and measuring processes on general von Neumann algebras [8, 9]. The author also developes quantum measurement theory in C*-algebraic quantum theory [11].

$\boldsymbol{T}(\mathcal{H})$ denotes the set of trace-class operators on a separable Hilbert space $\mathcal{H}$. $CP(\boldsymbol{T}(\mathcal{H}))$ denotes the set of completely positive linear maps on $\boldsymbol{T}(\mathcal{H})$. Let $(S, \mathcal{F})$ be a measurable space.

Definition 1 (CP instrument)**.** *A map $\mathcal{I} : \mathcal{F} \to CP(\boldsymbol{T}(\mathcal{H}))$ is called a CP instrument for $(\boldsymbol{B}(\mathcal{H}), S)$ if it satisfies the following two conditions:*
(1) *For all $\rho \in \boldsymbol{T}(\mathcal{H})$, $M \in \boldsymbol{B}(\mathcal{H})$ and mutually disjoint sequence $\{\Delta_j\} \subset \mathcal{F}$,*

$$\mathrm{Tr}[(\mathcal{I}(\cup_j \Delta_j)\rho)M] = \sum_{j=1}^{\infty} \mathrm{Tr}[(\mathcal{I}(\Delta_j)\rho)M]; \tag{5}$$

(2) $\mathrm{Tr}[\mathcal{I}(S)\rho] = \mathrm{Tr}[\rho]$ *for all $\rho \in \boldsymbol{T}(\mathcal{H})$.*

The dual map $\mathcal{I} : \boldsymbol{B}(\mathcal{H}) \times \mathcal{F} \to \boldsymbol{B}(\mathcal{H})$ of a CP instrument $\mathcal{I}$ for $(\boldsymbol{B}(\mathcal{H}), S)$ is defined by

$$\mathrm{Tr}[\rho\mathcal{I}(M, \Delta)] = \mathrm{Tr}[\mathcal{I}(\Delta)\rho M] \tag{6}$$

for all $\rho \in \boldsymbol{T}(\mathcal{H})$, $M \in \boldsymbol{B}(\mathcal{H})$ and $\Delta \in \mathcal{F}$.

The dual map of an instrument $\mathcal{I}$ for $(\boldsymbol{B}(\mathcal{H}), S)$ is characterized by the following conditions:
(i) For every $\Delta \in \mathcal{F}$, the map $M \mapsto \mathcal{I}(M, \Delta)$ is CP and linear;
(ii) For all $\rho \in \boldsymbol{T}(\mathcal{H})$, $M \in \boldsymbol{B}(\mathcal{H})$ and mutually disjoint sequence $\{\Delta_j\} \subset \mathcal{F}$,

$$\mathrm{Tr}[\rho\mathcal{I}(M, \cup_j\Delta_j)] = \sum_{j=1}^{\infty} \mathrm{Tr}[\rho\mathcal{I}(M, \Delta_j)]; \tag{7}$$

(iii) $\mathcal{I}(1, S) = 1$.
That is to say, a map $\mathcal{I} : \boldsymbol{B}(\mathcal{H}) \times \mathcal{F} \to \boldsymbol{B}(\mathcal{H})$ satisfying the above conditions is the dual map of a CP instrument $\mathcal{I}$ for $(\boldsymbol{B}(\mathcal{H}), S)$.

We shall define measuring process.

Definition 2 (Measuring process)**.** *A measuring process* $\mathbb{M}$ *for* $(\boldsymbol{B}(\mathcal{H}), S)$ *is a 4-tuple* $\mathbb{M} = (\mathcal{K}, \sigma, E, U)$ *consisting of*
(1) *a Hilbert space* $\mathcal{K}$*,*
(2) *a normal state* σ *on* $\boldsymbol{B}(\mathcal{K})$*,*
(3) *a spectral measure* $E : \mathcal{F} \to \boldsymbol{B}(\mathcal{K})$*, and*
(4) *a unitary operator* U *on* $\mathcal{H} \otimes \mathcal{K}$*.*

This is a generalization of von Neumann model of measurement [7].

Example 3. *Let* A *be an observable of* $\boldsymbol{B}(\mathcal{H})$ *to be measured. Von Neumann [7] discussed a model of measurement consisting of*
(1) *a Hilbert space* $L^2(\mathbb{R})$*,*
(2) *a unit vector* $\xi \in L^2(\mathbb{R})$*,*
(3) *a meter observable* $Q = \int q\, dE^Q(q)$*, and*
(4) *a unitary* $U = e^{i\gamma A\otimes P}$ *on* $\mathcal{H} \otimes L^2(\mathbb{R})$*,* $(0 \neq \gamma \in \mathbb{R}$*,* $[Q, P] = i1)$*,*
which defines a CP instrument $\mathcal{I}_{A,\mathrm{vN}}$ *for* $(\boldsymbol{B}(\mathcal{H}), \mathbb{R})$ *by*

$$\mathcal{I}_{A,\mathrm{vN}}(\Delta)\rho = \mathrm{Tr}_{L^2(\mathbb{R})}[U(\rho \otimes |\xi\rangle\langle\xi|)U^*(1 \otimes E^Q(\Delta))] \tag{8}$$

for all $\Delta \in \mathcal{B}(\mathbb{R})$ *and* $\rho \in \boldsymbol{T}(\mathcal{H})$*.*

By results of Stinespring's paper [21] and the uniqueness theorem of the irreducible normal representation of $\boldsymbol{B}(\mathcal{H})$, the following theorem holds:

Theorem 4 (Ozawa [12])**.** *Let* $\mathcal{H}$ *be a Hilbert space and* $(S, \mathcal{F})$ *a measurable space. Then there is a one-to-one correpondence between statistical equivalence classes of measuring processes* $\mathbb{M} = (\mathcal{K}, \sigma, E, U)$ *for* $(\boldsymbol{B}(\mathcal{H}), S)$ *and CP instruments* $\mathcal{I}$ *for* $(\boldsymbol{B}(\mathcal{H}), S)$*, which is given by the relation*

$$\mathcal{I}(\Delta)\rho = \mathrm{Tr}_{\mathcal{K}}[U(\rho \otimes \sigma)U^*(1 \otimes E(\Delta))] \tag{9}$$

for all $\Delta \in \mathcal{F}$ *and* $\rho \in \boldsymbol{T}(\mathcal{H})$*.*

Two measuring processes $\mathbb{M}_1 = (\mathcal{K}_1, \sigma_1, E_1, U_1)$ and $\mathbb{M}_2 = (\mathcal{K}_2, \sigma_2, E_2, U_2)$ for $(\mathcal{M}, S)$ are said to be statistically equivalent if $\mathcal{I}_{\mathbb{M}_1}(\Delta) = \mathcal{I}_{\mathbb{M}_2}(\Delta)$ for all $\Delta \in \mathcal{F}$.

III New uncertainty relations

For every CP instrument $\mathcal{I}$ for $(\boldsymbol{B}(\mathcal{H}), \mathbb{R})$, $\Pi_{\mathcal{I}}$ denotes a POVM on $\mathcal{H}$ defined by $\Pi_{\mathcal{I}}(\Delta) = \mathcal{I}(1, \Delta)$ for all $\Delta \in \mathcal{F}$. The n-th moment $\Pi_{\mathcal{I}}^{(n)}$ of $\Pi_{\mathcal{I}}$ is defined by

$$\Pi_{\mathcal{I}}^{(n)} = \int_{\mathbb{R}} x^n \, d\Pi_{\mathcal{I}}(x). \tag{10}$$

Here $\Pi_{\mathcal{I}}^{(n)}$ is defined on $\mathrm{dom}(\Pi_{\mathcal{I}}^{(n)})$ whose elements $\zeta \in \mathcal{H}$ are given by

$$\int_{\mathbb{R}} x^{2n} \, d\langle \zeta | \Pi_{\mathcal{I}}(x) \zeta \rangle < +\infty. \tag{11}$$

For every measuring process $\mathbb{M} = (\mathcal{K}, \sigma, E, U)$ for $(\boldsymbol{B}(\mathcal{H}), \mathbb{R})$, it is also denoted by $\mathbb{M} = (\mathcal{K}, \sigma, M, U)$, where M is an observable on $\mathcal{K}$ such that $M = \int m \, dE(m)$.

Definition 5. *Let A, B be observables of $\boldsymbol{B}(\mathcal{H})$. When a CP instrument $\mathcal{I}$ for $(\boldsymbol{B}(\mathcal{H}), \mathbb{R})$ in the state ρ, the error $\varepsilon_0(A)$ of A and the disturbance $\eta_0(B)$ of B, respectively, are defined as follows:*

$$\begin{aligned} \varepsilon_0(A)^2 = \varepsilon_0(A, \mathcal{I}, \rho)^2 &= \mathrm{Tr}[(\Pi_{\mathcal{I}}^{(2)} - A\Pi_{\mathcal{I}}^{(1)} - \Pi_{\mathcal{I}}^{(1)} A + A^2)(\rho)] \\ &= \mathrm{Tr}[(U^*(1 \otimes M)U - A \otimes 1)^2 (\rho \otimes \sigma)], \qquad (12) \\ \eta_0(B)^2 = \eta_0(B, \mathcal{I}, \rho)^2 &= \mathrm{Tr}[(\mathcal{I}(B^2, \mathcal{R}) - B\mathcal{I}(B, \mathbb{R}) - \mathcal{I}(B, \mathbb{R})B + B^2)(\rho)] \\ &= \mathrm{Tr}[(U^*(B \otimes 1)U - B \otimes 1)^2 (\rho \otimes \sigma)] \qquad (13) \end{aligned}$$

$\varepsilon_0(A)$ and $\eta_0(B)$ satisfies the following inequality called the Branciard-Ozawa error-disturbance relation:

Theorem 6.

$$\varepsilon_0(A)^2\sigma(B)^2 + \sigma(A)^2\eta_0(B)^2 + 2\varepsilon_0(A)\eta_0(B)\sqrt{\sigma(A)^2\sigma(B)^2 - D_{AB}^2} \geq D_{AB}^2, \tag{14}$$

where D_{AB} is defined by

$$D_{AB} = \frac{1}{2}\mathrm{Tr}|\sqrt{\rho}(-i[A, B])\sqrt{\rho}|. \tag{15}$$

Branciard [2] proved this inequality for vector states. For vector states $\rho = |\psi\rangle\langle\psi|$, D_{AB} is equal to

$$C_{AB} := \frac{1}{2}|\mathrm{Tr}[\rho(-i[A, B])]|. \tag{16}$$

After his study, Ozawa [18] showed Eq. (14) for any states.

It is known that there are many quantum generalizations of error and disturbance, which are not mutually equivalent in general. From now on, we show a similar uncertainty relation for another error and disturbance.

Let A be a discrete observable on $\mathcal{H}$ and $\mathcal{I}$ a CP instrument for $(\boldsymbol{B}(\mathcal{H}), \mathbb{R})$. We define a joint distribution of the successive measurement of A and $\mathcal{I}$ in this order by

$$\mu_\rho^{\mathcal{I},A}(\Delta \times \Gamma) = \mathrm{Tr}[\Pi_{\mathcal{I}}(\Gamma)E^A(\Delta)\mathcal{E}_A(\rho)] \tag{17}$$

for all $\Delta, \Gamma \in \mathcal{B}(\mathbb{R})$.

Definition 7. *Let A, B be discrete observables of $\boldsymbol{B}(\mathcal{H})$. When a CP instrument $\mathcal{I}$ for $(\boldsymbol{B}(\mathcal{H}), \mathbb{R})$ in the state ρ, the error $\varepsilon_m(A)$ of A and the disturbance $\eta_m(B)$ of B, respectively, are defined as follows:*

$$\begin{aligned}
\varepsilon_m(A)^2 = \varepsilon_m(A, \mathcal{I}, \rho)^2 &= \int_{\mathbb{R}^2} (a-m)^2 \, d\mu_\rho^{\mathcal{I},A}(m,a) \\
&= \mathrm{Tr}[(U^*(1 \otimes M)U - A \otimes 1)^2(\mathcal{E}_A(\rho) \otimes \sigma)], \qquad (18)\\
\eta_m(B)^2 = \eta_m(B, \mathcal{I}, \rho)^2 &= \int_{\mathbb{R}^2} (b'-b)^2 \, d\mathrm{Tr}[\mathcal{I}(E^B(b'), \mathbb{R})E^B(b)\mathcal{E}_B(\rho)] \\
&= \mathrm{Tr}[(U^*(B \otimes 1)U - B \otimes 1)^2(\mathcal{E}_B(\rho) \otimes \sigma)]. \qquad (19)
\end{aligned}$$

$\varepsilon_m(A)$ satisfies the following relation [20]:

$$\varepsilon_m(A)^2 = m_t(\varepsilon_m(A, \mathcal{I}, e^{-itA}\rho e^{itA})^2) \tag{20}$$

where $m = m_t$ is an invariant mean on $\mathbb{R}$. We define Gauss' error $\varepsilon_G(\mu)$ for a probability measure on $\mathbb{R}^2$ by

$$\varepsilon_G(\mu) = \left(\int_{\mathbb{R}^2} (x-y)^2 \, d\mu(x,y) \right)^{\frac{1}{2}}. \tag{21}$$

Let $(S_1, \mathcal{F}_1)$ and $(S_2, \mathcal{F}_2)$ be measurable spaces. Two POVMs $\Pi_1 : \mathcal{F}_1 \to \boldsymbol{B}(\mathcal{H})$ and $\Pi_2 : \mathcal{F}_2 \to \boldsymbol{B}(\mathcal{H})$ are commuting in ρ if

$$\Pi_1(\Delta)\Pi_2(\Gamma)\sqrt{\rho} = \Pi_2(\Gamma)\Pi_1(\Delta)\sqrt{\rho} \tag{22}$$

for all $\Delta \in \mathcal{F}_1$ and $\Gamma \in \mathcal{F}_2$. Let A be an observable of $\boldsymbol{B}(\mathcal{H})$. Π_1 and A are commuting in ρ if so are Π_1 and E^A in ρ. If a POVM $\Pi : \mathcal{B}(\mathbb{R}) \to \boldsymbol{B}(\mathcal{H})$ and an observable A are commuting in ρ, there exists a probability measure $\mu_\rho^{\Pi_{\mathcal{I}},A}$ on $\mathbb{R}^2$ such that

$$\mu_\rho^{\Pi_{\mathcal{I}},A}(\Delta \times \Gamma) = \mathrm{Tr}[\Pi(\Delta)E^A(\Gamma)\rho] \tag{23}$$

for all $\Delta, \mathcal{F} \in \mathcal{B}(\mathbb{R})$.

Theorem 8 ([20, Theorem 3])**.** *$\varepsilon_m(A)$ satisfies the following conditions:*
(1) *$\varepsilon_m(A)$ is defined by $\Pi_{\mathcal{I}}$, A and ρ (the operational definability).*
(2) *If $\Pi_{\mathcal{I}}$ and A are commuting in ρ, $\varepsilon_m(A)$ coincides with Gauss' error $\varepsilon_G(\mu_\rho^{\Pi_{\mathcal{I}},A})$.*
(3) *If $\mathcal{I}$ precisely measures A in ρ, then $\varepsilon_m(A) = 0$ (the soundness).*
(4) *If $\varepsilon_m(A) = 0$, then $\mathcal{I}$ precisely measures A in ρ (the completeness).*

$\varepsilon_0(A)$ does not satisfy (4). $\varepsilon_m(A)$ and $\eta_m(B)$ then satisfy the following relation:

Theorem 9.

$$\varepsilon_m(A)^2\sigma(B)^2 + \sigma(A)^2\eta_m(B)^2 + 2\varepsilon_m(A)\eta_m(B)\sqrt{\sigma(A)^2\sigma(B)^2 - E_{AB}^2} \geq E_{AB}^2,$$

where E_{AB} is defined by

$$E_{AB} = \frac{D_{AB}}{\sqrt{\#_\rho(A)\#_\rho(B)}} \tag{24}$$

and

$$\#_\rho(A) = |\{a \in \mathbb{R} \mid \Pr\{A \in \{a\} \| \rho\} = \mathrm{Tr}[E^A(\{a\})\rho] > 0\}|. \tag{25}$$

To show this theorem, the methods in [18, 10] are used.

Next, we give a similar inequality for simultaneous measurements. Let $\mathbb{M} = (\mathcal{K}, \sigma, E, U)$ be a measuring process for $(\boldsymbol{B}(\mathcal{H}), \mathbb{R}^2)$, and M_1, M_2 observables on $\mathcal{K}$ such that

$$M_1 = \int_{\mathbb{R}^2} m_1 \, dE(m_1, m_2), \quad M_2 = \int_{\mathbb{R}^2} m_2 \, dE(m_1, m_2). \tag{26}$$

For a measuring process $\mathbb{M} = (\mathcal{K}, \sigma, E, U)$ for $(\boldsymbol{B}(\mathcal{H}), \mathbb{R}^2)$, the q-rms errors $\varepsilon_m(A)$ and $\varepsilon_m(B)$ of discrete observables A and B are defined by

$$\varepsilon_m(A)^2 = \mathrm{Tr}[(U^*(1 \otimes M_1)U - A \otimes 1)^2(\mathcal{E}_A(\rho) \otimes \sigma)], \tag{27}$$

$$\varepsilon_m(B)^2 = \mathrm{Tr}[(U^*(1 \otimes M_2)U - A \otimes 1)^2(\mathcal{E}_B(\rho) \otimes \sigma)], \tag{28}$$

respectively. Similarly, we can define $\varepsilon_m(A)$ and $\varepsilon_m(B)$ for CP instruments $\mathcal{I}$ for $(\boldsymbol{B}(\mathcal{H}), \mathbb{R}^2)$. The q-rms errors $\varepsilon_m(A)$ and $\varepsilon_m(B)$ satisfies the following inequality:

Theorem 10.

$$\varepsilon_m(A)^2\sigma(B)^2 + \sigma(A)^2\varepsilon_m(B)^2 + 2\varepsilon_m(A)\varepsilon_m(B)\sqrt{\sigma(A)^2\sigma(B)^2 - E_{AB}^2} \geq E_{AB}^2. \tag{29}$$

Acknowledgements

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

参考文献

[1] W.B. Arveson, Analyticity in operator algebras. Amer. J. Math. **89**(3) (1967), 578–642.

[2] C. Branciard, Error-tradeoff and error-disturbance relations for incompatible quantum measurements, Proc. Nat. Acad. Sci. **110**(17) (2013), 6742–6747.

[3] E.B. Davies & J.T. Lewis, An operational approach to quantum probability, Commun. Math. Phys. **17** (1970), 239–260.

[4] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Z. Phy. **43** (1927), 172–198.

[5] G. Lüders, Über die Zustandsänderung durch den Meßprozeß, Ann. Phys. (Leipzig) **8** (1951), 322–328.

[6] M. Nakamura & H.Umegaki, On von Neumann's theory of measurements in quantum statistics, Math. Japon. **7** (1962), 151–157.

[7] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik*, (Springer, Berlin, 1932); *Mathematical Foundations of Quantum Mechanics*, (Princeton UP, Princeton, 1955).

[8] K. Okamura & M. Ozawa, Measurement theory in local quantum physics, J. Math. Phys. **57** (2016), 015209.

[9] K. Okamura, "Measuring processes and the Heisenberg picture," pp. 361–396 in Reality and Measurement in Algebraic Quantum Theory, edited by M. Ozawa et. al,, (Springer, Singapore, 2018).

[10] K. Okamura & M. Ozawa, Universally valid uncertainty relations in general quantum systems, (2018), arXiv:1808.10615.

[11] K. Okamura, A C^*-algebraic approach to quantum measurement, RIMS kokyuroku **2123** (2019), 170–176.

[12] M. Ozawa, Quantum measuring processes of continuous obsevables, J. Math. Phys. **25** (1984), 79–87.

[13] M. Ozawa, Measurement breaking the standard quantum limit for free-mass position, Phys. Rev. Lett. **60**(5) (1988), 385.

[14] M. Ozawa, Universally valid reformulation of the Heisenberg uncertainty principle on noise and disturbance in measurement, Phys. Rev. A **67**(4) (2003), 042105.

[15] M. Ozawa, Physical content of Heisenberg's uncertainty relation: limitation and reformulation. Phys. Lett. A **318**(1-2) (2003), 21-29.

[16] M. Ozawa, Uncertainty relations for joint measurements of noncommuting observables. Phys. Lett. A **320**(5-6) (2004), 367–374.

[17] M. Ozawa, Uncertainty relations for noise and disturbance in generalized quantum measurements, Ann. Phys. (N.Y.) **331** (2004), 350–416.

[18] M. Ozawa, Error-disturbance relations in mixed states, (2014), arXiv:1404.3388.

[19] M. Ozawa, Heisenberg's original derivation of the uncertainty principle and its universally valid reformulations, Curr. Sci. **109** (2015), 2006–2016.

[20] M. Ozawa, Soundness and completeness of quantum root-mean-square errors, npj Quantum Inf. **5**(1) (2019), 1–8.

[21] W.F. Stinespring, Positive functions on C^*-algebras. Proc. Amer. Math. Soc. **6** (1955), 211–216.

[22] H. Umegaki, Conditional expectation in an operator algebra, Tohoku Math. J. **6**(2) (1954), 177–181.