Hyperbolicity and Primitivity of Group Rings *

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A ring R is said to be *(right) primtive* if it contains a faithful irreducible (right) R-module. In order to show the primitivity of a group ring KG, Alexander and Nishinaka develop the following useful Property (*) for the group G [1].

> (*) For each subset M of G consisting of a finite number of elements not equal to 1, and for any positive integer $m \geq 2$, there exist distinct $a, b, c \in G$ so that if $(x_1^{-1} g_1 x_1)(x_2^{-1} g_2 x_2) \cdots (x_m^{-1} g_m x_m) = 1$, where $g_i \in M$ and $x_i \in \{a, b, c\}$ for all $i = 1, \ldots, m$, then $x_i = x_{i+1}$ for some i.

Equipped with Property (*), Alexander and Nishinaka obtain the following:

Theorem 1 ([1, Theorem 1.1]). *Let G be a group which has a non-Abelian free subgroup whose cardinality is the same as that of G, and suppose that G satisfies Property (*). Then, if R is a domain with $|R| \leq |G|$, the group ring RG of G over R is primitive. In particular, the group algebra KG is primitive for any field K.*

As seen in [1], Theorem 1 immediately implies the primitivity of group rings for a large class of groups, generalizing many classical results and obtaining several new ones. The present result, published in [9], shows that Theorem 1 also applies to the non-elementary torsion-free hyperbolic groups. Thus the class of groups satisfying Property (*) is indeed quite large and encompasses "almost all" groups in a particular statistical sense [7].

Let G be a group with finite generating set X. Recall that the *Cayley graph* $\Gamma_X(G)$ of G with respect to X is an X-digraph with vertex set G and an x-labelled edge directed from g to gx for all $g \in G$ and $x \in X$. We may promote $\Gamma_X(G)$ to a geodesic metric space by assigning each edge a length of one.

When there exists $\delta \geq 0$ such that each side of a geodesic triangle in $\Gamma_X(G)$ is contained in the δ-neighborhood of the remaining two sides, we say that $\Gamma_X(G)$ has *δ-thin triangles*. This thin triangle property is independent of the choice of

*A detailed version of this paper appears in Journal of Algebra 493 (2018) [9].

finite generating set, though δ may vary. When $\Gamma_X(G)$ has δ-thin triangles for some finite generating set X and $\delta \geq 0$, we say G is *hyperbolic*. A hyperbolic group is *non-elementary* if it is not virtually cyclic.

Hyperbolic groups enjoy a close relationship between their algebraic and geometric properties; see, for instance, [2, 4, 5]. For our purposes, we require the so called "big powers property" of torsion-free hyperbolic groups. The version stated here follows immediately from a more general version for certain relatively hyperbolic groups given in [6].

Theorem 2 (The big powers property [6])**.** *Let G be a torsion-free hyperbolic group. Let $u \in G$ be nontrivial and not a proper power. Let $g_1, \dots, g_k$ be elements of G which do not commute with u. Then there exists $N > 0$ such that if $|n_i| \geq N$ for $i = 0, \dots, k$ then*

$$u^{n_0} g_1 u^{n_1} g_2 \cdots u^{n_{k-1}} g_k u^{n_k} \neq 1.$$

The big powers property allows one to programmatically generate large sets of nontrivial elements of G, and has seen useful application towards residual properties, logic, and algebraic geometry [3, 8, 6].

Proposition 3. *If G is a non-elementary torsion-free hyperbolic group, then G satisfies Property (*).*

Proof. Let M be a finite subset of G not containing the identity. A classical result due to Gromov asserts that the subgroup generated by sufficiently high powers of elements of M must be free. Since nontrivial elements of a non-elementary torsion free hyperbolic group have maximal infinite cyclic centralizers, one can therefore find an element $u \in G$ which generates its own centralizer and commutes with no $g \in M$.

Let $m \geq 2$ be an integer and consider a finite sequence $g_1, \dots, g_m$ of elements from M. Since u commutes with none of the g_i and generates its own centralizer, the big powers property gives $N(g_1, \dots, g_m) > 0$ such that

$$u^{n_0} g_1 u^{n_1} g_2 \cdots g_{m-1} u^{n_{m-1}} g_m u^{n_m} \neq 1$$

whenever $|n_i| \geq N$ for all $i = 0, \dots, m$.

Since M is a finite set, there are finitely many m-tuples $(g_1, \dots, g_m)$ of elements from M. Therefore, let $N > \max\{N(g_1, \dots, g_m) \mid g_1, \dots, g_m \in M\}$.

We now define $a = u^N, b = u^{2N}$, and $c = u^{3N}$. Since G is torsion-free, these elements are distinct. Consider a product

$$w = (x_1^{-1} g_1 x_1)(x_2^{-1} g_2 x_2) \cdots (x_m^{-1} g_m x_m)$$

where $x_1, \dots, x_m \in \{u^N, u^{2N}, u^{3N}\}$. We then have

$$w = u^{n_0} g_1 u^{n_1} g_2 \cdots g_{m-1} u^{n_{m-1}} g_m u^{n_m},$$

where $u^{n_0} = x_1^{-1}, u^{n_m} = x_m, u^{n_i} = x_i x_{i+1}^{-1}$ and $n_i \in \{0, \pm N, \pm 2N\}$ for $i = 1, \dots, m-1$. Note that by choice of x_1 and x_m, we have $n_0 \neq 0$ and $n_m \neq 0$.

By the big powers property and choice of N, if $n_i \neq 0$ for all $i = 0, \ldots, m$, then $w \neq 1$. Therefore, if $w = 1$, then some $n_i = 0$. Since we cannot have $n_0 = 0$ or $n_{m+1} = 0$, we have $n_i = 0$ for some $i \in \{1, \ldots, m-1\}$, in which case we must have $1 = u^{n_i} = x_i x_{i+1}^{-1}$, and so $x_i = x_{i+1}$. □

We immediately obtain the following main result as a corollary to Theorem 1.

Theorem 4. *If G is a non-elementary torsion-free hyperbolic group, then for any countable domain R, the group ring RG of G over R is primitive. In particular, the group ring KG is primitive for any field K.*

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