Ehrhart rings of order and chain polytopes and traces of the canonical modules*

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1 Introduction

For a partially ordered set (poset for short) P, Stanley [Sta2] defined and studied two polytopes, the order polytope $\mathscr{O}(P)$ and the chain polytope $\mathscr{C}(P)$ associated to P. Although the definition of two polytopes are quite different, they share many properties in common, such as the Hilbert series of the corresponding Ehrhart rings.

About the same time, Hibi [Hib] released the notion of an algebra with straightening law (ASL for short) on a distributive lattice, which nowadays called a Hibi ring. Hibi used Birkhoff's structure theorem of a finite distributive lattice [Bir]: a distributive lattice D is isomorphic to the set of poset ideals $\mathscr{I}(P)$ of P, where P is the set of join-irreducible elements of D. Later, it turned out that the Hibi ring on a distributive lattice $D \cong \mathscr{I}(P)$ is identical with the Ehrhart ring of the order polytope $\mathscr{O}(P)$ of P.

On the other hand, Herzog, Hibi and Stamate [HHS] studied the trace of the canonical module of a Cohen-Macaulay graded ring and showed that the trace of the canonical module is a defining ideal of the non-Gorenstein locus of a Cohen-Macaulay graded ring, defined the nearly Gorenstein property and characterized nearly Gorenstein Hibi rings.

In this announcement paper, we state our recent results on the traces of the canonical modules of the Ehrhart rings of the order and chain polytopes of a poset.

*This paper is an announcement of our result and the detailed version will be submitted to somewhere.

2 Preliminaries

In this paper, all rings and algebras are assumed to be commutative with an identity element unless stated otherwise. Also, when discussing properties of Noetherian rings, such as Cohen-Macaulay or Gorenstein properties, we assume the rings under consideration are Noetherian.

First we state notations about sets used in this paper. We denote the set of nonnegative integers, the set of integers and the set of real numbers by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ respectively. We denote the cardinality of a set X by $\#X$. For sets X and Y, we define $X \setminus Y := \{x \in X \mid x \notin Y\}$. For nonempty sets X and Y, we denote the set of maps from X to Y by Y^X. If X is a finite set, we identify $\mathbb{R}^X$ with the Euclidean space $\mathbb{R}^{\#X}$. For a subset $\mathscr{X}$ of $\mathbb{R}^X$, we denote by $\operatorname{aff}\mathscr{X}$ the affine span of $\mathscr{X}$. Let A be a subset of X. We define the characteristic function $\chi_A \in \mathbb{R}^X$ by $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \in X \setminus A$. In order to clarify the domain of the characteristic function, we sometimes denote χ_A by χ_A^X. For ξ, $\xi' \in \mathbb{R}^X$ and $a \in \mathbb{R}$, we define maps $\xi \pm \xi'$ and $a\xi$ by $(\xi \pm \xi')(x) = \xi(x) \pm \xi'(x)$ and $(a\xi)(x) = a(\xi(x))$ for $x \in X$. For $\xi \in \mathbb{R}^X$, we set $\operatorname{supp}\xi := \{x \in X \mid \xi(x) \neq 0\}$.

Now we define a symbol which is frequently used in this paper.

Definition 2.1 Let X be a finite set and $\xi \in \mathbb{R}^X$. For $B \subset X$, we set $\xi^+(B) := \sum_{b \in B} \xi(b)$. We define the empty sum to be 0, i.e., if $B = \emptyset$, then $\xi^+(B) = 0$.

Note that if X is a finite set, B is a subset of X and $a \in \mathbb{R}$, then $(\xi \pm \xi')^+(B) = \xi^+(B) \pm (\xi')^+(B)$ and $(a\xi)^+(B) = a(\xi^+(B))$.

Next we recall some definitions concerning finite partially ordered sets (poset for short). Let Q be a finite poset. We denote the set of maximal (resp. minimal) elements of Q by $\max Q$ (resp. $\min Q$). If $\max Q$ (resp. $\min Q$) consists of one element z, we often abuse notation and write $z = \max Q$ (resp. $z = \min Q$). A chain in Q is a totally ordered subset of Q or an empty set. For a chain C in Q, we define the length of C as $\#C - 1$. The maximum length of chains in Q is called the rank of Q and denoted by $\operatorname{rank} Q$. If every maximal (with respect to the inclusion relation) chain of Q has the same length, we say that Q is pure. If a subset I of Q satisfies $x \in I$, $y \in Q$, $y \leq x \Rightarrow y \in I$, then we say that I is a poset ideal of Q.

Let ∞ (resp. $-\infty$) be a new element which is not contained in Q. We define a new poset Q^+ (resp. Q^-) whose base set is $Q \cup \{\infty\}$ (resp. $Q \cup \{-\infty\}$) and $x < y$ in Q^+ (resp. Q^-) if and only if $x, y \in Q$ and $x < y$ in Q or $x \in Q$ and $y = \infty$ (resp. $x = -\infty$ and $y \in Q$). We set $Q^\pm := (Q^+)^-$.

Let Q' be an arbitrary poset. (We apply the following definition for $Q' = Q$, Q^+, Q^- or $Q^\pm$.) If x, $y \in Q'$, $x < y$ and there is no $z \in Q'$

with $x < z < y$, we say that y covers x and denote by $x \lessdot y$ or $y \gtrdot x$. For x, $y \in Q'$ with $x \le y$, we set $[x,y]_{Q'} := \{z \in Q' \mid x \le z \le y\}$, $[x,y)_{Q'} := \{z \in Q' \mid x \le z < y\}$ and $(x,y]_{Q'} := \{z \in Q' \mid x < z \le y\}$. Further, for x, $y \in Q'$ with $x < y$, we set $(x,y)_{Q'} := \{z \in Q' \mid x < z < y\}$. We denote $[x,y]_{Q'}$ (resp. $[x,y)_{Q'}$, $(x,y]_{Q'}$ or $(x,y)_{Q'}$) as $[x,y]$ (resp. $[x,y)$, $(x,y]$ or (x,y)) if there is no fear of confusion.

Definition 2.2 Let Q' be an arbitrary finite poset and let x and y be elements of Q' with $x \le y$. A saturated chain from x to y is a sequence of elements z_0, z_1, ..., z_t of Q' such that

$$x = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_t = y.$$

Note that the length of the chain z_0, z_1, ..., z_t is t.

Definition 2.3 Let Q', x and y be as above. We define $\mathrm{dist}(x,y) := \min\{t \mid$ there is a saturated chain from x to y with length $t\}$ and call $\mathrm{dist}(x,y)$ the distance of x and y.

Note that $\mathrm{rank}([x,y]) = \max\{t \mid$ there is a saturated chain from x to y with length $t\}$.

Next we fix notation about Ehrhart rings. Let X be a finite set with $-\infty \notin X$ and $\mathscr{P}$ a integral convex polytope in $\mathbb{R}^X$, i.e., a convex polytope whose vertices are contained in $\mathbb{Z}^X$. Set $X^- := X \cup \{-\infty\}$ and let $\{T_x\}_{x \in X^-}$ be a family of indeterminates indexed by X^-. For $f \in \mathbb{Z}^{X^-}$, we denote the Laurent monomial $\prod_{x \in X^-} T_x^{f(x)}$ by T^f. We set $\deg T_x = 0$ for $x \in X$ and $\deg T_{-\infty} = 1$. Then the Ehrhart ring of $\mathscr{P}$ over a field $\mathbb{K}$ is the $\mathbb{N}$-graded subring

$$\mathbb{K}[T^f \mid f \in \mathbb{Z}^{X^-}, f(-\infty) > 0, \frac{1}{f(-\infty)} f|_X \in \mathscr{P}]$$

of the Laurent polynomial ring $\mathbb{K}[T_x^{\pm 1} \mid x \in X][T_{-\infty}]$. We denote the Ehrhart ring of $\mathscr{P}$ over $\mathbb{K}$ by $E_{\mathbb{K}}[\mathscr{P}]$. (We use $-\infty$ as the degree indicating element in order to be consistent with the case of Hibi ring.) If $E_{\mathbb{K}}[\mathscr{P}]$ is a standard graded algebra, i.e., generated as a $\mathbb{K}$-algebra by degree 1 elements, we denote $E_{\mathbb{K}}[\mathscr{P}]$ by $\mathbb{K}[\mathscr{P}]$.

It is known that the dimension (Krull dimension) of $E_{\mathbb{K}}[\mathscr{P}]$ is equal to $\dim \mathscr{P} + 1$.

Note that $E_{\mathbb{K}}[\mathscr{P}]$ is Noetherian since $\mathscr{P}$ is integral. Therefore normal by the criterion of Hochster [Hoc]. Further, by the description of the canonical module of a normal affine semigroup ring by Stanley [Sta1, p. 82], we see that the ideal

$$\bigoplus_{f \in \mathbb{Z}^{X^-}, f(-\infty) > 0, \frac{1}{f(-\infty)} f|_X \in \mathrm{relint} \mathscr{P}} \mathbb{K} T^f$$

of $E_{\mathbb{K}}[\mathscr{P}]$ is the canonical module of $E_{\mathbb{K}}[\mathscr{P}]$, where $\operatorname{relint}\mathscr{P}$ denotes the interior of $\mathscr{P}$ in the topological space $\operatorname{aff}\mathscr{P}$. We call this ideal the canonical ideal of $E_{\mathbb{K}}[\mathscr{P}]$.

From now on, we fix a finite poset P. First we recall the definitions of order and chain polytopes.

Definition 2.4 ([Sta2]) We set

$$\mathscr{O}(P) := \left\{ f \in \mathbb{R}^P \;\middle|\; \begin{array}{l} 0 \leq f(x) \leq 1 \text{ for any } x \in P \text{ and if} \\ x < y \text{ in } P, \text{ then } f(x) \geq f(y) \end{array} \right\}$$

and

$$\mathscr{C}(P) := \left\{ f \in \mathbb{R}^P \;\middle|\; \begin{array}{l} 0 \leq f(x) \text{ for any } x \in P \text{ and } f^+(C) \leq 1 \\ \text{for any chain in } P \end{array} \right\}.$$

$\mathscr{O}(P)$ (resp. $\mathscr{C}(P)$) is called the order (resp. chain) polytope of P.

Next we make the following definition (cf. [Miy1, Miy2]).

Definition 2.5 Let $n \in \mathbb{Z}$. We set

$$\mathcal{T}^{(n)}(P) := \left\{ \nu \in \mathbb{Z}^{P^-} \;\middle|\; \begin{array}{l} \nu(z) \geq n \text{ for any } z \in \max P \text{ and if} \\ x \lessdot y \text{ in } P^-, \text{ then } \nu(x) \geq \nu(y) + n \end{array} \right\}$$

and

$$\mathcal{S}^{(n)}(P) := \left\{ \xi \in \mathbb{Z}^{P^-} \;\middle|\; \begin{array}{l} \xi(x) \geq n \text{ for any } x \in P \text{ and } \xi(-\infty) \geq \\ \xi^+(C) + n \text{ for any maximal chain } C \text{ in} \\ P \end{array} \right\}.$$

If there is no fear of confusion, we abbreviate $\mathcal{T}^{(n)}(P)$ (resp. $\mathcal{S}^{(n)}(P)$) as $\mathcal{T}^{(n)}$ (resp. $\mathcal{S}^{(n)}$).

It is known the following.

Fact 2.6 ([Miy1, Miy2])

$$\begin{aligned}
\mathbb{K}[\mathscr{O}(P)] &= \bigoplus_{\nu \in \mathcal{T}^{(0)}(P)} \mathbb{K}T^\nu, \\
\omega_{\mathbb{K}[\mathscr{O}(P)]} &= \bigoplus_{\nu \in \mathcal{T}^{(1)}(P)} \mathbb{K}T^\nu, \quad \omega^{-1}_{\mathbb{K}[\mathscr{O}(P)]} = \bigoplus_{\nu \in \mathcal{T}^{(-1)}(P)} \mathbb{K}T^\nu, \\
\mathbb{K}[\mathscr{C}(P)] &= \bigoplus_{\xi \in \mathcal{S}^{(0)}(P)} \mathbb{K}T^\xi, \\
\omega_{\mathbb{K}[\mathscr{C}(P)]} &= \bigoplus_{\xi \in \mathcal{S}^{(1)}(P)} \mathbb{K}T^\xi, \quad \textit{and} \quad \omega^{-1}_{\mathbb{K}[\mathscr{C}(P)]} = \bigoplus_{\nu \in \mathcal{S}^{(-1)}(P)} \mathbb{K}T^\xi,
\end{aligned}$$

where $I^{-1} := \{x \in Q(R) \mid xI \subset R\}$ *for a ring* R *with total quotient ring* $Q(R)$ *and an ideal* I *of* R.

Now we recall the following.

Fact 2.7 ([Hib, Sta1]) $\mathbb{K}[\mathscr{O}(P)]$ *(resp.* $\mathbb{K}[\mathscr{C}(P)]$*) is Gorenstein if and only if* P *is pure.*

Finally, we recall the notion of the trace of a module.

Definition 2.8 Let R be a ring and M an R-module. We set

$$\mathrm{tr}(M) := \sum_{\varphi \in \mathrm{Hom}(M,R)} \varphi(M)$$

and call the trace of M.

Note that if $M \cong M'$, then $\mathrm{tr}(M) = \mathrm{tr}(M')$. For an ideal with positive grade, we have the following.

Fact 2.9 ([HHS, Lemma 1.1]) *If* I *is an ideal of a Noetherian ring* R *with* $\mathrm{grade} I > 0$*, then* $\mathrm{tr}(I) = I^{-1}I$.

The trace of the canonical module is especially important by the following fact.

Fact 2.10 ([HHS, Lemma 2.1]) *Let* R *be a Cohen-Macaulay local ring with a canonical module* ω_R *or an* $\mathbb{N}$*-graded Cohen-Macaulay ring with* R_0 *is a field and canonical module* ω_R*. Then for* $\mathfrak{p} \in \mathrm{Spec}(R)$,

$$R_{\mathfrak{p}} \textit{ is Gorenstein} \iff \mathfrak{p} \not\supset \mathrm{tr}(\omega_R).$$

3 Main results

Now we describe the radicals of the traces of the canonical modules of $\mathbb{K}[\mathscr{O}(P)]$ and $\mathbb{K}[\mathscr{C}(P)]$.

Theorem 3.1 *Let* $\nu \in \mathcal{T}^{(0)}$*. Then*

$$T^{\nu} \in \sqrt{\mathrm{tr}(\omega_{\mathbb{K}[\mathscr{O}(P)]})}$$

if and only if for any $a_1, \ldots, a_u, b_1, \ldots, b_u \in P^{\pm}$ *with*

($\star$) *$a_1 < b_1 > a_2 < b_2 > \cdots > a_u < b_u > a_1$, $a_i \not\leq a_j$ (resp. $b_i \not\leq b_j$) for any i and j with $i \neq j$ and $\sum_{i=1}^{u} \operatorname{rank}([a_i, b_i]) > \sum_{i=1}^{u-1} \operatorname{dist}(a_{i+1}, b_i) + \operatorname{dist}(a_1, b_u)$,*

it holds that $\sum_{i=1}^{u} \nu(a_i) > \sum_{i=1}^{u} \nu(b_i)$.

Note that if $a_i = -\infty$ or $b_i = \infty$ for some i, then $u = 1$.

Theorem 3.2 *Let $\xi \in \mathcal{S}^{(0)}$ and $d = \xi(-\infty)$. Then*

$$T^{\xi} \in \sqrt{\operatorname{tr}(\omega_{\mathbb{K}[\mathscr{C}(P)]})}$$

if and only if for any $a_1, \ldots, a_u, b_1, \ldots, b_u \in P^{\pm}$ with ($\star$) of Theorem 3.1 and for any chains $C_1, \ldots, C_u, C'_1, \ldots, C'_u$ in P such that

($\star\star$) *C_i (resp. C'_i) is a maximal chain in $(-\infty, a_i]$ (resp. $[b_i, \infty)$) for $1 \leq i \leq u$,*

it holds that $\sum_{i=1}^{u}(\xi^+(C_i)+\xi^+(C'_i)) < ud$, where we set $C_i = \emptyset$ (resp. $C'_i = \emptyset$) if $a_i = -\infty$ (resp. $b_i = \infty$).

Note that $\max C_i = a_i$ (resp. $\min C'_i = b_i$) for $1 \leq i \leq u$ if $C_i \neq \emptyset$ (resp. $C'_i \neq \emptyset$) and $a_i = -\infty$ (resp. $b_i = \infty$) if $C_i = \emptyset$ (resp. $C'_i = \emptyset$). In particular, $a_1, \ldots, a_u, b_1, \ldots, b_u$ are uniquely determined by $C_1, \ldots, C_u, C'_1, \ldots, C'_u$.

For $a_1, \ldots, a_u, b_1, \ldots, b_u \in P^{\pm}$ satisfying ($\star$) of Theorem 3.1, set

$$\mathcal{T}^{(0)}_{(a_1,\ldots,a_u,b_1,\ldots,b_u)} := \{\nu \in \mathcal{T}^{(0)} \mid \sum_{i=1}^{u} \nu(a_i) > \sum_{i=1}^{u} \nu(b_i)\}$$

and

$$\mathfrak{p}'_{(a_1,\ldots,a_u,b_1,\ldots,b_u)} := \bigoplus_{\nu \in \mathcal{T}^{(0)}_{(a_1,\ldots,a_u,b_1,\ldots,b_u)}} \mathbb{K}T^{\nu}.$$

Then $\mathfrak{p}'_{(a_1,\ldots,a_u,b_1,\ldots,b_u)}$ is a prime ideal of $\mathbb{K}[\mathscr{O}(P)]$. Moreover, we see by Theorem 3.1 that

$$\sqrt{\operatorname{tr}(\omega_{\mathbb{K}[\mathscr{O}(P)]})} = \bigcap_{(a_1,\ldots,a_u,b_1,\ldots,b_u)} \mathfrak{p}'_{(a_1,\ldots,a_u,b_1,\ldots,b_u)},$$

where $(a_1, \ldots, a_u, b_1, \ldots, b_u)$ runs through elements of $P^{\pm}$ satisfying ($\star$) of Theorem 3.1.

For chains $C_1, \ldots, C_u, C'_1, \ldots, C'_u$ in P satisfying $(\star\star)$ of Theorem 3.2 for some $a_1, \ldots, a_u, b_1, \ldots, b_u \in P^{\pm}$ with $(\star)$ of Theorem 3.1, we set

$$\mathcal{S}^{(0)}_{(C_1,\ldots,C_u,C'_1,\ldots,C'_u)} := \{\xi \in \mathcal{S}^{(0)} \mid \sum_{i=1}^{u}(\xi^+(C_i) + \xi^+(C'_i)) < u\xi(-\infty)\}$$

and

$$\mathfrak{p}_{(C_1,\ldots,C_u,C'_1,\ldots,C'_u)} := \bigoplus_{\xi \in \mathcal{S}^{(0)}_{(C_1,\ldots,C_u,C'_1,\ldots,C'_u)}} \mathbb{K}T^{\xi}.$$

Then $\mathfrak{p}_{(C_1,\ldots,C_u,C'_1,\ldots,C'_u)}$ is a prime ideal of $\mathbb{K}[\mathscr{C}(P)]$. Moreover, we see by Theorem 3.2 that

$$\sqrt{\operatorname{tr}(\omega_{\mathbb{K}[\mathscr{C}(P)]})} = \bigcap_{(C_1,\ldots,C_u,C'_1,\ldots,C'_u)} \mathfrak{p}_{(C_1,\ldots,C_u,C'_1,\ldots,C'_u)},$$

where $(C_1, \ldots, C_u, C'_1, \ldots, C'_u)$ runs through the sets of chains satisfying $(\star\star)$ of Theorem 3.2 for some $a_1, \ldots, a_u, b_1, \ldots, b_u \in P^{\pm}$ with $(\star)$ of Theorem 3.1.

For elements $a_1, \ldots, a_u, b_1, \ldots, b_u$ in $P^{\pm}$ satisfying $(\star)$ of Theorem 3.1, set

$$\mathscr{G}_{(a_1,\ldots,a_u,b_1,\ldots,b_u)} := \{f \in \mathscr{O}(P) \mid \sum_{i=1}^{u} f(a_i) = \sum_{i=1}^{u} f(b_j)\}.$$

Then $\mathscr{G}_{(a_1,\ldots,a_u,b_1,\ldots,b_u)}$ is a face of $\mathscr{O}(P)$ and the Ehrhart ring of $\mathscr{G}_{(a_1,\ldots,a_u,b_1,\ldots,b_u)}$ is isomorphic to $\mathbb{K}[\mathscr{O}(P)]/\mathfrak{p}'_{(a_1,\ldots,a_u,b_1,\ldots,b_u)}$.

For chains $C_1, \ldots, C_u, C'_1, \ldots, C'_u$ satisfying $(\star\star)$ of Theorem 3.2, set

$$\mathscr{F}_{(C_1,\ldots,C_u,C'_1,\ldots,C'_u)} := \{f \in \mathscr{C}(P) \mid \sum_{i=1}^{u}(f^+(C_i) + f^+(C'_i)) = u\}.$$

Then $\mathscr{F}_{(C_1,\ldots,C_u,C'_1,\ldots,C'_u)}$ is a face of $\mathscr{C}(P)$ and the Ehrhart ring of $\mathscr{F}_{(C_1,\ldots,C_u,C'_1,\ldots,C'_u)}$ is isomorphic to $\mathbb{K}[\mathscr{C}(P)]/\mathfrak{p}_{(C_1,\ldots,C_u,C'_1,\ldots,C'_u)}$.

Lemma 3.3 *Let $a_1, \ldots, a_u, b_1, \ldots, b_u$ be elements of $P^{\pm}$ with $(\star)$ of Theorem 3.1.*

(1) For any chains $C_1, \ldots, C_u, C'_1, \ldots, C'_u$ with $(\star\star)$ of Theorem 3.2 for $a_1, \ldots, a_u, b_1, \ldots, b_u$, it holds that

$$\dim \mathscr{G}_{(a_1,\ldots,a_u,b_1,\ldots,b_u)} \geq \dim \mathscr{F}_{(C_1,\ldots,C_u,C'_1,\ldots,C'_u)}.$$

(2) There are chains $C''_1, \ldots, C''_u, C'''_1, \ldots, C'''_u$ with $(\star\star)$ *of Theorem 3.2 for $a_1, \ldots, a_u, b_1, \ldots, b_u$ such that*

$$\dim \mathscr{G}_{(a_1,\ldots,a_u,b_1,\ldots,b_u)} = \dim \mathscr{F}_{(C''_1,\ldots,C''_u,C'''_1,\ldots,C'''_u)}.$$

Using this, we can also obtain the following result.

Theorem 3.4 *The non-Gorenstein loci of the Ehrhart rings of the chain and the order polytopes of a poset have the same dimension.*

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