Calculation of Rings of Invariants of Certain Reductive Algebraic Groups

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Abstract

Let G be an affine connected algebraic group acting regularly on an affine Krull scheme $X = \mathrm{Spec}(R)$ over the complex number field $\boldsymbol{C}$. We study on the equidimensionality of the inclusion $R^G \to R$ by the minimal calculation of the ring R^G of invariants of G in R by cutting prime semi-invariants which form free modules over R^G. Consequently e see that $\boldsymbol{C}[V]$ is free as a $\boldsymbol{C}[V]^G$-module, for any equidimensional representation V of a reductive algebraic group G with simple semisimple part,

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1 Preliminaries

Rings and algebras are assumed to be commutative and domains mean integral domains. For an ideal $\mathfrak{I}$ of a ring, let $\mathrm{ht}(\mathfrak{I})$ stands for the height of $\mathfrak{I}$. Let $\mathcal{Q}(A)$ denote the total quotient ring of a ring A and

$$\mathrm{Ht}_1(A) := \{\mathfrak{P} \in \mathrm{Spec}(A) \mid \mathrm{ht}(\mathfrak{P}) = 1\}.$$

In the case where A is Krull, let $\mathrm{v}_{A,\mathfrak{P}}$ be the discrete valuation defined by the minimal prime $\mathfrak{P} \in \mathrm{Ht}_1(A)$ of A and $\mathrm{Cl}(A)$ be the divisor class group of A.

A ring monomorphism $\rho : B \to A$ is said to be *equidimensional*, if

$$\dim(A/\mathfrak{Q}A) = \dim A - \dim B$$

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for any $\mathfrak{Q} \in \mathrm{Spec}B$. For a domain A with a subring B such that $B = \mathcal{Q}(B) \cap A$ and $\mathcal{Q}(B) \subseteq \mathcal{Q}(A)$, we denote by

$$\mathrm{Ht}_1(A, B) := \{\mathfrak{P} \in \mathrm{Ht}_1(A) \mid \mathfrak{P} \cap B \in \mathrm{Ht}_1(B)\},$$

$$\mathrm{Ht}_1^{(2)}(A, B) := \{\mathfrak{P} \in \mathrm{Ht}_1(A) \mid \mathrm{ht}(\mathfrak{P} \cap B) \geq 2)\}.$$

Furthermore, for $\mathfrak{p} \in \mathrm{Ht}_1(B)$, denote by

$$\mathrm{Over}_{\mathfrak{p}}(A) := \{\mathfrak{P} \in \mathrm{Ht}_1(A) \mid \mathfrak{P} \cap B = \mathfrak{p}\}.$$

Algebraic groups are assumed to be affine (i.e., linear). Let G be a reductive algebraic group over the complex number field $\boldsymbol{C}$. For example, the algebraic torus $(\boldsymbol{C}^\times)^r$, a direct product of r-copies of the multiplicative group $\boldsymbol{C}^\times = \boldsymbol{C} \backslash \{0\}$, the special linear group SL_n (simple algebraic groups), etc. Let G' be the commutator subgroup of G. Then G' is a semi-simple algebraic group and the factor group G/G' is an algebraic torus.

For the n-dimensional vector space V over $\boldsymbol{C}$, let $\rho : G \to GL(V)$ be a rational representation of any algebraic group G (a morphism as algebraic groups) and V is said to be a rational G-module (e.g., [6]).

Denote by $V^\vee = \oplus_{i=1}^n \boldsymbol{C} X_i$ the dual space of V on which G acts. Let $\boldsymbol{C}[V]$ be the $\boldsymbol{C}$-algebra consisting of polynomial functions on V and $= \boldsymbol{C}[X\_] := \boldsymbol{C}[X_1, \ldots, X_n]$ the polynomial ring over C. Then G acts on $\boldsymbol{C}[V]$ as $\boldsymbol{C}$-algebra automorphisms and $\boldsymbol{C}[V]$ is a union of finite-dimensional rational G-modules.

Definition 1.1 *For an algebra R over $\boldsymbol{C}$, we say that an algebraic group G acts regularly on R, if the action of G on R induces*
(1) $\boldsymbol{C}$-algebra automorphisms on R.
(2) R is a union of finite-dimensional rational G-modules (e.g., [6]).
In this case, we shortly say (R, G) (or $(\mathrm{Var}(R), G)$) is regular or a regular action where $\mathrm{Var}(R)$ denotes the affine scheme associated with R over $\boldsymbol{C}$.

Define $\mathfrak{X}(G)$ to be the group of morphisms $G \to \boldsymbol{C}^\times$ (rational characters of G) (the rational character group of G).

Set $R^G := \{f \in R \mid \sigma(f) = f \ (\forall \sigma \in G)\}$ as a subring of R which is called the ring of invariants of G in R. For a $\chi \in \mathfrak{X}(G)$, put

$$R_\chi := \{f \in R \mid \sigma(f) = \chi(\sigma) f \ (\forall \sigma \in G)\}$$

called the module of relative invariants of G in R. The set R_χ is regarded as an R^G-module.

For a regular action (R, G) on an affine R (finitely generated as a $\boldsymbol{C}$-algebra), we have the classical result as follows. If G is reductive, then R^G is an affine $\boldsymbol{C}$-algebra and unless G is reductive, this conclusion does not hold in general.

We now define some properties on (R, G) in theory of group actions, invariant theory and commutative algebra.

Definition 1.2 *A regular action* (R, G) *(or* $(\mathrm{Var}(R), G)$*) of a reductive G on an affine $\boldsymbol{C}$-domain R is said to be respectively*

(i) equidimensional if $R^G \to R$ is equidimensional.

(ii) cofree if R is a free R^G-module.

(iii) coregular if R^G is a polynomial ring over $\boldsymbol{C}$.

(iv) coCI if R^G is a global complete intersection (GCI) over $\boldsymbol{C}$ (i.e., R^G is isomorphic to $\boldsymbol{C}[Y_1, \ldots, Y_{\dim R^G + d}]/(f_1, \ldots, f_d)$ as $\boldsymbol{C}$-algebras).

(v) stable if $\mathrm{Var}(R)$ *contains a non-empty open subset consisting of closed G-orbits.*

(vi) relatively stable if $(R^{G'}, G)$ is stable. (Recall : G' the commutator subgroup of G and so G on $R^{G'}$ is an action of the torus G/G')

(vi) effective if $\mathrm{Ker}(G \to \mathrm{Aut} R)$ *is not finite.*

(vii) relatively effective if $\mathrm{Ker}(G/G' \to \mathrm{Aut} R^{G'})$ *is not finite.*

The next result can be immediately obtained.

Lemma 1.3 *Suppose that (R, G) is a regular action of a reductive G on an affine $\boldsymbol{C}$-domain.*

(i) (R, G) : relatively stable if and only if the condition $R_\chi \neq \{0\}$ implies $R_{-\chi} \neq \{0\}$ for any $\chi \in \mathfrak{X}(G)$.

(ii) For a closed normal subgroup N of G :

(a) (R, G) : equidimensional if and only if (R, N) and $(R^N, G/N)$: equidimensional.

(b) (R, G) : stable if and only if (R, N) and $(R^N, G/N)$: stable

The purpose of this paper is to study on :

Russian Conjecture (1976 V.G. Kac, V.L. Popov, e.g., [1, 2, 6]) *Suppose that G is (connected) reductive and $\rho : G \to GL(V)$ is a rational representation over $\boldsymbol{C}$. Then does the condition that (V, G) is equidimensional imply that (V, G) is cofree (and so) coregular?* V.G. Kac notes that he does not have examples for equidimensional representations of non-reductive connected algebraic groups which are not cofree. Thus we may drop the assumption "reductivity" of G in this conjecture.

Remark 1.4 We have the following results to this conjecture:
(1) (G.W. Shwartz [7]) *For G : a simple algebraic group, this conjecture holds.*
(2) (D. Wehlau [9]) *For irreducible (V, G) of semi-simple algebraic group with* 2-*simple components, this conjecture holds.*
(3) (D. Wehlau [8]) *For G : an algebraic torus, this conjecture holds.*
(4) ([4]) *For a normal positively-graded algebra R, this conjecture does not hold even if G is an algebraic torus.*

2 Module of relative invariants

Consider an action of a group G on a ring R as automorphisms. Let $Z^1(G, \mathrm{U}(R))$ be the group of 1-cocycles of G on the unit group $\mathrm{U}(R)$ of R as an additive group For a 1-cocycle χ,

$$R_\chi := \{x \in R \mid \sigma(x) = \chi(\sigma)x \ (\sigma \in G)\},$$

which is a module over R^G. This is a generalization of modules of relative invariants with respect to rational characters of algebraic groups. The next result is fundamental:

Theorem 2.1 *([3])] Let R be a Krull domain acted by a group G as automorphisms. For a cocycle $\chi \in Z^1(G, \mathrm{U}(R))$,*
R_χ is a free R^G-module if and only if the following conditions are satisfied:

(i) $\dim \mathcal{Q}(R^G) \otimes_{R^G} R_\chi = 1$

(ii) $\exists\ f \in R_\chi$ *satisfying*

$$\forall \mathfrak{p} \in \mathrm{Ht}_1(R^G) \Rightarrow \exists \mathfrak{P} \in \mathrm{Over}_{\mathfrak{p}}(R) \ \textit{such that}\ \mathrm{v}_{R,\mathfrak{P}}(f) < \mathrm{v}_{R,\mathfrak{P}}(\mathfrak{p}R).$$

Here the condition *(i)* holds, if $R_\chi \cdot R_{-\chi} \neq \{0\}$.

Suppose that a domain R contains $\boldsymbol{C}$ as a subring. Let $\mathrm{U}_{\boldsymbol{C}}(R)$ denote the quotient group of $\mathrm{U}(R)$ by the multiplicative group $\mathrm{U}(\boldsymbol{C}) = \boldsymbol{C}^\times$. For a regular action (R, G) of an algebraic group on this R, we say an R-module M a twisted rational RG-module, if M has a rational G-module structure such that $\sigma(a \cdot x) = \sigma(a) \cdot \sigma(x)$ for any $\sigma \in G$, $a \in R$ and $x \in M$. An R-module M is said to be a *torsion* R-module, if, for its arbitrary element x, there exists a non-zero $a \in R$ satisfying $a \cdot x = 0$ (i.e., x is R-torsion). Denote by $1_{\mathcal{Q}(R)} \otimes_R M$ the image of the natural R-morphism $R \otimes_R M \to \mathcal{Q}(R) \otimes_R M$. If M is a rational RG-module, then $1_{\mathcal{Q}(R)} \otimes_R M$ is also rational.

Theorem 2.2 *Suppose that G is connected. Let M be a twisted rational RG-module not torsion as R-module. Let z be a non-zero element of $\mathcal{Q}(R) \otimes_R M$. Then the R-submodule Rz of $\mathcal{Q}(R) \otimes_R M$ generated by z is G-invariant if and only if so is the $\boldsymbol{C}$-subspace $\boldsymbol{C}z$ of $\mathcal{Q}(R) \otimes_R M$ generated by z.*

Hereafter in the section, suppose that (R, G) is a regular action of a connected algebraic group G on a Krull $\boldsymbol{C}$-domain R. We must have

Corollary 2.3 *Let f be a nonzero element of $\mathcal{Q}(R)$. If Rf is invariant under the action of G, then $\boldsymbol{C}f$ is G-invariant and, moreover if $\mathfrak{P} \cap R^G \neq \{0\}$ for any $\mathfrak{P} \in \mathrm{Ht}_1(R)$ such that $\mathrm{v}_{R,\mathfrak{P}}(f) < 0$, then*

$$G \ni \sigma \mapsto \frac{\sigma(f)}{f} \in \boldsymbol{C}^\times$$

is a rational character of G.

By this corollary, for a nonzero $f \in R$ satisfying that Rf is G-invariant, let $\delta_{f,G}$ be the rational character

$$\delta_{f,G} : G \ni \sigma \mapsto \frac{\sigma(f)}{f} \in \boldsymbol{C}^\times \in \mathfrak{X}(G).$$

Using valuations of R, we have:

Lemma 2.4 *Suppose that (R, G) is a regular action of a connected G on a Krull $\boldsymbol{C}$-domain R. Then*

(i) If $\bigcup_{\mathfrak{p} \in \Lambda} \mathrm{Over}_{\mathfrak{p}}(R)$ consists of principal ideals, then this set is finite, where $\Lambda := \{\mathfrak{p} \in \mathrm{Ht}_1(R^G) \mid |\mathrm{Over}_{\mathfrak{p}}(R)| \geq 2\}$.

(ii) If $\mathrm{Ht}_1^{(2)}(R, R^G)$ consists of principal ideals, then this set is finite.

Moreover we assume

The Principal Assumtion *Suppose that the both sets of Lemma 2.4 consist of principal ideals of R. (For example, R : factorial.)*

Definition 2.5 *By the Principal Assumption, there are non-associated prime elements $f_1, \ldots, f_d, g_1, \ldots, g_e$ of R such that*

$$\bigcup_{\mathfrak{p}\in\Lambda} \mathrm{Over}_{\mathfrak{p}}(R) \supseteq \{Rg_1, \ldots, Rg_e\}$$

,

$$|\{Rg_1, \ldots, Rg_e\} \cap \mathrm{Over}_{\mathfrak{p}}(R)| = |\mathrm{Over}_{\mathfrak{p}}(R)| - 1 \ (\mathfrak{p} \in \Lambda)$$

for every $\mathfrak{p} \in \mathrm{Ht}_1(R^G)$. We say these $f_1, \ldots, f_d, g_1, \ldots, g_e$ the removable prime elements for the action (R, G). and

$$\{Rf_1, \ldots, Rf_d\} = \mathrm{Ht}_1^{(2)}(R, R^G).$$

Definition 2.6 *Let H be the stabilizer*

$$\mathrm{Stab}(G : f_1, \ldots, f_d) = \cap_{i=1}^{d} G_{f_i} = \cap_{i=1}^{d} \mathrm{Ker}(\delta_{f_i,G})$$

of G at the set $\{f_1, \ldots, f_d\}$ and put

$$I := \mathrm{Stab}(G : f_1, \ldots, f_d, g_1, \ldots, g_e) = H \cap (\cap_{j=1}^{e} G_{g_j}) = \cap_{j=1}^{e} \mathrm{Ker}(\delta_{g_j,H}).$$

We say H and I the reduction groups for the action (R, G). Then $G \rhd H \rhd I$ and the factor groups G/H, H/I, G/I are algebraic tori.

We have the relative structure theorem of toric invariants of Krull $\boldsymbol{C}$-domains as follows:

Theorem 2.7 *Suppose that (R, G) is a regular action of a connected G on a Krull $\boldsymbol{C}$-domain R. Under the the Principal Assumption, we have*

(i) $\mathrm{Cl}(R^I) \cong \mathrm{Cl}(R^G)$ *and* $R^I/(f_1 - 1, \ldots, f_d - 1, g_1 - 1, \ldots, g_e - 1) \cong R^G$.

(ii) R^I *is a GCI over* R^H *and the action* (R^I, H) *is cofree.*

(iii) If R is affine and (R, G) is equidimensional, then $I = H$.

This is a partially stated in [5].

3 Representations

For representations of algebraic tori, the concepts in the previous section can be easily treated in the following way:

Example 3.1 *Let* $R := \boldsymbol{C}[X_1, X_2, X_3, X_4, X_5, X_6]$ *be the* 6*-dimensional polynomial ring and* $G := (\boldsymbol{C}^\times)^3$ *the algebraic torus of rank* 3*. For an element* $\sigma = (s, t, u) \in G$*, we define the action of* σ *on* R *by*

$$\sigma(X_1, \ldots, X_6) = (X_1, \ldots, X_6) \cdot \mathrm{diag}(s^{-1}, s^{-1}, s, t, u, (tu)^{-1}).$$

Then we can see that $f_1 = X_3$*,* $g_1 = X_4$*,* $g_2 = X_5$ *in Definition 2.5. Moreover the reduction groups are*

$$H = G_{X_3} = \{(1, t, u) \in G\} \cong (\boldsymbol{C}^\times)^2$$

and $I = H_{X_4, X_5} = \{1\}$*.*

We say that R is positively graded algebra over $\boldsymbol{C}$, if $R = \oplus_{i \geqq 0} R_i$ of a direct sum of $\boldsymbol{C}$-spaces, $R_0 = \boldsymbol{C}$ and $R_i \cdot R_j \subseteq R_{i+j}$. A regular action (R, G) is said to be conical, if R is positively graded and the action of G on R preserves the graded structure of R. From Theorem 2.1 we obtain

Theorem 3.2 *([4]) Suppose that* (R, G) *is a conical regular action of a connected* G *on an affine normal graded domain over* $\boldsymbol{C}$ *such that* R^G *is affine. If* (R, G) *is equidimensional, then* $\mathrm{Cl}(R^G)$ *is finite.*

We now study on the Russian conjecture for representations V of non-semisimple reductive algebraic group G with its commutator subgroup G' which is the semisimple part. For $R = \boldsymbol{C}[V]^{G'}$, R is an affine factorial domain over $\boldsymbol{C}$ and so we have the removable prime elements $\{f_1, \ldots, f_d, g_1, \ldots, g_1, \ldots, g_e\}$ and the reduction groups H, I defined by these elements.

Let V be a finite-dimensional rational representation of G such that $V^{G'} = \{0\}$. Combining Theorem 2.7 with Theorem3.2, we can show

Theorem 3.3 *Suppose that* G' *is simple and* $(\boldsymbol{C}[V]^I, H)$ *is stable effective. If* $\mathrm{Cl}(\boldsymbol{C}[V]^G)$ *is finite, then* $(\boldsymbol{C}[V]^I, H)$ *is cofree and* (V, G') *is coregular.*

Corollary 3.4 *Suppose that* G' *is simple and* (V, G) *is relatively effective and relatively stable.*

(i) If (V, G) *is relatively equidimensional, then* (V, G') *is coregular.*

(ii) If (V, G) *is equidimensional, then* (V, G) *is cofree.*

The relative *effectiveness* and *stability* follow from the action (V, G) of *non-semisimlicity* of G and are not technical assumptions. The second assertion of Corollary 3.4 shows the Russian conjecture holds for this case.

Closing our study on invariant theory, we gather the further problems related to the generalization of the present approach.

Problem 3.5 *Is the Russian Conjecture for representations of reductive algebraic groups with non-simple commutator subgroups hold?*

The proof of Corollary 3.4 depends essentially on the classification of coregular representations of simple algebaraic groups. So it is difficult to generalize this without the assumption on simple G''s. In fact we do not have a list of coregular reducible representations of semi-simple algebraic groups.

Problem 3.6 *Can we generalize Theorem 2.7 (The Relative Structure of Toric Invariants of Krull Domains)* *without the Principal Assumption?*

The Principal Assumption is used in defining the reduction groups H and I via rational characters of G. However Theorem 2.1, fundamental in the proof of Theorem 2.7, is stated in term of cocyles generalizing rational characters.

Problem 3.7 *Is Theorem 3.2 true, without graded structure of R? I.e., suppose that (R, G) is a regular action of an algebraic torus on an affine normal $\boldsymbol{C}$-domain R. If (R, G) is equidimensional, then is* $\mathrm{Cl}(R^G)$ *finite?*

Obviously the proof of Theorem 3.2, which is a consequence of Theorem 2.1, depends essentially on theory of the associated cones (i.e., the deformation theory of conical structure, cf. [4]). Thus, for this problem, it seems to be needed the alternating sight of the proof.

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