Non-noetherian groups without free subgroups and their group algebras

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Our interest is in group algebras of non-noetherian groups. In particular, we have studied about primitivity of group algebras and showed that they are often primitive if base groups have non-abelian free subgroups. Our main method includes using two edge-colored graphs. Actually, this method is effective for group algebras of groups with non-abelian free subgroups. On the other hand, there exist some important infinite groups which are non-noetherian but have no non-abelian free subgroups; e.g. Free Burnside groups and the Thompson's group F. In this talk, we first see a brief history on primitivity problem of group algebras of groups with non-abelian free subgroups. Next we introduce Thompson's group F and a problem on group algebras of it. Finally, we improve our graph theory in order to enable to investigate group algebras of Thompson's group F and apply our new graph theory to the problem.

1 Introduction

A ring R is said to be (right) primitive if it contains a faithful irreducible (right) R-module, or equivalently, if there exists a maximal (right) ideal in R which includes no non-trivial ideal of R. A primitive ring which is commutative is simply a field. Primitivity is a generalization of simplicity in the sense that a simple ring is always primitive. As for group algebras, because a group algebra has the non-trivial augmentation ideal, they are never primitive if the base group is non-trivial and either finite or abelian.

A group G is noetherian if every subgroup of it is finitely generated. If a solvable group G is noetherian, then G is called a polycyclic group. In 1978, by a series of studies by Domanov [7], Farkas-Passman [8], and Roseblade [15], a complete classification of the primitivity of group algebras of polycyclic-by-finite groups was given. In particular, it was determined that, for a polycyclic-by-finite group G, the group algebra KG is primitive if and only if its FC-center is trivial and K is not an

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absolute field.

Since it is difficult to find a noetherian group which is not polycyclic-by-finite (see [14]), almost all other known infinite groups belong to the class of non-noetherian groups, including free groups, locally free groups, free products, amalgamated free products, HNN-extensions, Fuchsian groups, one relator groups, free Burnside groups and Thompson's group F.

In 1973, Formanek [9] showed that every group ring RG of a free product G of non-trivial groups (except $G = \mathbb{Z}_2 * \mathbb{Z}_2$) over a domain R is primitive provided that the cardinality of R is not larger than that of G. After that, the primitivity of some interesting rings and algebras has been studied. For example, the primitivity of group rings of amalgamated free products by Jordan [10] and Balogun [3], and the primitivity of semigroup algebras of free products by Chaudhry, Crabb and McGregor [6]. Our results also followed; for proper ascending HNN extensions of free groups [12] and locally free groups [13].

In 2017, we found a property which is often satisfied by groups with free subgroups and showed that group algebras of a group satisfying the property are primitive [2]. Recently, by making use of our result, Solie [16] gave the primitivity of group rings of non-elementary torsion-free hyperbolic groups. More recently, motivated by his result, Abbott and Dahmani [1] have generalized his result to one for acylindrically hyperbolic groups. Most groups with free subgroups are in the class of hyperbolic groups (more generally acylindrically hyperbolic groups).

But the free Burnside group $B(m, n)$ of exponent n generated by m elements and Thompson's group F are not in the case. They are non-noetherian but do not have any free subgroups. We would like to know whether group algebras of these groups is primitive or not. However we do not know almost anything about their group algebras.

In the present note, in order to investigate group algebras of Thompson's group F, we will improve our graph theoretical method used in [2], and then apply it to a problem on group algebras of F.

2 Thompson's group F

Originally Thompson's groups $F \subseteq T \subseteq V$ were defined by Richard Thompson in 1965 to construct finitely-presented groups with unsolvable word problems [11]. The Thompson's group F was rediscovered by homotopy theorists in connection with work on homotopy, and then Brin and Squier [4] proved that F does not contain a free group of rank greater than one. After that, many papers on F have been produced until today. We refer the reader to Cannon, Floyd, and Parry [5] for a more detailed discussion of the Thompson's groups (F, T and V).

Thompson's group F is defined as a group of piecewise linear maps of the interval $[0, 1]$ as follows:

Definition 2.1. Thompson's group F is the group (under composition) of those homeomorphisms of the interval $[0, 1]$, which satisfy the following conditions:

1. they are piecewise linear and orientation-preserving,
2. in the pieces where the maps are linear, the slope is always a power of 2, and
3. the breakpoints are dyadic, i.e., they belong to the set $D \times D$, where $D = [0, 1] \cap \mathbb{Z}[\frac{1}{2}]$.

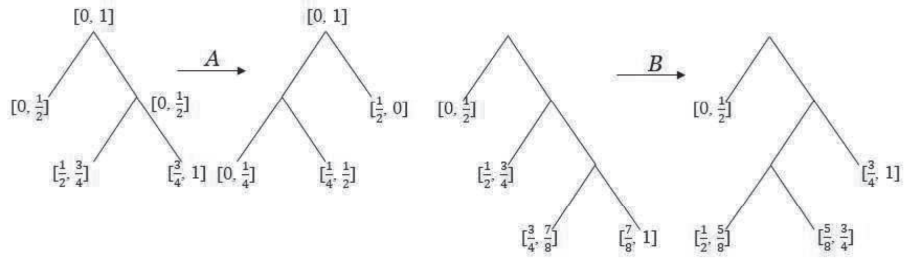
Example 2.2. The following two functions A and B are elements in Thompson's group F.

$$A(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4} & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x - 1 & \frac{3}{4} \leq x \leq 1 \end{cases} \qquad B(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{8} & \frac{3}{4} \leq x \leq \frac{7}{8} \\ 2x - 1 & \frac{7}{8} \leq x \leq 1 \end{cases}$$

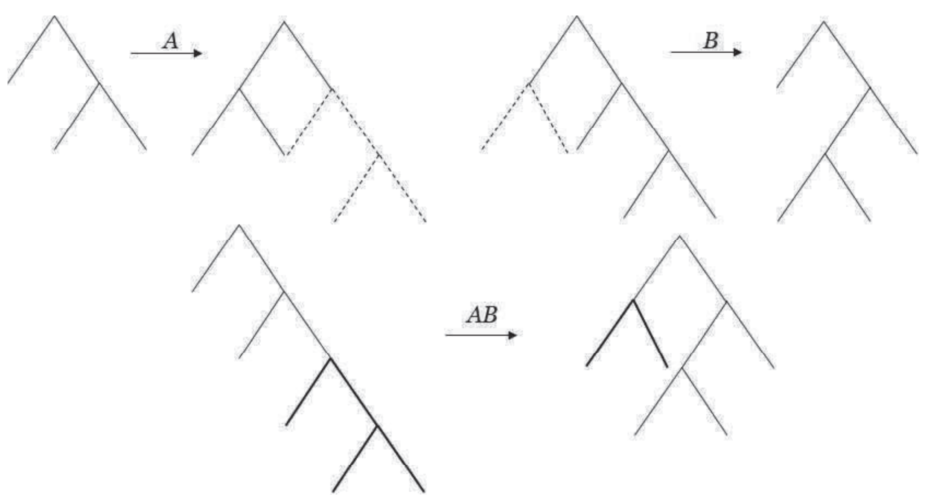
An element of F can be represented by a tree pair diagram which is a pair of binary trees with the same number of leaves.

Formally, a tree pair diagram is an ordered pair (R, S) of τ-trees such that R and S have the same number of leaves, where τ is defined as follows. The vertices of τ are the standard dyadic intervals in $[0, 1]$. An edge of τ is pair (I, J) of standard dyadic intervals I and J such that either I is the left half of J, in which case (I, J) is a left edge, or I is the right half of J, in which case (I, J) is a right edge.

For example, A and B described above are as follows:



Group multiplication in F corresponds to composition of homeomorphisms, which we can interpret on the level of tree pair diagrams as well. For example, the composition of A and B is as follows:



Actually, Thompson's group F is generated by A and B above, and so F is finitely generated. Moreover, F is finitely presented. For example, it is known the following presentation:

$$\langle a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2] \rangle,$$

where $[x, y]$ denotes the commutator of x and y. On the other hand, F has the following presentation:

$$F = \langle x_0, x_1, x_2, \cdots x_n, \cdots, \mid x_i^{-1} x_j x_i = x_{j+1}, \text{ for } i < j \rangle.$$

For the above presentation, every non-trivial element of F can be expressed in unique normal form

$$x_0^{\alpha_0} x_1^{\alpha_0} \cdots x_n^{\alpha_n} x_n^{-\beta_n} \cdots x_1^{-\beta_1} x_0^{-\beta_0},$$

where n, $\alpha_0, \ldots, \alpha_n, \beta_0, \cdots, \beta_n$ are non-negative integers such that

1. exactly one of α_n and β_n is non-zero and
2. if $\alpha_k > 0$ and $\beta_k > 0$ for some integer k with $0 \leq k < n$, then $\alpha_{k+1} > 0$ or $\beta_{k+1} > 0$.

As is mentioned above, F is finitely generated and finitely presented. In addition, it is known that F is torsion free and has no non-abelian free subgroup.

On the other hand, the Burnside group $B(m, n)$ is clearly a torsion group and has no non-abelian free subgroup. It seems that a group algebra of a torsion group is more difficult to deal with than one of a torsion free group. That is why we would first like to investigate group algebras of Thompson's group F and get some results.

3 A problem on group algebras of F

Let KG be the group algebra of a group G over a field K. KG has common right multipliers if it satisfies the following condition:

(C) For any A and B in KG^*, there exist X and Y in KG^* such that $AX = BY$,

where $KG^* = KG \setminus \{0\}$, the set of non-zero elements in KG.

Problem 3.1. *For Thompson's group F, does KF satisfies the condition* (C)*?*

This problem is strongly connected to a well known problem; the amenability problem for Thompson's group F which is a long standing open problem. Generally, it is known that if a group G is amenable, then KG satisfies the condition (C) for any field K.

Definition 3.2. (Amenable) A group G is amenable if for $P(G) = \{S \mid S \subseteq G\}$, there exists $\mu : P(G) \longrightarrow [0,1]$ such that

1. $\mu(G) = 1$,
2. if S and T are disjoint subsets of G, then $\mu(S \cup T) = \mu(S) + \mu(T)$,
3. if $S \in P(G)$ and $g \in G$, then $\mu(gS) = \mu(S)$.

If G has a non-abelian free subgroup, then we can see that KG fails to satisfy the condition (C) for any field K; in particular G is not amenable.

In fact, in this case, G has a subgroup freely generated by infinitely many elements; say a_1, a_2, b_1, b_2, $\cdots$. We let here $A = a_1 + a_2$ and $B = b_1 + b_2$ and suppose, to the contrary, that $AX + BY = 0$ for some X and Y in KG^*. Since X and Y in KG, they are expressed as follows:

$$X = \sum_{x \in S_X} \alpha_x x, \quad Y = \sum_{y \in S_Y} \beta_y y,$$

where $\alpha_x, \beta_y \in K \setminus \{0\}$, $S_X = Supp(X)$ and $S_Y = Supp(Y)$. Since $AX + BY = 0$, we have

$$\sum_{x \in S_X} \alpha_x(a_1 x + a_2 x) + \sum_{y \in S_Y} \beta_y(b_1 y + b_2 y) = 0. \tag{1}$$

We would like to regard these elements $a_i x$ and $b_i y$ as vertices and construct the graph (V, E, F) with two edge sets E and F. The graph is called an SR-graph in [2] which is a special case of a two-edge coloured graph. We therefore distinguish all of these elements $a_i x$ and $b_i y$ even if for $i \neq j$, $a_i x = a_j x'$, $b_i y = b_j y'$ or $a_i x = b_j y$ in G, and define the vertex set as $V = \{(a_i, x), (b_i, y) \mid i = 1, 2, x \in S_X, y \in S_Y\}$. Two edge sets are defined as follows:

$$E = \{vw \mid v, w \in V; v \neq w, \tilde{v} = \tilde{w} \text{ in } G\}, \text{ where } \tilde{v} = ax \text{ if } v = (a, x).$$
$$F = \{vw \mid v, w \in V; v \neq w, \text{ either } v = (a_1, x), w = (a_2, x) \text{ or } v = (b_1, y), w = (b_2, y)\}.$$

Because of (1), all elements of G in the left side of the equation (1) are cancelled each other. That is, for each $v_1 \in V$, there exists $w_1 \in V$ with $v_1 \neq w_1$ such that $v_1w_1 \in E$, and then, by the definition of F, there exists $v_2 \in V$ such that $w_1v_2 \in F$. We can continue with this procedure. We have $v_1w_1 \in E$, $w_1v_2 \in F$, $\cdots$. On the other hand, since V is a finite set, we may assume that $v_mw_m \in E$ and $w_mv_1 \in F$. We call such a cycle an SR-cycle; a cycle in an SR-graph (V, E, F) is called an SR-cycle if its edges belong alternatively to E and not to E. Since $v_i = c_{2i-1}z_i$, $w_i = c_{2i}z_{i+1}$ and $z_i \in S_X \cup S_Y$ for $c_i \in \{a_i, b_i \mid i = 1, 2\}$, This cycle implies that $c_1^{-1}c_2 \cdots c_{2m-1}^{-1}c_{2m} = 1$; a contradiction, because $\{a_i, b_i \mid i = 1, 2\}$ is a free basis. □

Our graph theory used in the above does not seem to be effective to show that KF fails to satisfy the condition (C) for Thompson's group F. We will therefore improve our graph theory to be effective for KF.

4 Improvement of SR-graph

As we saw in the previous section, the application of SR-graph theory needs free generators. But Thompson's group F has no such elements, and so we change a part of an SR-graph which is undirected into a directed graph. We call it a DSR-graph and define as follows:

Definition 4.1. Let $\mathcal{G} := (V, E)$ and $\mathcal{H} := (V, F)$. If every component of $\mathcal{G}$ is a complete graph, $\mathcal{H}$ is a simple directed graph and if $E \cap F = \emptyset$, then we call the triple $\mathcal{D} = (V, E, F)$ a DSR-graph.

Definition 4.2. A cycle in an DSR-graph (V, E, F) is called an DSR-cycle if its edges belong alternatively to E and F; more formally, we call cycle (V', E') an DSR-cycle if there is labeling $V' = \{v_1, v_2, \ldots, v_c\}$

and $E' = \{v_1v_2, v_2v_3, \ldots, v_{2m-1}v_{2m}, v_{2m}v_1\}$ so that $v_{2i-1}v_{2i} \in E$ and $(v_{2i}, v_{2i+1}) \in F$.

We might be able to get a desired cycle which induce a equation containing only positive words by using a DSR-graph. This means that our new method does not always require a free subgroup in a group. In fact, by making use of our new graph theory, we can get the following result:

Theorem 4.3. *Let F be a Thompson's group F. If there exist elements a_i, b_i ($i \in [3]$) in F such that for $u_i \in \{a_1a_2^{-1}, a_2a_3^{-1}, a_3a_1^{-1}, b_1b_2^{-1}, b_2b_3^{-1}, b_3b_1^{-1}\}$, $u_1 \cdots u_n = 1$ implies that $u_i = c_jc_k^{-1}$ and $u_{i+1} = c_kc_l^{-1}$ for some $i \in [3]$ and $c_i \in \{a_i, b_i \mid i \in [3]\}$, then two elements $A = \sum_{i=1}^{3} a_i$ and $B = \sum_{i=1}^{3} b_i$ of KF satisfy $AX + BY \neq 0$ for any $X, Y \in KG^*$.*

References

[1] C. R. Abbott and F. Dahmani, *Property P_{naive} for acylindrically hyperbolic groups* Math. Z. **291**(2019), 555–568.

[2] J. Alexander and T. Nishinaka, *Non-noetherian groups and primitivity of their group algebras*, J. Algebra **473** (2017), 221-246

[3] B. O. Balogun, *On the primitivity of group rings of amalgamated free products* Proc. Amer. Math. Soc., **106**(1)(1989), 43-47

[4] M. Brin and C. C. Squier, *Groups of piecewise linear homeomorphisms of the real line*, Invent . math. **79** (1985), 485–498.

[5] J. W. Cannon, W. J. Floyd, and W. R.Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. **42(2)** (1996), 215–256.

[6] M. A. Chaudhry, M. J. Crabb and M. McGregor, *The primitivity of semigroup algebras of free products* Semigroup Forum, **54**(2)(1997), 221-229

[7] O. I. Domanov, *Primitive group algebras of polycyclic groups*, Sibirsk. Mat. Ž., 19(1)(1978), 37-43

[8] D. R. Farkas and D. S. Passman, *Primitive Noetherian group rings*, Comm. Algebra, 6(3)(1978), 301-315.

[9] E. Formanek, *Group rings of free products are primitive* J. Algebra, **26**(1973), 508-511

[10] C. R. Jordan, *Group Rings of Generalised Free Products*, Ph D thesis (1975), 79-81

[11] R. McKenzie and R. J. Thompson, *An elementary construction of unsolvable word problems in group theory*,Word Problems, Studies in Logic and the Foundations of Mathematics, **71** (1973), North-Holland, Amsterdam, 457–478.

[12] T. Nishinaka, *Group rings of proper ascending HNN extensions of countably infinite free groups are primitive*, J. Algebra, 317(2007), 581-592

[13] T. Nishinaka, *Group rings of countable non-abelian locally free groups are primitive*,Int. J. algebra and computation, **21**(3) (2011), 409-431

[14] A. Ju. Ol'shanskiĭ, *An infinite simple torsion-free Noetherian group*, Izv. Akad. Nauk BSSR, Ser. Mat., 43(1979), 1328-1393.

[15] J. E. Roseblade, *Prime ideals in group rings of polycyclic groups*, Proc. London Math. Soc., 36(3)(1978), 385-447. Corrigenda: Proc. London Math. Soc., 36(3)(1979), 216-218.

[16] B. B. Solie, *Primitivity of group rings of non-elementary torsion-free hyperbolic groups*, J. Algebra **493** (2018), 438-443.