Injective hulls of bi S-sets

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In this paper, we study the injective hull of bi S-sets. In particular, we discuss descriptions of the injective hull of bi S-sets.

1 Injective hulls of bi S-sets

Let S be a semigroup and S^1 a semigroup S adjoined with an identity element.
A set M is a *bi S-set M* if M has associative operations of S on both sides.
Let $Map(S^1 \times S^1, M)$ denote the set of all mappings $f : S^1 \times S^1 \to M$ is a S-biset as follows :
$(sft)((a, b)) = f((as, tb))$ for all $a, b, s, t \in S$.
Define the map $\Phi : M \to Map(S^1 \times S^1, M)$ $(m \longmapsto f_m)$, where $f_m((a, b)) = amb$ for all $a, b \in S$ and $m \in M$. Then Φ is an S-isomorphism and M is identified with $\Phi(M)$ as bi S-sets.

A bi S-set M is *injective* if for any S-homomorphism ξvof a bi S-set A to M and an injective S-homomorphism α of A to a bi S-set B, there exists an S-homomorphism σ of B to M with $\alpha\sigma = \xi$.

Result [1, Theorem 6]. *$Map(S^1 \times S^1, M)$ is an injective bi S-set.*

Let M, N be bi S-sets such that M is a bi S-subset of N. Then M is *large* in N if any congruence σ of N with the restriction of $\sigma$to M being the identity relation is the identity relation itself.

By Theorem 10 of [1], $Map(S^1 \times S^1, M)$ contains a maximal large bi S-set $I(M)$ of M. Then $I(M)$ is the injective hull of M. $I(M)$ is a retraction of $Map(S^1 \times S^1, M)$. Actually, there exists an S-homomorphism α of $Map(S^1 \times S^1, M)$ to $I(M)$ with the restriction of α to $I(M)$ is an identity map of $I(M)$. In other words, there exists a congruence ξ on $Map(S^1 \times S^1, M)$ such that $Map(S^1 \times S^1, M)/\xi$ is S-isomorhic to $I(M)$ and the restriction of ξ to M is the identity relation of M.

Here we consider a description of ξ.

Define a relation ξ' on $Map(S^1 \times S^1, M)$ as follows :
$f\xi' g$ if and only if (i) $I_f = \{(s, t) \in S^1 \times S^1 \mid sft \in M\}$ and I_g are equal to each other and (ii) for any $(s, t) \in I_f = I_g$, $sft = sgt$.

Then ξ' is a congruence and the restriction of ξ_M of ξ to M is the identity relatiion. In particular, the set $\{f \in Map(S^1 \times S^1, M) \mid I_f$ is empty$\}$ is a single ξ'-class and is dented by O.

If M does not contain any element m with $SmS = \{m\}$, then O is a single ξ-class. $O \cup M$ is a large extension of M.
Suppose that M contains an element m with $SmS = \{m\}$. Let $\xi'' = \xi' \cup \{(m, x), (x, m) | x \in O\}$. Then ξ'' is a congruence and $\xi' \subset \xi'' \subseteq \xi$.

Example Let $X = \{1, 2\}$. Then $\mathcal{T}(X) = \left\{x = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, 1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, g = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\right\}$.
We use notation $\mathcal{T}_2$ in stead of $\mathcal{T}(\{1, 2\})$.
Then for any $f \in Map(\mathcal{T}_2 \times \mathcal{T}_2, \mathcal{T}_2)$ and $s \in \mathcal{T}_2$, we have the following (1), (2) :

(1) $xfs \in \mathcal{T}_2$ [$yfs \in \mathcal{T}_2$] implies $xfs = x$ [$yfs = y$].

(2) if $fx \in \mathcal{T}_2$ [$yfs \in \mathcal{T}_2$] then $fx = x$ or $fx = y$ [$fy = x$ or $fy = y$] .

Let $f, h \in Map(\mathcal{T}_2 \times \mathcal{T}_2, \mathcal{T}_2)$ with $fy = x$, $xf \notin \mathcal{T}_2$ and $xh = x$, $hy = x$. Then $(f, h) \notin \xi'$ but by Theorem 7 of [1] and (i), (ii), $(f, h) \in \xi$.
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Consequently, we conclude that ξ'' is properly contained in ξ.

We will continue to study the congruence ξ in a subsequent paper.

References

[1] P. Berthiaume, *The Injective Envelope of S-Sets*, Canadian Mathematical Bulletin**10**(2), 261-273.