Non-singularity and primitivity of Fully Prime Rings

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In 2016 at the same workshop, a brief survey of results on the structure of fully prime rings was presented. This is a continuation of the presentation for further introduction to the subject.

Recall that a ring in which every ideal is prime is called a fully prime ring. (This means every <u>proper</u> ideal is a prime <u>ideal</u>, and hence R itself is a prime <u>ring</u>.) The center of a fully prime ring is a field. Every ideal of a fully prime ring is an idempotent, and the set of the ideals is linearly ordered (but not necessarily well-ordered.) Examples of fully prime rings and other properties of such rings can be found in [2] and [6]. In this presentation, we particularly consider non-singularity and primitivity of fully prime rings.

Throughout, a ring R is an associative ring with identity. A ring R is called (right) primitive if there is a simple faithful right R-module, and an ideal P is called primitive if R/P is a primitive ring. It is evident from the definition that R is primitive if and only if R contains a maximal right ideal that does not contain nonzero two-sided ideals. For a commutative ring, a primitive ring is therefore a field, and a primitive ideal is a maximal ideal. A ring in which every prime ideal is an intersection of primitive ideals is called a Jacobson ring. Note that if a ring is commutative, Jacobson ring is in close connection with Hilbert's Nullstellensatz.

For $x \in R$, we define $\mathrm{Ann}(x) = \{r \in R \mid xr = 0\}$. Note that $\mathrm{Ann}(x)$ is a right ideal of R. A right ideal I of a ring R is called essential if $I \cap J \neq 0$ for any nonzero right ideal J of R. It is evident that for a prime ring R, every nonzero two-sided ideal is essential. A ring R is called (right) nonsingular if $\operatorname{Sing}(R)=\{x \in R \mid \operatorname{Ann}(x)$ is an essential right ideal of $R\}=0$. Note that $\operatorname{Sing}(R)$ is a two-sided ideal. The sum of all (nonzero) minimal right ideals of R is called the socle of R, and will be denoted by $\operatorname{Soc}(R)$. $\operatorname{Soc}(R)$ is a two-sided ideal and if R has no minimal right ideals, then $\operatorname{Soc}(R)$ is defined to be zero. Jacobson radical of R, denoted by $J(R)$, is the intersection of all primitive ideals of R. If $J(R)=0$, then R is called a semiprimitive ring.

In the literature, rings with certain conditions imposed on their right, left, or two-sided ideals have studied ubiquitously. For example, a ring in which every right ideal is an intersection of maximal right ideals is called a right V-ring. A prime right V-ring R is either primitive, or a ring with $\operatorname{Soc}(R)=0$ such that every non-zero right ideal contains a non-zero nilpotent element.

Proposition 1. A fully prime ring R is either a nonsingular primitive ring with minimum nonzero ideal, or a ring with $\operatorname{Soc}(R)=0$.

Proof: If $\operatorname{Soc}(R) \neq 0$, then, since R is prime, R is primitive. Further, since $\operatorname{Soc}(R)$ is the intersection of all essential right ideals; every two sided ideal of a prime ring is essential; and every ideal of a fully prime ring is linearly ordered, $0 \neq \operatorname{Soc}(R) \subseteq \bigcap_{0 \neq P_i \lhd R} P_i$. Thus, $\operatorname{Soc}(R)$ is the minimum nonzero ideal of R. As $\operatorname{Sing}(R) \cdot \operatorname{Soc}(R)=0$, $\operatorname{Sing}(R)=0$. □

If a fully prime ring R is not primitive, then $J(R) \supseteq \bigcap_{0 \neq P_i \lhd R} P_i$. Hence, the proposition below is evident.

Proposition 2. Let R be a fully prime ring with $\operatorname{Soc}(R)=0$. Suppose that $\bigcap_{0 \neq P_i \lhd R} P_i \neq 0$. Then R is primitive if and only if R is semiprimitive.

Example 1. The last example in [2] (P5399-5400) gives an example of a non-semiprimitive fully prime ring with $\operatorname{Soc}(R)=0$.

A ring R is called (right) strongly prime if for every $0 \neq r \in R$, there is a finite set $S(r)$ such that for $t \in R$, $\{rst \mid s \in S(r)\} = 0$ implies $t = 0$. An ideal P is called strongly prime if R/P is a strongly prime ring. Clearly, a strongly prime ring is a prime ring. Similarly, one can define a left strongly prime ring.

A right Noetherian prime ring is a right (and left) strongly prime ring. A strongly prime ring R is nonsingular, and $\mathrm{Soc}(R)$ is either 0 or R. Detailed explanations about these facts can be found in [3].

Proposition 3. Let R be a right Noetherian fully prime ring with $\mathrm{Soc}(R) \neq 0$. Then R is a simple Artinian ring.

Proof: Since $R = \mathrm{Soc}(R)$, one can show that R is Artinian. But a prime Artinian ring is a simple Artinian ring. □

Example 2. Theorem 4.6 in [2] gives an example of a primitive right Noetherian fully prime ring with $\mathrm{Soc}(R) = 0$ that is not a simple ring.

Proposition 4. Let R be a right Noetherian fully prime ring with $\mathrm{Soc}(R) = 0$. Then R is a nonsingular Jacobson ring.

Proof: Since R is strongly prime, it is nonsingular. Since $J(R)$ is an idempotent and finitely generated, $J(R) = 0$. As every factor ring of a right Noetherian fully prime ring R is right Noetherian and fully prime, $J(R/P)$ for every ideals P in R. Hence every (prime) ideal is an intersection of primitive ideals. □

Is a fully prime ring nonsingular in general? This question was originally asked by Professor K. Masaike during the 29th Symposium on Ring Theory held in Kashikojima, Mie in 1996.

Lemma 1.31 in [5] and Lemma 4.3 in [1] both claim the answer to be affirmative with less assumptions. The former only assumes that every ideal is idempotent and showing the conclusion directly from the definition. The latter assumes that every ideal is idempotent and the ring is prime, showing the conclusion using Torsion theory. Unfortunately, both proofs contain an incorrect step. We also note that Lemma 3.3 (d) in [1] is incorrect due to our Example in [2] (P5399-5400).

Example 3. A ring R that is not nonsingular is given in (11.21) Example in [4]. R has exactly one nonzero proper ideal and the ideal is idempotent. Hence R is a fully prime ring that is not nonsingular. Professor Masaike's question is hereby answered negatively.

Proposition 5. A fully prime ring R with exactly one nonzero proper ideal P is nonsingular if for any $x \in P$, $x \in Px$.

Proof: Since $\mathrm{Ann}(1) = \{r \in R \mid 1 \cdot r = 0\} = 0$ is not an essential right ideal, $1 \notin \mathrm{Sing}(R)$. Hence R is either nonsingular, or $\mathrm{Sing}(R) = P$.

Since P is simple as a ring, by Theorem 1.4 [2], every left ideal of P is prime. Hence every left ideal of P is idempotent.

If $x \in \mathrm{Sing}(R) \subseteq P$, then since $Px = (PxP)x$, $x = yx$ for some $y \in \mathrm{Sing}(R)$. Thus, if $a \in \mathrm{Ann}(y) \cap xP$, $a = xr$, $r \in P$ and $ya = 0$. But then $a = xr = yxr = ya = 0$. As $y \in \mathrm{Sing}(R)$, $\mathrm{Ann}(y)$ is essential, and we have $x = 0$. □

References

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