

On \tilde{D} -separable polynomials in skew polynomial rings of derivation type

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Abstract

The notion of $(\tilde{\rho}, \tilde{D})$ -separable polynomials in skew polynomial rings was introduced by S. Ikehata, and X. Lou gave a characterization of $\tilde{\rho}$ -separable polynomials in skew polynomial rings of automorphism type. In this paper, we shall give a new characterization of \tilde{D} -separable polynomials in skew polynomial rings of derivation type.

1 Introduction and Preliminaries

Let A/B be a ring extension with common identity. A/B is said to be *separable* if the A - A -homomorphism of $A \otimes_B A$ onto A defined by $z \otimes w \mapsto zw$ ($z, w \in A$) splits. It is well known that A/B is separable if and only if there exists $\sum_i z_i \otimes w_i \in (A \otimes_B A)^A$ such that $\sum_i z_i w_i = 1$, where $(A \otimes_B A)^A = \{\theta \in A \otimes_B A \mid u\theta = \theta u \ (\forall u \in A)\}$.

Throughout this paper, let B be an associative ring with identity element 1, and D a derivation of B . By $B[X, D]$ we denote the skew polynomial ring in which the multiplication is given by $\alpha X = X\alpha + D(\alpha)$ for any $\alpha \in B$. Moreover, by $B[X; D]_{(0)}$, we denote the set of all monic polynomials f in $B[X; D]$ such that $fB[X; D] = B[X; D]f$. From now on, let $f = \sum_{i=0}^m X^i a_i \in B[X; D]_{(0)}$ ($m \geq 1, a_m = 1$), $A = B[X; D]/fB[X; D]$, and $x = X + fB[X; D]$. As was shown in [3, Lemma 1.6], we see that f is in $B[X; D]_{(0)}$ iff $a_i \in B^D$ ($0 \leq i \leq m-1$) and

$$a_i \alpha = \sum_{j=i}^m \binom{j}{i} D^{j-i}(\alpha) a_j \quad (\forall \alpha \in B, 0 \leq i \leq m-1). \quad (1.1)$$

Since $f \in B^D[X]$, there is a derivation \tilde{D} of A which is naturally induced by D (that is, \tilde{D} is defined by $\tilde{D}\left(\sum_{j=0}^{m-1} x^j c_j\right) = \sum_{j=0}^{m-1} x^j D(c_j)$ ($c_j \in B$)). Now we consider the following A - A -homomorphisms:

$$\begin{cases} \mu : A \otimes_B A \rightarrow A, & \mu(z \otimes w) = zw \\ \xi : A \otimes_B A \rightarrow A \otimes_B A, & \xi(z \otimes w) = \tilde{D}(z) \otimes \tilde{\rho}(w) + z \otimes \tilde{D}(w) \end{cases} \quad (z, w \in A)$$

We say that f is a *separable polynomial* in $B[X; D]$ if A is a separable extension of B , namely, there exists an A - A -homomorphism $\nu : A \rightarrow A \otimes_B A$ such that $\mu\nu = 1_A$ (the identity map of A). Moreover, f is called a \tilde{D} -*separable polynomial* in $B[X; D]$

if there exists an A - A -homomorphism $\nu : A \rightarrow A \otimes_B A$ such that $\mu\nu = 1_A$ and $\xi\nu = \nu\tilde{D}$. The notion of \tilde{D} -separable polynomials was introduced by S. Ikehata (cf. [3]). By [7, Theorem 2.1] and [3, Theorem 2.1], the following lemma is already known.

Lemma 1.1. *Let $f = \sum_{i=0}^m X^i a_i$ ($m \geq 1, a_m = 1$) be in $B[X; D]_{(0)}$. The following are equivalent.*

- (1) f is \tilde{D} -separable in $B[X; D]$.
- (2) f is separable in $C(B^D)[X]$, where $C(B^D)$ is the center of $B^D = \{b \in B \mid D(b) = 0\}$.
- (3) $\delta(f)$ is invertible in $C(B^D)$, where $\delta(f)$ is the discriminant of f .

In this paper, we shall characterize \tilde{D} -separable polynomials in $B[X; D]$. In section 2, we define a D -matrix over B , and we shall mention briefly on it. In section 3, we shall give a new characterization of \tilde{D} -separable polynomial by making use of the trace map. Moreover, we shall show the “distance” between separability and \tilde{D} -separability.

2 D -matrix

In this section, let $B^D = \{b \in B \mid D(b) = 0\}$, and $C(B^D)$ the center of B^D . For any $b \in B$, I_b will represent the inner derivation effected by b (i.e. $I_b(\alpha) = \alpha b - b\alpha$ for any $\alpha \in B$). In [12], X. Lou defined the ρ -matrix, and he characterized $\tilde{\rho}$ -separable polynomials in $B[X; \rho]$ by making use of it. In constant, we shall define the D -matrix as follows:

Definition 2.1. (1) An element b in B is called a D -element if $I_b(B) \subset D(B)$ and $B^D \subset \text{Ker } I_b$, where $\text{Ker } I_b$ is the kernel of I_b .

- (2) A matrix P over B is called a D -matrix if every entry of P is a D -element.

Remark 1. In Definition 2.1 (1), if a D -element b is in B^D then the condition $B^D \subset \text{Ker } I_b$ implies that $b \in C(B^D)$.

Lemma 2.2. *Let b and c be D -elements in B .*

- (1) $b + c$ is also a D -element.
- (2) If either b or c is in B^D , then bc is also a D -element.
- (3) If $b \in B^D$ and b is invertible in B , then b^{-1} is also a D -element.

Proof. Let α and β be arbitrary elements in B and B^D , respectively. Assume that b and c are D -elements. Then there exist $b', c' \in B$ such that

$$I_b(\alpha) = \alpha b - b\alpha = D(b'), \quad I_c(\alpha) = \alpha c - c\alpha = D(c'). \quad (2.1)$$

Moreover, we see that $\beta b = b\beta$ and $\beta c = c\beta$.

(1) Since the equation (2.1), we have $I_{b+c}(\alpha) = \alpha(b+c) - (b+c)\alpha = (\alpha b - b\alpha) + (\alpha c - c\alpha) = D(b') + D(c') = D(b' + c')$. Therefore $I_{b+c}(B) \subset D(B)$. It is obvious that $\beta(b+c) - (b+c)\beta = 0$, and hence $B^D \subset \text{Ker } I_{b+c}$.

(2) Assume that b is in B^D . Note that $bc = cb$ and $I_b(\alpha c) = D(b'')$ for some $b'' \in B$. So it follows from the equation (2.1) that $I_{bc}(\alpha) = \alpha bc - bc\alpha = \alpha cb - b\alpha c + b\alpha c - bc\alpha = I_b(\alpha c) + bI_c(\alpha) = D(b'') + bD(c') = D(b'' + bc')$. Thus $I_{bc}(B) \subset D(B)$. Clearly, we see that $\beta bc - bc\beta = 0$. Therefore $B^D \subset \text{Ker } I_{bc}$.

(3) Assume that $b \in B^D$ and b is invertible. Since $bb^{-1} = 1$ and $b \in B^D$, we obtain

$$0 = D(1) = D(bb^{-1}) = bD(b^{-1}).$$

This implies that $b^{-1} \in B^D$. Since the equation (2.1), we have $b^{-1}\alpha - \alpha b^{-1} = b^{-1}D(b')b^{-1} = D(b^{-1}b'b^{-1})$. Hence $I_{b^{-1}}(B) \subset D(B)$. In addition, $\beta b = b\beta$ means that $\beta b^{-1} = b^{-1}\beta$. Thus $B^D \subset \text{Ker } I_{b^{-1}}$. \square

Lemma 2.3. Let P be a D -matrix over B^D .

- (1) $\det(P)$ is a D -element in $C(B^D)$.
- (2) If P is an invertible matrix, then the inverse matrix of P is also a D -matrix.
- (3) Assume that $P = P^T$ (the transpose of P). If P has a left (or right) inverse matrix which is a D -matrix, then $\det(P)$ is invertible in $C(B^D)$.

Proof. Let $P = [p_{ij}]_{n \times n}$ be a D -matrix over B^D for some positive integer n . In particular, P is a matrix over $C(B^D)$ by Remark 1.

(1) It is obvious by Lemma 2.2 (1) (2).

(2) Assume that P is an invertible matrix, and P^* be the cofactor matrix of P . It follow from Lemma 2.2 (1) (2) that every entry of P^* is a D -element (i.e. P^* is a D -matrix). By Lemma 2.2 (3) and the assertion (1), moreover, $\det(P)^{-1}$ is also a D -element. Therefore $P^{-1} = \det(P)^{-1}P^*$ is a D -matrix.

(3) Assume that $P = P^T$ and there exists a D -matrix $Q = [q_{ij}]_{n \times n}$ such that $QP = E$ (the identity matrix). Since q_{ij} is a D -element and $p_{ij} \in B^D$, we have $q_{kl}p_{ij} = p_{ij}q_{kl}$ ($1 \leq i, j, k, \ell \leq n$). Then we see that $QP = E$ iff $\sum_{j=1}^n q_{ij}p_{jk} = \delta_{ik}$ (the Kronecker's delta) iff $\sum_{j=1}^n p_{jk}q_{ij} = \delta_{ik}$ iff $P^T Q^T = E$ iff $PQ^T = E$. Hence $Q = Q^T$ is the inverse matrix of P . We put here $D(Q) = [D(q_{ij})]_{n \times n}$. Since $p_{ij} \in B^D$, we obtain

$$0 = D(\delta_{ik}) = D\left(\sum_{j=1}^n q_{ij}p_{jk}\right) = \sum_{j=1}^n D(q_{ij}p_{jk}) = \sum_{j=1}^n D(q_{ij})p_{jk}.$$

This implies that $D(Q)P = O$ (the zero matrix), and hence $D(Q) = O$ (i.e. Q is a matrix over B^D). Therefore Q is a matrix over $C(B^D)$ by Remark 1. Since P and Q are matrices over $C(B^D)$ such that $PQ = QP = E$, we see that $\det(P)$ is invertible in $C(B^D)$. \square

3 \tilde{D} -separability in $B[X; D]$

The conventions and notations employed in the preceding section will be used in this section. We shall use the following conventions:

- $A^{\tilde{D}} = \{z \in A \mid \tilde{D}(z) = 0\}$.
- $C(A^{\tilde{D}})$ is the center of $A^{\tilde{D}}$.
- $\pi_i : A \rightarrow A$ is the projection map defined by

$$\pi_i \left(\sum_{j=0}^{m-1} x^j c_j \right) = c_i \quad (c_i \in B, 0 \leq i \leq m-1).$$

- $\tau : A \rightarrow B$ is the trace map defined by

$$\tau(z) = \sum_{i=0}^{m-1} \pi_i(x^i z) \quad (z \in A).$$

- $T_f = [\tau(x^i x^j)]_{m \times m}$, where $m = \deg f$ (i.e. T_f is a $m \times m$ symmetric matrix whose $(i+1, j+1)$ element is $\tau(x^i x^j)$).
- $\delta(f) = \det(T_f)$ (the discriminant of f).

Remark 2. (1) Clearly, π_i ($0 \leq i \leq m-1$) and τ are left $C(B^D)$ - right B -homomorphisms.

(2) It is easy to see that

$$C(A^{\tilde{D}}) = \{C(B^D)[X] + fB[X; D]\} / f[B[X; D] \cong C(B^D)[X] / fC(B^D)[X].$$

First we shall show the following.

Lemma 3.1. (1) Every a_i ($0 \leq i \leq m-1$) is a D -element in $C(B^D)$.

(2) $T_f = [\tau(x^i x^j)]_{m \times m}$ is a D -matrix over $C(B^D)$.

Proof. (1) Since the equation (1.1), we have

$$\begin{aligned} a_i \alpha &= \alpha a_i + \sum_{j=i+1}^m \binom{j}{i} D^{j-i}(\alpha) a_j \\ &= \alpha a_i + D \left(\sum_{j=i+1}^m \binom{j}{i} D^{j-i-1}(\alpha) a_j \right). \end{aligned}$$

This implies that $I_{a_i}(B) \subset D(B)$ and $B^D \subset \text{Ker } I_{a_i}$ ($0 \leq i \leq m-1$). Hence every a_i is a D -element.

(2) It is easy to see that every $\tau(x^i x^j)$ ($0 \leq i, j \leq m-1$) is generated by a_k ($0 \leq k \leq m-1$). Then, by Lemma 2.2 and the assertion (1), every $\tau(x^i x^j)$ is a D -element. Therefore T_f is a D -matrix. \square

So we shall show the following lemma which gives a new equivalent condition of \tilde{D} -separability.

Lemma 3.2. *The following are equivalent.*

- (1) f is \tilde{D} -separable in $B[X; D]$.
- (2) T_f has a left (or right) inverse matrix which is a D -matrix.

Proof. (1) \implies (2) Let f be \tilde{D} -separable in $B[X; D]$. So, by Lemma 1.1, $\delta(f) = \det(T_f)$ is invertible in $C(B^D)$, and hence T_f has an inverse matrix T_f^{-1} . Noting that T_f is a D -matrix, T_f^{-1} is also a D -matrix by Lemma 2.3 (2).

(2) \implies (1) Assume that T_f has a left inverse matrix which is a D -matrix. Then, by Lemma 2.3 (3), $\delta(f) = \det(T_f)$ is invertible in $C(B^D)$. Thus f is \tilde{D} -separable by Lemma 1.1. \square

The following theorem at some extent shows the “distance” between separability and \tilde{D} -separability in $B[X; D]$.

Theorem 3.3. *The following are equivalent.*

- (1) f is \tilde{D} -separable in $B[X; D]$.
- (2) f is separable in $B[X; D]$ with a separable set $\{z_i, w_i\}$ of A/B such that $\sum_i z_i \tau(w_i) = 1$.

Proof. (1) \implies (2) Let f be \tilde{D} -separable in $B[X; D]$. So, by Lemma 1.1, f is separable in $C(B^D)[X]$. As was shown in [1, chapter III, Theorem 2.1], there exists separable system $\{z_i, w_i\}$ of $C(A^{\tilde{D}})/C(B^D)$ such that $\sum_i z_i \tau(w_i u) = u$ for any $u \in C(A^{\tilde{D}})$. So we can see that $\{z_i, w_i\}$ is still a separable system of A/B , and $\sum_i z_i \tau(w_i) = 1$.

(2) \implies (1) Assume that f is separable in $B[X; D]$ with a separable set $\{z_i, w_i\}$ of A/B such that $\sum_i z_i \tau(w_i) = 1$. We put here $z_i = \sum_{j=0}^{m-1} x^j c_{ij}$ and $w_i = \sum_{k=0}^{m-1} d_{ik} x^k$. We obtain then

$$\begin{aligned} \sum_i z_i \otimes w_i &= \sum_i \left(\sum_{j=0}^{m-1} x^j c_{ij} \otimes \sum_{k=0}^{m-1} d_{ik} x^k \right) \\ &= \sum_{j=0}^{m-1} x^j \otimes \left(\sum_{k=0}^{m-1} \sum_i c_{ij} d_{ik} x^k \right). \end{aligned}$$

We set $e_{jk} = \sum_i c_{ij} d_{ik}$ and $u_j = \sum_{k=0}^{m-1} e_{jk} x^k$. Clearly, $\{x^j, u_j\}$ is a still separable system of A/B such that $\sum_{j=0}^{m-1} x^j \tau(u_j) = 1$. Let $\hat{\tau}$ be a map from $A \otimes_B A$ to A defined by $\hat{\tau}(r_1 \otimes r_2) = r_1 \tau(r_2)$ ($r_1, r_2 \in A$). So we have

$$\begin{aligned} x^\ell &= x^\ell \sum_{j=0}^{m-1} x^j \tau(u_j) \\ &= \hat{\tau} \left(x^\ell \sum_{j=0}^{m-1} x^j \otimes u_j \right) \\ &= \hat{\tau} \left(\sum_{j=0}^{m-1} x^j \otimes u_j x^\ell \right) \\ &= \sum_{j=0}^{m-1} x^j \tau \left(\sum_{k=0}^{m-1} e_{jk} x^{k+\ell} \right) \\ &= \sum_{j=0}^{m-1} x^j \left(\sum_{k=0}^{m-1} e_{jk} \tau(x^{k+\ell}) \right) \end{aligned}$$

This implies that $\sum_{k=0}^{m-1} e_{jk} \tau(x^{k+\ell}) = \delta_{j\ell}$. By setting $P = [e_{j+1, k+1}]_{m \times m}$, we have $PT_f = E$. Noting that $\alpha \sum_{j=0}^{m-1} x^j \otimes u_j = \sum_{j=0}^{m-1} x^j \otimes u_j \alpha$ for any $\alpha \in B$, we obtain $\sum_{j=\ell}^{m-1} \binom{j}{\ell} D^{j-\ell}(\alpha) u_j = u_j \alpha$. This implies that e_{jk} is a D -element, and hence P is a D -matrix. Therefore f is \tilde{D} -separable by Lemma 3.2. □

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