Strong Convergence Theorems under Hybrid Methods for Two Nonlinear Mappings in Banach Spaces

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Abstract. In this article, using the hybrid method defined by Nakajo and Takahashi [17], we first obtain a strong convergence theorem for two noncommutative nonlinear mappings in a Banach space. Next, using the shrinking projection method defined by Takahashi, Takeuchi and Kubota [25], we prove another strong convergence theorem for the mappings in a Banach space. Using these results, we get well-known and new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space and a Banach space.

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. In 2010, Kocourek, Takahashi and Yao [11] defined a broad class of nonlinear mappings in a Hilbert space: A mapping $T : C \to H$ is called *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2, \quad \forall x, y \in C. \tag{1.1}$$

Such a mapping T is called (α, β)-*generalized hybrid*. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is *nonspreading* [14, 15] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [22] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous [8]. Recently, by using the hybrid method of Nakajo and Takahashi [17], Hojo and Takahashi [2] obtained a strong convergence theorem for two noncommutative generalized hybrid mappings in a Hilbert space. Furthermore, by using the shrinking projection method of Takahashi, Takeuchi and Kubota [25], they proved another strong convergence theorem in a Hilbert space.

In this article, using the hybrid method defined by Nakajo and Takahashi [17], we first obtain a strong convergence theorem for two noncommutative nonlinear mappings in a Banach space. Next, using the shrinking projection method defined by Takahashi, Takeuchi and Kubota [25], we prove another strong convergence theorem for the mappings in a Banach space. Using these results, we get well-known and new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space and a Banach space.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}, \quad \forall \epsilon \in \mathbb{R} \text{ with } 0 \le \epsilon \le 2.$$

A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to E$ is *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$. If C is a nonempty, closed and convex subset of a strictly convex Banach space E and $T : C \to E$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [9]. For a Banach space E,. the duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called *smooth*. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A Banach space E is called *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm-to-weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous. If E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E. For more details, see [19, 20, 21]. Let E be a smooth Banach space. The function $\phi\colon E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E,$$

where J is the duality mapping of E; see [1] and [10]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E. \tag{2.2}$$

From $(\|x\| - \|y\|)^2 \le \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \ge 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \quad \forall x, y, z, w \in E. \tag{2.3}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.4}$$

Let E be a smooth, strictly convex and reflexive Banach space. Let $\phi_*\colon E^* \times E^* \to \mathbb{R}$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2, \quad \forall x^*, y^* \in E^*,$$

where J is the duality mapping of E. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx), \quad \forall x, y \in E. \tag{2.5}$$

The following results are in Xu [28] and Kamimura and Takahashi [10].

Lemma 2.1 ([28]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \le \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \le \lambda \le 1$, where $B_r = \{z \in E : \|z\| \le r\}$.

Lemma 2.2 ([10]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \le \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \le r\}$.

Lemma 2.3 ([10]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called *generalized nonexpansive* [5] if $F(T) \ne \emptyset$ and

$$\phi(Tx, y) \le \phi(x, y), \quad \forall x \in C,\ y \in F(T).$$

Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be *sunny* [18] if

$$R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E,\ t \ge 0.$$

A mapping $R : E \to D$ is said to be a *retraction* or a *projection* if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a *generalized nonexpansive retract* (resp. *sunny generalized nonexpansive retract*) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [4, 5] for more details. The following results are in Ibaraki and Takahashi [5].

Lemma 2.4 ([5]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([5]). *Let C be a nonempty and closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) $z = Rx$ *if and only if* $\langle x - z, Jy - Jz \rangle \leq 0$ *for all* $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

In 2007, Kohsaka and Takahashi [13] proved the following results:

Lemma 2.6 ([13]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty and closed subset of E. Then the following are equivalent:*

(a) C is a sunny generalized nonexpansive retract of E;
(b) C is a generalized nonexpansive retract of E;
(c) JC is closed and convex.

Lemma 2.7 ([13]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) $z = Rx$;
(ii) $\phi(x, z) = \min_{y \in C} \phi(x, y)$.

Ibaraki and Takahashi [7] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Lemma 2.8 ([7]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Lemmas 2.6 and 2.8.

Lemma 2.9 ([7]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then $F(T)$ is a sunny generalized nonexpansive retract of E.*

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \neq \emptyset\}$. A multi-valued mapping A on E is said to be *monotone* if $\langle x - y, u^* - v^* \rangle \geq 0$ for all $x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on E. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [21]. Let E be a smooth, strictly convex and reflexive Banach space and let B be a maximal monotone operator of E^* into 2^E. For each $r > 0$ and $x \in E$, consider the set

$$J_r x = \{z \in E : x \in z + rBJz\}.$$

Then $J_r x$ consists of one point. Such J_r is called the *sunny generalized resolvent* of B and is denoted by $J_r = (I + BJ)^{-1}$. It follows that for any $x, y \in E$ and $r > 0$,

$$\langle x - J_r x - (y - J_r y), JJ_r x - JJ_r y \rangle \geq 0. \tag{2.6}$$

See [5] for more details.

3 Strong convergence theorems by hybrid methods

In this section, using the hybrid method by Nakajo and Takahashi [17], we first prove a strong convergence theorem for two noncommutative generic skew 2-generalized nonspreading mappings in a Banach space. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping $T : C \to C$ is called *generic 2-generalized nonspreading* [23] if there exist $\alpha_2, \alpha_1, \alpha_0, \beta_2, \beta_1, \beta_0, \gamma_2, \gamma_1, \gamma_0, \delta_2, \delta_1, \delta_0 \in \mathbb{R}$ such that
$\alpha_2 + \alpha_1 + \alpha_0 + \beta_2 + \beta_1 + \beta_0 \geq 0, \quad \alpha_2 + \alpha_1 + \alpha_0 > 0 \quad$ and

$$\begin{aligned}
\alpha_2\phi(T^2x,Ty) &+ \alpha_1\phi(Tx,Ty) + \alpha_0\phi(x,Ty) \\
&+ \beta_2\phi(T^2x,y) + \beta_1\phi(Tx,y) + \beta_0\phi(x,y) \\
&\leq \gamma_2\{\phi(Ty,T^2x) - \phi(Ty,Tx)\} + \gamma_1\{\phi(Ty,Tx) - \phi(Ty,x)\} \\
&\quad + \gamma_0\{\phi(Ty,x) - \phi(Ty,T^2x)\} + \delta_2\{\phi(y,T^2x) - \phi(y,Tx)\} \\
&\quad\quad + \delta_1\{\phi(y,Tx) - \phi(y,x)\} + \delta_0\{\phi(y,x) - \phi(y,T^2x), \quad \forall x, y \in C.
\end{aligned} \tag{3.1}$$

A mapping $T : C \to E$ is called *generic generalized nonspreading* [26] if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \beta > 0$ and

$$\begin{aligned}
\alpha\phi(Tx,Ty)+&\beta\phi(x,Ty) + \gamma\phi(Tx,y) + \delta\phi(x,y) \\
&\leq \varepsilon\{\phi(Ty,Tx) - \phi(Ty,x)\} + \zeta\{\phi(y,Tx) - \phi(y,x)\}, \quad \forall x, y \in C.
\end{aligned} \tag{3.2}$$

We call such a mapping a *generic* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$*-generalized nonspreading mapping.* A generic $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-generalized nonspreading mapping $T : C \to E$ is *generalized nonspreading* in the sense of Kocourek, Takahashi and Yao [12] if $\alpha + \beta = -\gamma - \delta = 1$ in (3.2). In particular, putting $\alpha = 1$, $\beta = \delta = 0$, $\gamma = \varepsilon = -1$ and $\zeta = 0$ in (3.2), we obtain that

$$\phi(Tx,Ty) + \phi(Ty,Tx) \leq \phi(Tx,y) + \phi(Ty,x), \quad \forall x, y \in C.$$

Such a mapping is *nonspreading* in the sense of Kohsaka and Takahashi [15]. A nonspreading mapping is obtained from a resolvent of a maximal monotone operator in a Banach space; see [15]. A mapping $T : C \to C$ is called *generic skew 2-generalized nonspreading* [23] if there exist $\alpha_2, \alpha_1, \alpha_0, \beta_2, \beta_1, \beta_0, \gamma_2, \gamma_1, \gamma_0, \delta_2, \delta_1, \delta_0 \in \mathbb{R}$ such that

$\alpha_2 + \alpha_1 + \alpha_0 + \beta_2 + \beta_1 + \beta_0 \geq 0, \quad \alpha_2 + \alpha_1 + \alpha_0 > 0$ and

$$\begin{aligned}\alpha_2\phi(Ty,T^2x) &+ \alpha_1\phi(Ty,Tx) + \alpha_0\phi(Ty,x)\\ &+ \beta_2\phi(y,T^2x) + \beta_1\phi(y,Tx) + \beta_0\phi(y,x)\\ &\leq \gamma_2\{\phi(T^2x,Ty) - \phi(Tx,Ty)\} + \gamma_1\{\phi(Tx,Ty) - \phi(x,Ty)\}\\ &\quad + \gamma_0\{\phi(x,Ty) - \phi(T^2x,Ty)\} + \delta_2\{\phi(T^2x,y) - \phi(Tx,y)\}\\ &\quad + \delta_1\{\phi(Tx,y) - \phi(x,y)\} + \delta_0\{\phi(x,y) - \phi(T^2x,y)\}, \quad \forall x,y \in C.\end{aligned} \tag{3.3}$$

A mapping $T : C \to E$ is called *generic skew generalized nonspreading* [26] if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \beta > 0$ and

$$\begin{aligned}\alpha\phi(Ty,Tx) &+ \beta\phi(Ty,x) + \gamma\phi(y,Tx) + \delta\phi(y,x)\\ &\leq \varepsilon\{\phi(Tx,Ty) - \phi(x,Ty)\} + \zeta\{\phi(Tx,y) - \phi(x,y)\}, \quad \forall x,y \in C.\end{aligned} \tag{3.4}$$

We call such a mapping a *generic* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$*-skew generalized nonspreading mapping*. For example, a generic $(1, 0, -1, 0, -1, 0)$-skew generalized nonspreading mapping is a *skew nonspreading* mapping in the sense of Ibaraki and Takahashi [6], i.e.,

$$\phi(Tx,Ty) + \phi(Ty,Tx) \leq \phi(x,Ty) + \phi(y,Tx), \quad \forall x,y \in C.$$

A skew nonspreading mapping is obtained from a sunny generalized resolvent of a maximal monotone operator in a Banach space; see [15]. Let $T : C \to E$ be a generic skew generalized nonspreading mapping satisfying (3.4). Putting $x = u \in F(T)$ in (3.4), we have that

$$\phi(Ty,u) \leq \phi(y,u), \quad \forall y \in C,\ u \in F(T). \tag{3.5}$$

This implies that T is generalized nonexpansive [5]. The following proposition was proved by Takahashi [23].

Proposition 3.1 ([23]). *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty, closed and convex subset of E and let T be a generic 2-generalized nonspreading mapping of C into C. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup z$, $x_n - Tx_n \to 0$ and $x_n - T^2x_n \to 0$, then $z \in F(T)$.*

Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty subset of E. Let T be a mapping of C into E. Define a mapping T^* as follows:

$$T^*x^* = JTJ^{-1}x^*, \quad \forall x^* \in JC,$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. A mapping T^* is called the *duality mapping* of T; see also [27] and [3]. It is easy to show that if T is a mapping of C into itself, then T^* is a mapping of JC into itself. In fact, for any $x^* \in JC$, we have $J^{-1}x^* \in C$ and hence $TJ^{-1}x^* \in C$ from the property of T. So we have

$$T^*x^* = JTJ^{-1}x^* \in JC.$$

Then T^* is a mapping of JC into itself.

Lemma 3.2 ([24]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let T be a generic skew 2-generalized nonspreading mapping of C into itself such that $F(T) \neq \emptyset$. Then, for any bounded sequence $\{z_n\}$ of C such that $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$ and $\lim_{n\to\infty} \|z_n - T^2z_n\| = 0$, every weak cluster point of $\{Jz_n\}$ belongs to $JF(T)$.*

Theorem 3.3 ([24]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let S and T be generic skew 2-generalized nonspreading mappings of C into C such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n S x_n + (1-\lambda_n) T x_n\big) + c_n\big(\mu_n S^2 x_n + (1-\mu_n) T^2 x_n\big), \\ C_n = \{z \in C : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ Q_n = \{z \in C : \langle x_1 - x_n, Jz - Jx_n \rangle \leq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_n \cap Q_n}$ is the sunny generalized nonexpansive retraction of E onto $C_n \cap Q_n$, $a, b, c, d \in \mathbb{R}$, $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \lambda_n, \mu_n \leq b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = R_{F(S)\cap F(T)} x_1$, where $R_{F(S)\cap F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$.

Next, we prove a strong convergence theorem by the shrinking projection method [25] for two noncommutative generic skew-generalized nonspreading mappings in a Banach space.

Theorem 3.4 ([24]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let S and T be generic skew 2-generalized nonspreading mappings of C into C such that $F(S) \cap F(T) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n S x_n + (1-\lambda_n) T x_n\big) + c_n\big(\mu_n S^2 x_n + (1-\mu_n) T^2 x_n\big), \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ is the sunny generalized nonexpansive retraction of E onto C_{n+1}, $a, b, c, d \in \mathbb{R}$, $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \lambda_n, \mu_n \leq b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges strongly to $z_0 = R_{F(S)\cap F(T)} x_1$, where $R_{F(S)\cap F(T)}$ is the sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$.

4 Applications

In this section, using Theorems 3.3 and 3.4, we get well-known and new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space and a Banach space. As a direct result of Theorem 3.3, we have the following theorem for generic skew 2-generalized nonspreading mappings in a Banach space.

Theorem 4.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty and closed subset of E such that JC is closed and convex. Let S be a generic skew 2-generalized nonspreading mapping of C into E such that $F(S) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n S x_n + c_n S^2 x_n, \\ C_n = \{z \in C : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ Q_n = \{z \in C : \langle x_1 - x_n, Jz - Jx_n \rangle \leq 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_n \cap Q_n}$ is the sunny generalized nonexpansive retraction of E onto $C_n \cap Q_n$, $c, d \in \mathbb{R}$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ satisfy the following:

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = R_{F(S)} x_1$, where $R_{F(S)}$ is the sunny generalized nonexpansive retraction of E onto $F(S)$.

In a Hilbert space H, we have that $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. Using this and from (3.3), we obtain that T is a normally 2-generalized hybrid mapping in the sense of Kondo and Takahashi [16], i.e., there exist $\alpha_2, \alpha_1, \alpha_0, \beta_2, \beta_1, \beta_0 \in \mathbb{R}$ such that $\alpha_2 + \alpha_1 + \alpha_0 + \beta_2 + \beta_1 + \beta_0 \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$ and

$$\begin{aligned} &\alpha_2 \|T^2 x - Ty\|^2 + \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 \\ &\quad + \beta_2 \|T^2 x - y\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x - y\|^2 \leq 0, \quad \forall x, y \in C. \end{aligned}$$

Theorem 4.2. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to C$ be normally 2-generalized hybrid mappings with $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n \big(\lambda_n S x_n + (1 - \lambda_n) T x_n\big) + c_n \big(\mu_n S^2 x_n + (1 - \mu_n) T^2 x_n\big), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$, $a, b, c, d \in \mathbb{R}$, $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \lambda_n, \mu_n \leq b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S) \cap F(T)} x_1$, where $P_{F(S) \cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Theorem 4.3. *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty and closed subset of E such that JC is closed and convex. Let S and T be generic skew generalized nonspreading mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$.*

Let $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n S x_n + (1-\lambda_n) T x_n\big) + c_n\big(\mu_n S^2 x_n + (1-\mu_n) T^2 x_n\big), \\ C_n = \{z \in C : \phi(y_n, z) \le \phi(x_n, z)\}, \\ Q_n = \{z \in C : \langle x_1 - x_n, Jz - Jx_n\rangle \le 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_n \cap Q_n}$ *is the sunny generalized nonexpansive retraction of* E *onto* $C_n \cap Q_n$, $a, b, c, d \in \mathbb{R}$, $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$0 < a \le \lambda_n, \mu_n \le b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \le a_n, b_n, c_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = R_{F(S)\cap F(T)} x_1$, *where* $R_{F(S)\cap F(T)}$ *is the sunny generalized nonexpansive retraction of* E *onto* $F(S) \cap F(T)$.

Using Theorem 3.3, we have the following strong convergence theorem for finding a common null point of two maximal monotone operators in a Banach space.

Theorem 4.4. *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* B *and* G *be maximal monotone operators of* E^* *into* 2^E *and let* J_r *and* Q_s *be sunny generalized resolvents for* $r > 0$ *and* $s > 0$ *of* B *and* G, *respectively. Suppose that* $B^{-1}0 \cap G^{-1}0 \ne \emptyset$. *Let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n J_r x_n + (1-\lambda_n) Q_s x_n\big) + c_n\big(\mu_n (J_r)^2 x_n + (1-\mu_n)(Q_s)^2 x_n\big), \\ C_n = \{z \in C : \phi(y_n, z) \le \phi(x_n, z)\}, \\ Q_n = \{z \in C : \langle x_1 - x_n, Jz - Jx_n\rangle \le 0\}, \\ x_{n+1} = R_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_n \cap Q_n}$ *is the sunny generalized nonexpansive retraction of* E *onto* $C_n \cap Q_n$, $a, b, c, d \in \mathbb{R}$, $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$0 < a \le \lambda_n, \mu_n \le b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \le a_n, b_n, c_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = R_{(BJ)^{-1}0 \cap (GJ)^{-1}0} x_1$, *where* $R_{(BJ)^{-1}0 \cap (GJ)^{-1}0}$ *is the sunny generalized nonexpansive retraction of* E *onto* $(BJ)^{-1}0 \cap (GJ)^{-1}0$.

Similarly, using Theorem 3.4, we have the following results.

Theorem 4.5. *Let* E *be a uniformly convex and uniformly smooth Banach space and let* C *be a nonempty and closed subset of* E *such that* JC *is closed and convex. Let* S *be a generic skew-generalized nonspreading mapping of* C *into* E *such that* $F(S) \ne \emptyset$. *Let* $C_1 = C$ *and let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n S x_n + c_n S^2 x_n, \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \le \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ *is the sunny generalized nonexpansive retraction of* E *onto* C_{n+1}*,* $c, d \in \mathbb{R}$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \le a_n, b_n, c_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = R_{F(S)}x_1$*, where* $R_{F(S)}$ *is the sunny generalized nonexpansive retraction of* E *onto* $F(S)$*.*

Theorem 4.6. *Let* H *be a real Hilbert space and let* C *be a nonempty, closed and convex subset of* H*. Let* S *and* T *be normally 2-generalized hybrid mappings such that* $F(S) \cap F(T) \neq \emptyset$*. Let* $C_1 = C$ *and let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n S x_n + (1-\lambda_n)T x_n\big) + c_n\big(\mu_n S^2 x_n + (1-\mu_n)T^2 x_n\big), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ *is the metric projection of* H *onto* C_{n+1}*,* $a, b, c, d \in \mathbb{R}$*,* $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$0 < a \le \lambda_n, \mu_n \le b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \le a_n, b_n, c_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ *converges strongly to* $z_0 = P_{F(S)\cap F(T)}x_1$*, where* $P_{F(S)\cap F(T)}$ *is the metric projection of* H *onto* $F(S) \cap F(T)$*.*

Theorem 4.7. *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* C *be a nonempty and closed subset of* E *such that* JC *is closed and convex. Let* S *and* T *be generic skew generalized nonspreading mappings of* C *into itself such that* $F(S) \cap F(T) \neq \emptyset$*. Let* $C_1 = C$ *and let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n S x_n + (1-\lambda_n)T x_n\big) + c_n\big(\mu_n S^2 x_n + (1-\mu_n)T^2 x_n\big), \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \le \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ *is the sunny generalized nonexpansive retraction of* E *onto* C_{n+1}*,* $a, b, c, d \in \mathbb{R}$*,* $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$0 < a \le \lambda_n, \mu_n \le b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \le a_n, b_n, c_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = R_{F(S)\cap F(T)}x_1$*, where* $R_{F(S)\cap F(T)}$ *is the sunny generalized nonexpansive retraction of* E *onto* $F(S) \cap F(T)$*.*

Theorem 4.8. *Let* E *be a uniformly convex and uniformly smooth Banach space. Let* B *and* G *be maximal monotone operators of* E^* *into* 2^E *and let* J_r *and* Q_s *be sunny generalized resolvents for* $r > 0$ *and* $s > 0$ *of* B *and* G*, respectively. Suppose that* $B^{-1}0 \cap G^{-1}0 \neq \emptyset$*. Let* $C_1 = C$ *and let* $\{x_n\} \subset C$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\lambda_n J_r x_n + (1-\lambda_n)Q_s x_n\big) + c_n\big(\mu_n (J_r)^2 x_n + (1-\mu_n)(Q_s)^2 x_n\big), \\ C_{n+1} = \{z \in C_n : \phi(y_n, z) \le \phi(x_n, z)\}, \\ x_{n+1} = R_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $R_{C_{n+1}}$ *is the sunny generalized nonexpansive retraction of* E *onto* C_{n+1}, $a, b, c, d \in \mathbb{R}$, $\{\mu_n\}, \{\lambda_n\} \subset (0,1)$ *and* $\{a_n\}, \{b_n\}, \{c_n\} \subset (0,1)$ *satisfy the following:*

$$0 < a \leq \lambda_n, \mu_n \leq b < 1,$$

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < c \leq a_n, b_n, c_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = R_{(BJ)^{-1}0 \cap (GJ)^{-1}0} x_1$, *where* $R_{(BJ)^{-1}0 \cap (GJ)^{-1}0}$ *is the sunny generalized nonexpansive retraction of* E *onto* $(BJ)^{-1}0 \cap (GJ)^{-1}0$.

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