A relationship between possibility-theoretical comparison indices for fuzzy sets and set relations

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Abstract

This paper presents a new characterization of possibility-theoretical comparison indices for fuzzy sets. The indices dealt with here are a vector-ordering version of those originated by Dubois and Prade and extended by Inuiguchi, Ichihashi, and Kume. In three forms with different assumptions, it is shown that the indices can be related to six types of set relations used in the area of set optimization.

1 Introduction

There are a lot of studies that deal with how to compare fuzzy sets or particularly fuzzy numbers (see, e.g., [1] for an overview of them). Such studies are important in fuzzy optimization, giving criteria to determine the optimal solutions or to specify the constraints of problems. In 1983, Dubois and Prade [2] proposed four types of comparison indices for fuzzy numbers by using Zadeh's possibility theory [9] and the usual orders $\leq$ and $<$ on $\mathbb{R}$. These indices were highly evaluated in some papers because the set of them can suitably describe many different situations. Inuiguchi, Ichihashi, and Kume [5] then extended the indices to a more general setting: for fuzzy sets in any space and using any fuzzy preference relation on the space. Thus six types of possibility-theoretical comparison indices, expressed as fuzzy relations between general fuzzy sets, were given. Investigations of their properties and applications of them to fuzzy mathematical programming can be seen in [6, 7].

Set relations are order-like binary relations between crisp sets in a preordered vector space. They are widely used in the area of set optimization mainly to compare the values of objective set-valued mappings. There exist various types of set relations in the literature, but here we focus only on the six types (including the most famous two ones) proposed by Kuroiwa, Tanaka, and Ha [8].

In this paper, we present a new characterization of the above indices of Inuiguchi et al. in a vector-ordering case, showing that the indices can be completely related to the six types of set relations. In particular, certain cone-compactness and cone-continuity assumptions contribute to deriving practical equivalences involving the indices and the set relations.

2 Preliminaries

Throughout the paper, let Z be a real Hausdorff topological vector space and C a convex cone in Z with $0_Z \in C \neq Z$. A preorder $\leq_C$ on Z is defined by

$$z \leq_C z' \;:\Longleftrightarrow\; z' - z \in C$$

for $z, z' \in Z$. The power set of Z is denoted by $\mathcal{P}(Z)$.

Definition 1. Six types of *set relations* $\preceq_C^{(*)}$ $(* = 1, 2L, 2U, 3L, 3U, 4)$ are defined by

$$\begin{aligned}
A \preceq_C^{(1)} B &:\Longleftrightarrow \forall a \in A\ \forall b \in B\colon a \leq_C b,\\
A \preceq_C^{(2L)} B &:\Longleftrightarrow \exists a \in A\ \forall b \in B\colon a \leq_C b,\\
A \preceq_C^{(2U)} B &:\Longleftrightarrow \exists b \in B\ \forall a \in A\colon a \leq_C b,\\
A \preceq_C^{(3L)} B &:\Longleftrightarrow \forall b \in B\ \exists a \in A\colon a \leq_C b,\\
A \preceq_C^{(3U)} B &:\Longleftrightarrow \forall a \in A\ \exists b \in B\colon a \leq_C b,\\
A \preceq_C^{(4)} B &:\Longleftrightarrow \exists a \in A\ \exists b \in B\colon a \leq_C b
\end{aligned}$$

for $A, B \in \mathcal{P}(Z)$.

These set relations were originally proposed in [8], but the numbering and the form employed here are based on the definition in [4].

When A and B are nonempty, we have

$$\begin{aligned}
A \preceq_C^{(1)} B \implies A \preceq_C^{(2L)} B \implies A \preceq_C^{(3L)} B \implies A \preceq_C^{(4)} B,\\
A \preceq_C^{(1)} B \implies A \preceq_C^{(2U)} B \implies A \preceq_C^{(3U)} B \implies A \preceq_C^{(4)} B.
\end{aligned}$$

The next proposition shows the monotonicity of the set relations over set inclusion.

Proposition 1. *Let $A, A', B, B' \in \mathcal{P}(Z)$. If*

- *$A \supset A'$ and $B \supset B'$ for $* = 1$,*
- *$A \subset A'$ and $B \supset B'$ for $* = 2L, 3L$,*
- *$A \supset A'$ and $B \subset B'$ for $* = 2U, 3U$,*
- *$A \subset A'$ and $B \subset B'$ for $* = 4$,*

then

$$A \preceq_C^{(*)} B \implies A' \preceq_C^{(*)} B'.$$

It is well known that any (crisp) set A in Z can be identified with its characteristic function $\chi_A\colon Z \to \{0, 1\}$. By analogy with this fact, a *fuzzy set* $\tilde{A}$ in Z is defined by a function $\mu_{\tilde{A}}\colon Z \to [0, 1]$, called the *membership function* of $\tilde{A}$ (see Figure 1). For each $\alpha \in [0, 1]$, the *α-cut* (*α-level set*) of $\tilde{A}$ is defined as

$$[\tilde{A}]_\alpha := \begin{cases} \{z \in Z \mid \mu_{\tilde{A}}(z) \geq \alpha\} & (\alpha \in (0, 1]) \\ \operatorname{cl}\{z \in Z \mid \mu_{\tilde{A}}(z) > 0\} & (\alpha = 0). \end{cases}$$

The fuzzy set $\tilde{A}$ is said to be

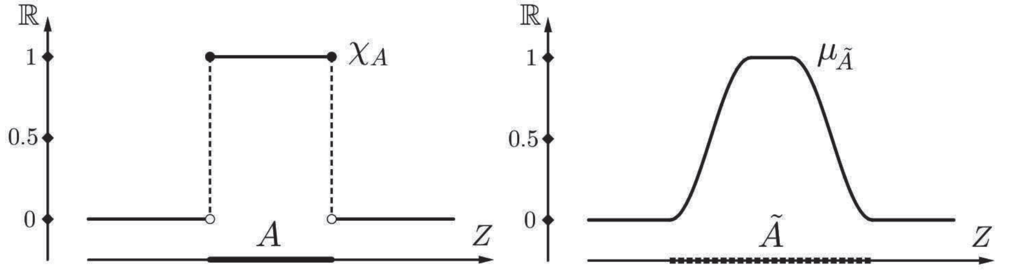

Figure 1: Illustration of a fuzzy set $\tilde{A}$ and its membership function $\mu_{\tilde{A}}$ versus a crisp set A and its characteristic function χ_A.

- *normal* if there exists $z \in Z$ such that $\mu_{\tilde{A}}(z) = 1$.
- *compact* if its α-cut is compact for all $\alpha \in [0,1]$.
- *strictly convex* if for any $z, z' \in Z$ with $z \neq z'$ and $\lambda \in (0,1)$,

$$\begin{aligned} \min\{\mu_{\tilde{A}}(z), \mu_{\tilde{A}}(z')\} \in (0,1) &\implies \mu_{\tilde{A}}(\lambda z + (1-\lambda)z') > \min\{\mu_{\tilde{A}}(z), \mu_{\tilde{A}}(z')\}, \\ \min\{\mu_{\tilde{A}}(z), \mu_{\tilde{A}}(z')\} = 1 &\implies \mu_{\tilde{A}}(\lambda z + (1-\lambda)z') = 1. \end{aligned}$$

Next we introduce possibility-theoretical comparison indices for fuzzy sets, expressed as *fuzzy relations* on $\mathcal{F}(Z)$, where $\mathcal{F}(Z)$ denotes the set of all fuzzy sets in Z and any fuzzy relation on $\mathcal{F}(Z)$ is defined to be a fuzzy set in the product $\mathcal{F}(Z) \times \mathcal{F}(Z)$.

Definition 2. Six types of fuzzy relations $\precsim_C^{(*)}$ $(* = 1, 2L, 2U, 3L, 3U, 4)$ on $\mathcal{F}(Z)$ are defined by

$$\begin{aligned} \mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) &:= \inf_{\substack{a,b \in Z \\ a \not\leq_C b}} \max\{1 - \mu_{\tilde{A}}(a), 1 - \mu_{\tilde{B}}(b)\}, \\ \mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) &:= \sup_{a \in Z} \inf_{\substack{b \in Z \\ a \not\leq_C b}} \min\{\mu_{\tilde{A}}(a), 1 - \mu_{\tilde{B}}(b)\}, \\ \mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B}) &:= \sup_{b \in Z} \inf_{\substack{a \in Z \\ a \not\leq_C b}} \min\{1 - \mu_{\tilde{A}}(a), \mu_{\tilde{B}}(b)\}, \\ \mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) &:= \inf_{b \in Z} \sup_{\substack{a \in Z \\ a \leq_C b}} \max\{\mu_{\tilde{A}}(a), 1 - \mu_{\tilde{B}}(b)\}, \\ \mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B}) &:= \inf_{a \in Z} \sup_{\substack{b \in Z \\ a \leq_C b}} \max\{1 - \mu_{\tilde{A}}(a), \mu_{\tilde{B}}(b)\}, \\ \mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}) &:= \sup_{\substack{a,b \in Z \\ a \leq_C b}} \min\{\mu_{\tilde{A}}(a), \mu_{\tilde{B}}(b)\} \end{aligned}$$

for $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$.

Let us remark that these indices are a vector-ordering version of those in [5] and that the numbering is changed into the same one as the set relations for the understandability of later results.

When $\tilde{A}$ and $\tilde{B}$ are normal, we find

$$\mu_{\precsim_C^{(1)}}(\tilde{A},\tilde{B}) \le \mu_{\precsim_C^{(2L)}}(\tilde{A},\tilde{B}) \le \mu_{\precsim_C^{(3L)}}(\tilde{A},\tilde{B}) \le \mu_{\precsim_C^{(4)}}(\tilde{A},\tilde{B}),$$
$$\mu_{\precsim_C^{(1)}}(\tilde{A},\tilde{B}) \le \mu_{\precsim_C^{(2U)}}(\tilde{A},\tilde{B}) \le \mu_{\precsim_C^{(3U)}}(\tilde{A},\tilde{B}) \le \mu_{\precsim_C^{(4)}}(\tilde{A},\tilde{B}).$$

In the framework of possibility theory (the details are omitted here), the value $\mu_{\precsim_C^{(*)}}(\tilde{A},\tilde{B})$ can be interpreted as

- the necessity of $\begin{cases}\tilde{A}\\ \tilde{B}\end{cases}$ being necessarily $\begin{cases}\text{less than } \tilde{B}\\ \text{greater than } \tilde{A}\end{cases}$ for $* = 1$.
- the possibility of $\tilde{A}$ being necessarily less than $\tilde{B}$ for $* = 2L$.
- the possibility of $\tilde{B}$ being necessarily greater than $\tilde{A}$ for $* = 2U$.
- the necessity of $\tilde{B}$ being possibly greater than $\tilde{A}$ for $* = 3L$.
- the necessity of $\tilde{A}$ being possibly less than $\tilde{B}$ for $* = 3U$.
- the possibility of $\begin{cases}\tilde{A}\\ \tilde{B}\end{cases}$ being possibly $\begin{cases}\text{less than } \tilde{B}\\ \text{greater than } \tilde{A}\end{cases}$ for $* = 4$.

This is why we say the indices are possibility-theoretical.

3 Main results

In this section, we show three results that relate the indices in Definition 2 to the set relations in Definition 1. The first result is below, which requires no assumptions on the fuzzy sets.

Theorem 1. *For $\tilde{A},\tilde{B} \in \mathcal{F}(Z)$, the following equalities hold:*

$$\mu_{\precsim_C^{(1)}}(\tilde{A},\tilde{B}) = \sup\left\{\alpha\in[0,1] \,\middle|\, [\tilde{A}]_{1-\alpha} \preceq_C^{(1)} [\tilde{B}]_{1-\alpha}\right\},$$
$$\mu_{\precsim_C^{(2L)}}(\tilde{A},\tilde{B}) = \sup\left\{\alpha\in[0,1] \,\middle|\, [\tilde{A}]_{\alpha} \preceq_C^{(2L)} [\tilde{B}]_{1-\alpha}\right\},$$
$$\mu_{\precsim_C^{(2U)}}(\tilde{A},\tilde{B}) = \sup\left\{\alpha\in[0,1] \,\middle|\, [\tilde{A}]_{1-\alpha} \preceq_C^{(2U)} [\tilde{B}]_{\alpha}\right\},$$
$$\mu_{\precsim_C^{(3L)}}(\tilde{A},\tilde{B}) = \sup\left\{\alpha\in[0,1] \,\middle|\, [\tilde{A}]_{\alpha} \preceq_C^{(3L)} [\tilde{B}]_{1-\alpha}\right\},$$
$$\mu_{\precsim_C^{(3U)}}(\tilde{A},\tilde{B}) = \sup\left\{\alpha\in[0,1] \,\middle|\, [\tilde{A}]_{1-\alpha} \preceq_C^{(3U)} [\tilde{B}]_{\alpha}\right\},$$
$$\mu_{\precsim_C^{(4)}}(\tilde{A},\tilde{B}) = \sup\left\{\alpha\in[0,1] \,\middle|\, [\tilde{A}]_{\alpha} \preceq_C^{(4)} [\tilde{B}]_{\alpha}\right\}$$

where $\sup\emptyset := 0$.

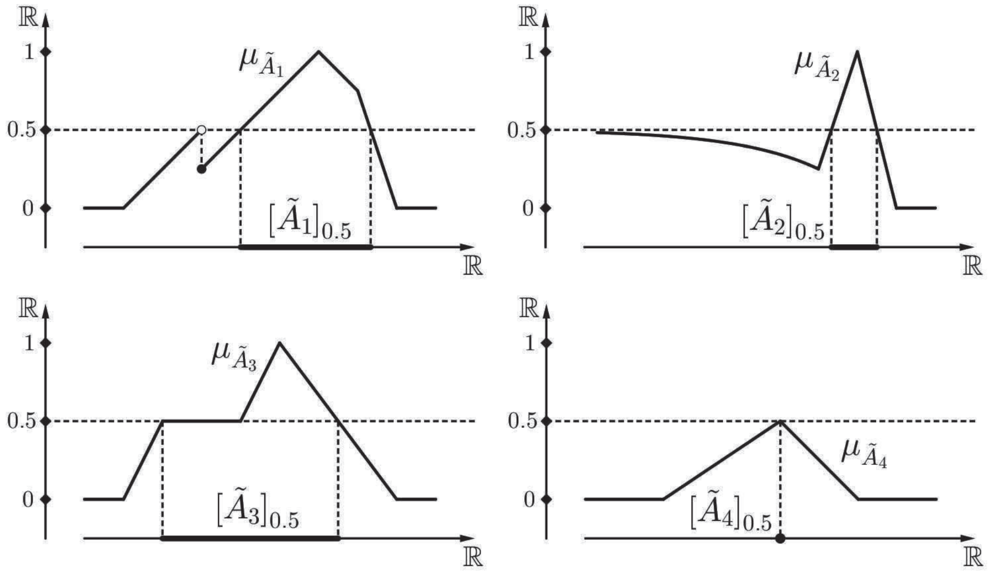

Figure 2: Illustration of the membership functions and the 0.5-cuts of the fuzzy sets $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4$ in Example 1.

Next we give the definitions of some cone-notions (e.g., [3]) and then use them to describe the second result.

A set A in Z is said to be *C-compact* if any cover of A of the form $\{O_s + C\}_{s \in S}$ for open sets O_s $(s \in S)$ admits a finite subcover. A fuzzy set $\tilde{A}$ in Z is said to be *C-compact* if its α-cut is C-compact for all $\alpha \in [0,1]$. A set-valued mapping $F\colon X \to \mathcal{P}(Z)$, where X is a topological space, is said to be

- *C-upper continuous at $x_0 \in X$* if for any open set O in Z with $F(x_0) \subset O$, there exists a neighborhood U of x_0 such that $F(x) \subset O + C$ for all $x \in U$.

- *C-lower continuous at $x_0 \in X$* if for any open set O in Z with $F(x_0) \cap O \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap (O - C) \neq \emptyset$ for all $x \in U$.

- *C-upper continuous* (*C-lower continuous*) if F is so at every $x \in X$.

For each $\tilde{A} \in \mathcal{F}(Z)$, the mapping $[0,1] \ni \alpha \mapsto [\tilde{A}]_\alpha \in \mathcal{P}(Z)$ is a set-valued mapping. We refer to it as the *cut mapping* of $\tilde{A}$.

Example 1. Consider four fuzzy sets $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4$ in $\mathbb{R}$ shown in Figure 2 and the set of all nonnegative real numbers $\mathbb{R}_+$, which is a convex cone in $\mathbb{R}$. One can check that the cut mappings of $\tilde{A}_1$ and $\tilde{A}_2$ are $\mathbb{R}_+$-lower continuous but not $\mathbb{R}_+$-upper continuous at 0.5 and that those of $\tilde{A}_3$ and $\tilde{A}_4$ are $\mathbb{R}_+$-upper continuous but not $\mathbb{R}_+$-lower continuous at 0.5.

Theorem 2. *Let Z be locally convex, C be closed, $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$, and $\alpha \in (0,1]$.*

- *If the cut mapping of $\tilde{A}$ is $(-C)$-lower continuous and the cut mapping of $\tilde{B}$ is C-lower continuous, then*

$$\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) \geq \alpha \iff [\tilde{A}]_{1-\alpha} \preceq_C^{(1)} [\tilde{B}]_{1-\alpha}.$$

- *If $\tilde{A}$ is C-compact, the cut mapping of $\tilde{A}$ is C-upper continuous, and the cut mapping of $\tilde{B}$ is C-lower continuous, then*

$$\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) \geq \alpha \iff [\tilde{A}]_{\alpha} \preceq_C^{(2L)} [\tilde{B}]_{1-\alpha},$$
$$\mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) \geq \alpha \iff [\tilde{A}]_{\alpha} \preceq_C^{(3L)} [\tilde{B}]_{1-\alpha}.$$

- *If the cut mapping of $\tilde{A}$ is $(-C)$-lower continuous, $\tilde{B}$ is $(-C)$-compact, and the cut mapping of $\tilde{B}$ is $(-C)$-upper continuous, then*

$$\mu_{\precsim_C^{(2U)}}(\tilde{A}, \tilde{B}) \geq \alpha \iff [\tilde{A}]_{1-\alpha} \preceq_C^{(2U)} [\tilde{B}]_{\alpha},$$
$$\mu_{\precsim_C^{(3U)}}(\tilde{A}, \tilde{B}) \geq \alpha \iff [\tilde{A}]_{1-\alpha} \preceq_C^{(3U)} [\tilde{B}]_{\alpha}.$$

- *If $\tilde{A}$ is C-compact, the cut mapping of $\tilde{A}$ is C-upper continuous, $\tilde{B}$ is $(-C)$-compact, and the cut mapping of $\tilde{B}$ is $(-C)$-upper continuous, then*

$$\mu_{\precsim_C^{(4)}}(\tilde{A}, \tilde{B}) \geq \alpha \iff [\tilde{A}]_{\alpha} \preceq_C^{(4)} [\tilde{B}]_{\alpha}.$$

The assumptions on the fuzzy sets in this theorem are a little complicated although they are appropriate. Replacing the assumptions with simple ones by use of the next proposition, we obtain the third result.

Proposition 2. *For $\tilde{A} \in \mathcal{F}(Z)$, the following statements hold:*

- *If $\tilde{A}$ is compact, then $\tilde{A}$ is C-compact.*
- *If $\tilde{A}$ is compact, then the cut mapping of $\tilde{A}$ is C-upper continuous.*
- *If $\tilde{A}$ is normal and strictly convex, then the cut mapping of $\tilde{A}$ is C-lower continuous.*

Corollary 1. *Let Z be locally convex, C be closed, $\tilde{A}, \tilde{B} \in \mathcal{F}(Z)$, and $\alpha \in (0,1]$.*

- *If $\tilde{A}$ and $\tilde{B}$ are normal and strictly convex, then*

$$\mu_{\precsim_C^{(1)}}(\tilde{A}, \tilde{B}) \geq \alpha \iff [\tilde{A}]_{1-\alpha} \preceq_C^{(1)} [\tilde{B}]_{1-\alpha}.$$

- *If $\tilde{A}$ is compact and $\tilde{B}$ is normal and strictly convex, then*

$$\mu_{\precsim_C^{(2L)}}(\tilde{A}, \tilde{B}) \geq \alpha \iff [\tilde{A}]_{\alpha} \preceq_C^{(2L)} [\tilde{B}]_{1-\alpha},$$
$$\mu_{\precsim_C^{(3L)}}(\tilde{A}, \tilde{B}) \geq \alpha \iff [\tilde{A}]_{\alpha} \preceq_C^{(3L)} [\tilde{B}]_{1-\alpha}.$$

- *If $\tilde{A}$ is normal and strictly convex and $\tilde{B}$ is compact, then*

$$\mu_{\precsim_C^{(2U)}}(\tilde{A},\tilde{B}) \geq \alpha \iff [\tilde{A}]_{1-\alpha} \preceq_C^{(2U)} [\tilde{B}]_\alpha,$$
$$\mu_{\precsim_C^{(3U)}}(\tilde{A},\tilde{B}) \geq \alpha \iff [\tilde{A}]_{1-\alpha} \preceq_C^{(3U)} [\tilde{B}]_\alpha.$$

- *If $\tilde{A}$ and $\tilde{B}$ are compact, then*

$$\mu_{\precsim_C^{(4)}}(\tilde{A},\tilde{B}) \geq \alpha \iff [\tilde{A}]_\alpha \preceq_C^{(4)} [\tilde{B}]_\alpha.$$

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