NOTE ON SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR FRACTIONAL ORDER BEAM EQUATIONS

TOSHIKAZU WATANABE
SCHOOL OF INTERDISCIPLINARY MATHEMATICAL SCIENCES,
MEIJI UNIVERSITY

1. Introduction

Throughout this paper, we denote by $\mathbb{R}$ the set of all real numbers. In [19], we consider the boundary value problem for fractional order differential equation

$$
\begin{cases} D^{\beta}_{0+}D^{\alpha}_{0+}u(t) - f(t, u(t), D^{\alpha}_{0+}u(t)) = 0, t \in [0, 1] \\ u(0) = A, u(1) = B, D^{\alpha}_{0+}u(0) = C, D^{\alpha}_{0+}u(1) = D, \end{cases} \tag{1.1}
$$

where D^{α}_{0+} is the Riemann-Liouville derivative of order α with respect to t, $1 < \alpha, \beta \leq 2$, A, B, C, D are constants, and f is a continuous function of $[0, 1] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. In this paper we propose the following differential equation (1.2) of order α, $3 < \alpha \leq 4$ with the two point boundary condition involving the form (1.1). For simplicity, we consider the cases of $A = B = C = D = 0$.

$$
\begin{cases} D^{\alpha}_{0+}u(t) = f(t, u(t), D^{\alpha-3}_{0+}u(t), D_{0+}D^{\alpha-3}_{0+}u(t), D_{0+}D_{0+}D^{\alpha-3}_{0+}u(t)), \\ \quad 0 < t < 1, \\ u(0) = u(1) = 0, D_{0+}D^{\alpha-3}_{0+}u(0) = D_{0+}D^{\alpha-3}_{0+}u(1) = 0, \end{cases} \tag{1.2}
$$

where D^{α}_{0+} is the Riemann-Liouville fractional derivative and f is a function of $[0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Let $\alpha > 0$. The Riemann-Liouville fractional integral of order α of u, denoted $I^{\alpha}_{0+}u$, is defined by

$$
I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}u(s)ds,
$$

provided the right-hand side exists. The Riemann-Liouville fractional derivative of order α of a function u of $(0, \infty)$ into $\mathbb{R}$ is given by

$$
D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n-\alpha-1}u(s)ds,
$$

where $n = [\alpha] + 1$ ($[\alpha]$ denotes the integer part of α) and $\Gamma(\alpha)$ denotes the gamma function; see [11, 18]. Note that for $\alpha > \beta > 0$, we have

$$
D^{\alpha}_{0+}t^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta-\alpha}.
$$

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A function $u \in C[0,1]$ is called a solution of problem (1.2) if $D^{\alpha}_{0+}u \in C[0,1]$, $D^{\alpha-3}_{0+}u \in L^1[0,1]$, $D^{\alpha-2}_{0+}u \in L^1[0,1]$, $D^{\alpha-1}u \in L^1[0,1]$, u satisfies the boundary conditions and equality in (1.2) a.e. on [0, 1].

Many researchers have considered the differential equation (1.2) with $\alpha = 4$; see [1, 2, 9, 10, 14, 15, 21, 22, 23, 25, 26]. Equation (1.2) with $\alpha = 4$ can be used to model the deformations of an elastic beam; see [21, 22] and the references therein. The boundary conditions in (1.2) with respect to normal derivative ensures that both endpoints are simply supported. Meanwhile, fractional differential equations have been of interest recently; see [3, 5, 6, 7, 8, 11, 12, 17, 18, 19, 24]. In particular, for higher order boundary problems, see [17, 19, 20, 24]. However, to the best of our knowledge, there are no results for the boundary value problem represented by (1.2) for $3 < \alpha \leq 4$, which we consider in the present paper. We use the several methods to prove the existence and uniqueness of solutions. Moreover we consider the properties of Green function given by (1.2).

2. Lemmas

For a continuous mapping h of $[0,1]$ into $\mathbb{R}$, we consider the following fractional differential boundary problems defined by

$$\begin{cases} D^{\alpha}_{0+}u(t) = h(t), \quad 0 < t < 1, \\ u(0) = u(1) = 0, D_{0+}D^{\alpha-3}_{0+}u(0) = D_{0+}D^{\alpha-3}_{0+}u(1) = 0, \end{cases} \tag{2.1}$$

where $3 < \alpha \leq 4$. In this section, we show the unique solution to the boundary value problem represented by (2.1). A mapping u of $[0,1]$ into $\mathbb{R}$ is a solution of that boundary value problem if u is continuous on $[0,1]$ and u satisfies (2.1). The following lemma can be found in [6]; see [11] also. We denoted by $C(0,1)$ the set of all continuous mappings of $(0,1)$ into $\mathbb{R}$ and by $L(0,1)$ the set of all Lebesgue integrable mappings of $[0,1]$ into $\mathbb{R}$.

Lemma 2.1. *Let $\alpha > 0$. If $u(t) \in C(0,1) \cap L(0,1)$ satisfying $D^{\alpha}_{0+}u(t) \in C(0,1) \cap L(0,1)$, then there exist constants $C_1, C_2, \ldots, C_n \in \mathbb{R}$ such that*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

where $n = [\alpha] + 1$ and $I^{\alpha}_{0+}u$ is the Riemann-Liouville fractional integral of order α of a function u defined by

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds.$$

Using Lemma 2.1, we obtain the following.

Lemma 2.2. *Let h be a continuous mapping of $[0,1]$ into $\mathbb{R}$. Let $3 < \alpha \leq 4$. Then the unique solution of the boundary value problem represented by (2.1) is*

$$u(t) = \int_0^1 G(t,s)h(s)ds,$$

where

(2.2)

$$G(t,s)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\left((t-s)^{\alpha-1}-(1-s)t^{\alpha-1}-(1-s)^{\alpha-1}t^{\alpha-3}+(1-s)t^{\alpha-3}\right)\\(0\le s\le t\le 1),\\-\dfrac{1}{\Gamma(\alpha)}\left((1-s)t^{\alpha-1}+(1-s)^{\alpha-1}t^{\alpha-3}-(1-s)t^{\alpha-3}\right)\\(0\le t\le s\le 1).\end{cases}$$

Remark 2.3. If $\alpha=4$, then

$$G(t,s)=\begin{cases}\frac{1}{6}\left(s(1-t)(2t-t^2-s^2)\right)(0\le s\le t\le 1),\\\frac{1}{6}\left(t(1-s)(2s-s^2-t^2)\right)(0\le t\le s\le 1).\end{cases}$$

Lemma 2.4. *Let $3<\alpha\le 4$. The function $G(t,s)$ in Lemma 2.2 satisfies the following conditions:*

(i) *$m\le G(t,s)\le M$, where*

$$m=\begin{cases}s(t(1-t)(1-t)-(t-s)(t-s)) \text{ if } (s\le t),\\0 \text{ if } (t\le s),\end{cases}$$

$$M=\begin{cases}(t-s)^2+(1-s)(2t-t^2-s^2) \text{ if } (s\le t),\\(1-t)(2s-t^2-s^2) \text{ if } (t\le s).\end{cases}$$

(ii) *If s and t satisfy $0\le t\le\frac{3-\sqrt{5}}{2}$, or $1\ge t\ge\frac{3-\sqrt{5}}{2}$ and $t-\sqrt{t}(1-t)\le s\le t$, then $G(t,s)\ge 0$.*

We consider the following.

$$\int_0^t G(t,s)h(s)ds=[-G(t,s)v(s)]_0^t+\int_0^t G_1(t,s)v(s)ds,$$

and

$$\int_t^1 G(t,s)h(s)ds=[-G(t,s)v(s)]_t^1+\int_t^1 G_1(t,s)v(s)ds,$$

where $v(s)=\int_t^1 h(s)ds$ and $G_1(t,s)=\frac{\partial G}{\partial s}(t,s)$

(2.3)

$$G_1(t,s)=\begin{cases}\frac{1}{\Gamma(\alpha)}\left(-(\alpha-1)(t-s)^{\alpha-2}+t^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}t^{\alpha-3}-t^{\alpha-3}\right)\\\text{if } 0\le s\le t\le 1\\\frac{1}{\Gamma(\alpha)}t^{\alpha-1}+(\alpha-1)(1-s)^{\alpha-2}t^{\alpha-3}-t^{\alpha-3}\\\text{if } 0\le t\le s\le 1.\end{cases}$$

Moreover

$$\int_0^1 G(t,s)h(s)ds=\int_0^1 G_1(t,s)v(s)ds.$$

We also have

$$\int_0^1 D_{0+}^{\alpha-3} G(t,s)h(s)ds = \int_0^t G_2(t,s)v(s)ds,$$

$$\int_0^1 D_{0+} D_{0+}^{\alpha-3} G(t,s)h(s)ds = \int_0^1 G_3(t,s)v(s)ds,$$

where

$$G_2(t,s) = \begin{cases} \frac{1}{\Gamma(2)}((t-s) - (1-s)t^{\alpha-3}) & (0 \le s \le t \le 1), \\ -\frac{1}{\Gamma(2)}(1-s)t^{\alpha-3} & (0 \le t \le s \le 1), \end{cases} \tag{2.4}$$

and

$$G_3(t,s) = \frac{\partial G_2}{\partial s}(t,s) = \begin{cases} t^{\alpha-3} - 1 \ (0 \le s \le t \le 1) \\ t^{\alpha-3} \ (0 \le t \le s \le 1). \end{cases} \tag{2.5}$$

Remark 2.5. The function $\int_0^1 G(\cdot, s)ds$ is continuous on $[0,1]$. In fact, we have

$$\int_0^1 G(t,s)ds = \frac{1}{\alpha\Gamma(\alpha)} \left(t^\alpha - \frac{\alpha}{2} t^{\alpha-1} + \left(1 - \frac{\alpha}{2}\right) t^{\alpha-3} \right) \text{ for all } 0 \le t \le 1.$$

3. Main result

Next we use the method of order reduction to transform (1.2) to a nonlinear integral equation. To do this, let

$$T_1 v(t) = I_{0+}^{\alpha-3} T_2 v(t) = \int_0^1 G_1(t,s)v(s)ds,$$

$$T_2 v(t) = \int_0^1 G_2(t,s)v(s)ds, \ T_3 v(t) = \int_0^1 G_3(t,s)v(s)ds,$$

where $G_1(t,s)$, $G_2(t,s)$ and $G_3(t,s)$ are given by (2.3), (2.4) and (2.5). From the above formulas, it follows that

$$D_{0+} D_{0+} D_{0+}^{\alpha-3} T_1 v(t) = D_{0+} D_{0+} T_2 v(t) = D_{0+} T_3 v(t) = -v(t).$$

Note that since

$$T_1 v(t) = \int_0^1 G_1(t,s)v(s)ds = \int_0^1 G(t,s)f(s)ds,$$

we have

$$T_1 v(0) = T_1 v(1) = 0.$$

Moreover by definition,

$$T_2 v(0) = \int_0^1 G_2(0,s)ds = 0, T_2 v(1) = \int_0^1 G_2(1,s)ds = 0.$$

Boundary value problem (1.1) can be converted into a ternminal value problem

$$D_{0+} v(t) = -f(t, T_1 v(t), T_2 v(t), T_3 v(t), -v(t)), \int_0^1 v(s)ds = 0.$$

From the above formulas, it follows that

$$D_{0+}v(t) = f(t, T_1v(t), T_2v(t), T_3v(t), -v(t))$$

where

$$D_{0+}T_3v(t) = -v(t),\ D_{0+}T_2v(t) = T_3v(t),\ D_{0+}^{\alpha-3}T_1v(t) = T_2v(t).$$

Then we have the following lemma.

Lemma 3.1. *Let $3 < \alpha \le 4$. The boundary value problem (1.2) is equivalent to the following integral equations forms;*

$$\begin{cases} v(t) = \int_t^1 f(s, T_1v(s), T_2v(s), T_3v(s), -v(s))ds, \\ T_1v(t) = \int_0^1 G_1(t,s)v(s)ds, \\ T_2v(t) = \int_0^1 G_2(t,s)v(s)ds, \\ T_3v(t) = \int_0^1 G_3(t,s)v(s)ds, \end{cases}$$

where $G_1(t,s)$, $G_2(t,s)$ and $G_3(t,s)$ are given by (2.3), (2.4) and (2.5).

Lemma 3.2. *$G_1(t,s)$, $G_2(t,s)$ and $G_3(t,s)$ satisfy the followings.*

(i) $-8 \le 3s+3s^3-3t-4t^2-1 \le G_1(t,s) \le 3t^2-2s^2+3-4s \le 6$, *if* $0 \le s \le t \le 1$
$-5 \le 3t^3 - 4s^2 + 2t - 1 \le G_1(t,s) \le t^2 + 3 - 4t \le 3$, *if* $0 \le t \le s \le 1$.

(ii) $-1 \le G_2(t,s) \le 0$, $-1 \le G_3(t,s) \le 1$.

Next we define an operator A from $C[0,1]$ into $C[0,1]$ by

$$Av(t) = \int_t^1 f(s, T_1v(s), T_2v(s), T_3v(s), -v(s))ds$$

,

where $v \in C[0,1]$. Then the solution of boundary value problem (1.2) is a fixed point of mapping A. Also let

$$(T_1v)(t) = \int_0^1 G_1(t,s)v(s)ds,\ (T_2v)(t) = \int_0^1 G_2(t,s)v(s)ds,\ (T_3v)(t) = \int_0^1 v(s)ds.$$

Then the existence of solution of the boundary value problem (1.2) is equivalent to the existence of fixed point of A on $C[0,1]$. Take $u_0(t) = 1 - t$.

$$\int_t^1 (T_1u_0)(t)dt \le \frac{1}{\Gamma(\alpha+2)}u_0(t),\ \int_t^1 (T_2u_0)(t)dt \le \left(\frac{1}{3(\alpha-2)} - \frac{1}{8}\right)u_0(t).$$

$$\int_t^1 (T_3u_0)(t)dt \le 0.0276515u_0.(\alpha = 3.5,$$

$$t = 1/6(8 - 8/(-109 + 27\sqrt{17})^{1/3} + (-109 + 27\sqrt{17})^{1/3}) \sim 0.5474636625659386)$$

Moreover if $\alpha = 4$,

$$\int_t^1 (T_3u_0)(t)dt \le \frac{3}{32}u_0 = 0.09375u_0,\ (t = 1/4).$$

If $\alpha = 3$, $\int_t^1 (T_3u_0)(t)dt \le \frac{1}{6}u_0(t) = 0.166667u_0(t)$ $(t = 0)$.

$$C_1 = \frac{1}{\Gamma(\alpha+2)},\ C_2 = \frac{1}{3(a-2)} - \frac{1}{8},\ C_3 = 0.0276515,\ (\alpha = 3.5). \tag{3.1}$$

If $\alpha = 4$, then we have

$$C_1 = \frac{1}{120}, \, C_2 = \frac{1}{24}.$$

However if $\alpha = 4$, calculate directly, then we can take

$$C_1 = \frac{291}{30720} > \frac{1}{120}, \, C_2 = \frac{1}{9\sqrt{6}} = 0.04536090 > \frac{1}{24} = 0.0416667,$$
$$C_3 = \frac{3}{32} = 0.09375 > 0.0276515.$$

Now we have the following theorem, which is the version of [27, Theorem 1].

Theorem 3.3. *Suppose that there exist four nonnegative constants M_i $(i = 1, 2, 3, 4)$ with $M_1C_1 + M_2C_2 + M_3C_3 + M_4 < 1$ such that*

$$|f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \leq \sum_{i=1}^{4} M_i|x_i - y_i|, x_i, y_i \in \mathbb{R}.$$

Then the boundary value problem (1.2) has a unique solution.

Next we consider the Banach contraction principal. Then, there exist constants

$$D_1 = \frac{4}{\Gamma(\alpha)} \left(\frac{\alpha - 1}{\alpha(\alpha - 2)} \right), \, D_2 = \frac{1}{6} + \frac{1}{2(\alpha - 2)}, \, D_3 = \frac{1}{2} - \frac{3 - \alpha}{(\alpha - 1)(\alpha - 2)}.$$

If $\alpha = 4$, then

$$D_1 = \frac{1}{4}, \, D_2 = \frac{5}{12}, \, D_3 = \frac{2}{3}.$$

Note that for $\alpha = 4$, if we calculate directly, result is same. In this case we also have the theorem, which is the version of [27, Theorem 3].

Theorem 3.4. *Suppose that there exist four nonnegative constants M_i $(i = 1, 2, 3, 4)$ with $M_1D_1 + M_2D_2 + M_3D_3 + M_4 < 1$ such that*

$$|f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \leq \sum_{i=1}^{4} M_i|x_i - y_i|, x_i, y_i \in \mathbb{R}.$$

Then the boundary value problem (1.2) has a unique solution.

We also have the theorem, which is the version of [27, Theorem 3]. In this section, we consider the existence and uniqueness of solutions of the boundary value problem represented by (1.2). It seems that there are few uniqueness results if the norm of related linear operator is greater than 1. In fact, Theorems 3.3, conclude that $r(T)$ is less than 1, where $r(T)$ is the spectral radius of linear operator T. Note that $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$. By Theorem 3.3, since for any $v \in C[0.1]$, $T^n v(t) \leq NM^n u_0(t)$, we have $\|T^n\|^{\frac{1}{n}} \leq M < 1$, thus we have $r(T) < 1$.

Theorem 3.5. *Suppose that there exist four nonnegative constants M_i $(i = 1, 2, 3, 4)$ such that*

$$|f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \leq \sum_{i=1}^{4} M_i|x_i - y_i|, x_i, y_i \in \mathbb{R},$$

and $r(T) < 1$. Then the boundary value problem (1.2) has a unique solution.

Finally we consider the method in [28]. In order to do this, we give several lemmas. First put $H_1(t,s) = D_{0+}^{\alpha-3}G(t,s)$. Thus we have

(3.2)
$$H_1(t,s) = \begin{cases} \frac{1}{2}(t-s)^2 - \frac{1}{2}(1-s)t^2 + \frac{1}{(\alpha-1)(\alpha-2)}(-(1-s)^{\alpha-1} + (1-s))) \text{ if } 0 \le s \le t \le 1, \\ -\frac{1}{2}(1-s)t^2 + \frac{1}{(\alpha-1)(\alpha-2)}(-(1-s)^{\alpha-1} + (1-s)) \text{ if } 0 \le t \le s \le 1. \end{cases}$$

Also put $H_2(t,s) = D_{0+}D_{0+}^{\alpha-3}G(t,s)$.

$$H_2(t,s) = \begin{cases} st - s = s(t-1) \text{ if } 0 \le s \le t \le 1, \\ s - 1 \text{ if } 0 \le t \le s \le 1. \end{cases} \tag{3.3}$$

$$H_3(t,s) = \begin{cases} s \text{ if } 0 \le s \le t \le 1, \\ 0 \text{ if } 0 \le t \le s \le 1. \end{cases} \tag{3.4}$$

Lemma 3.6. *$H_1(t,s)$ satisfies the following.*

(i) $-\frac{5}{6}s(1-s) \le H_1(t,s) \le \frac{1}{2}\left((t-s)^2 + (1-s)(s(1-s) + t(1-t))\right)$ *if* $0 \le s \le t \le 1$,
$\frac{1}{6}(1-s)t(1-3t) \le H_1(t,s) \le \frac{1}{2}(1-s)(s(1-s) + t(1-t))$ *if* $0 \le t \le s \le 1$.

(ii) *If* $0 \le t \le 1 - \sqrt{1-s} \le s$, *then* $H_1(t,s) \ge 0$.

For $u \in C^\alpha[0,1]$, put $\varphi(t) = f(t, u(t), D_{0+}^{\alpha-3}u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-1}u(t))$. Then the boundary value problem (1.2) becomes

$$\begin{cases} D_{0+}^{\alpha}u\varphi(t) = \varphi(t) \\ u(0) = u(1) = 0, D_{0+}^{\alpha-2}u(0) = D_{0+}^{\alpha-2}u(1) = 0, \end{cases}$$

where

$$u(t) = \int_0^1 G(t,s)\varphi(s)ds,\ D_{0+}^{\alpha-3}u(t) = \int_0^1 H_1(t,s)\varphi(s)ds,$$
$$D_{0+}^{\alpha-2}u(t) = \int_0^1 H_2(t,s)\varphi(s)ds,\ D_{0+}^{\alpha-1}u(t) = \int_0^1 H_3(t,s)\varphi(s)ds.$$

For φ, we have the equation $A\varphi = \varphi$, where A is a non-linear operator defined by

$$A\varphi(t) = f(t, u_\varphi(t), v_\varphi(t), w_\varphi(t), x_\varphi(t)),$$

with

$$v_\varphi(t) = D_{0+}^{\alpha-3}u(t),\ w_\varphi(t) = D_{0+}^{\alpha-2}u(t),\ x_\varphi(t) = D_{0+}^{\alpha-1}u(t).$$

Thus we have the following lemma.

Lemma 3.7. *Let $3 < \alpha \leq 4$. The boundary problem (1.2) is equivalent to the following integral equations forms;*

$$\begin{cases} \varphi(t) = f(t, u_\varphi(t), v_\varphi(t), w_\varphi(t), x_\varphi(t)), \\ v_\varphi(t) = \int_0^1 H_1(t,s)\varphi(s)ds, \\ w_\varphi(t) = \int_0^1 H_2(t,s)\varphi(s)ds, \\ x_\varphi(t) = \int_0^1 H_3(t,s)\varphi(s)ds, \end{cases}$$

where $H_1(t,s)$, $H_2(t,s)$ and $H_3(t,s)$ are given by (3.2), (3.3) and (3.4).

By (3.2) and Lemma 3.6, there exists E_1, E_2, E_3 and E_4 such that

$$E_1 = \sup_{t\in[0,1]} \int_0^1 |G(t,s)|ds, E_2 = \sup_{t\in[0,1]} \int_0^1 |H_1(t,s)|ds,$$

$$E_3 = \sup_{t\in[0,1]} \int_0^1 |H_2(t,s)|ds, E_4 = \sup_{t\in[0,1]} \int_0^1 |H_3(t,s)|ds.$$

In this case following theorem holds. It is a version of [28].

Theorem 3.8. *Suppose that there exist four nonnegative constants M_i $(i = 1,2,3,4)$ with $M_1E_1 + M_2E_2 + M_3E_3 + M_4E_4 < 1$ such that*

$$|f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \leq \sum_{i=1}^{4} M_i|x_i - y_i|, x_i, y_i \in \mathbb{R}.$$

Then the boundary value problem (1.2) has a unique solution.

For the case that $\alpha = 4$, we have the following; see Dang and Ngo [28].

Corollary 3.9. *Let f be a continuous function of $[0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$. Let g be a Lipschitz continuous function of $\mathbb{R}$ into itself with a nonnegative constant L. Assume that there exists nonnegative constants M_1, M_2, M_3 and M_4 with*

$$\frac{1}{120}M_1 + \frac{1}{6}M_2 + \frac{5}{12}M_3 + M_4 < 1$$

such that

$$|f(t, x_1, x_2, x_3, x_4) - f(t, y_1, y_2, y_3, y_4)| \leq \sum_{i=1}^{4} M_i|x_i - y_i|, x_i, y_i \in \mathbb{R}.$$

Then the boundary value problem represented by (1.2) has a unique solution.

References

[1] A. R. Aftabizadeh, *Existence and uniqueness theorems for fourth-order boundary value problems*, J. Math. Anal. Appl., 116 (1986), 415–426.

[2] R. P. Agarwal, *On fourth order boundary value problems arising in beam analysis*, Differential Integral Equations, 2 (1989), 91–110.

[3] R. P. Agarwal and B. Ahmad, *Existence theory for anti-periodic boundary value problems of fractional differential equations and inclusions*, Comput. Math. Appl. 62 (2011), 1200–1214.

[4] E. Alves, T. F. Ma, M. L. Pelicer, *Monotone positive solutions for a fourth order equation with nonlinear boundary conditions*, Nonlinear Anal., 71 (2009) 3834–3841.

[5] C. Bai, *Triple positive solutions for a boundary value problem of nonlinear fractional differential equation*, Electron. J. Qual. Theory Differ. Equ., 24 (2008), 1–10.

[6] Z. Bai and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl., 311 (2005), 495–505.

[7] G. Chai, *Existence results of positive solutions for boundary value problems of fractional differential equations*, Bound. Value Probl., 2013, 2013:109.

[8] K. Diethelm and N. J. Ford, *Analysis of fractional differential equations*, J. Math. Anal. Appl., 265 (2002), 229–248.

[9] C. P. Gupta, *Existence and uniqueness results for the bending of an elastic beam equation at resonance*, J. Math. Anal. Appl., 135 (1988), 208–225.

[10] C. P. Gupta, *A nonlinear boundary value problem associated with the static equilibrium of an elastic beam supported by sliding clamps*, Int. J. Math. Math. Sci., 12 (1989), 697–711.

[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, In North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[12] S. Liang and J. Zhang, *Positive solutions for boundary value problems of nonlinear fractional differential equation*, Nonlinear Anal., 71 (2009), 5545–5550.

[13] R. Liu and R. Ma, *Existence of positive solutions for an elastic beam equation with nonlinear boundary conditions*, J. Appl. Math., 2014 (2014), Article ID 972135.

[14] R. Ma, *Multiple positive solutions for a semipositone fourth-order boundary value problem*, Hiroshima Math. J., 33 (2003), 217–227.

[15] R. Ma, *Existence of positive solutions of a fourth-order boundary value problem*, Appl. Math. Comput., 168 (2005), 1219–1231.

[16] T. F. Ma and J. da Silva, *Iterative solutions for a beam equation with nonlinear boundary conditions of third order*, Appl. Math. Comput., 159 (2004), 11–18.

[17] N. Nyamoradi and M. Javidi, *Existence of multiple positive solutions for fractional differential inclusions with m-point boundary conditions and two fractional orders*, Electron. J. Differential Equations, 187 (2012), 1–26.

[18] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Translated from the 1987 Russian original, Gordon and Breach Science Publishers, Switzerland, 1993.

[19] M. Toyoda and T. Watanabe, *Existence and uniqueness theorem for fractional order differential equations with boundary conditions and two fractional order*, J. Nonlinear Convex Anal., 17 (2016), 267–273.

[20] M. Toyoda and T. Watanabe, *Note on solutions of boundary value problems involving a fractional differential equation*, Linear and Nonlinear Analysis Volume 3, Number 3, 2017, 449–455.

[21] R. A. Usmani, *A uniqueness theorem for a boundary value problems*, Proc. Amer. Math. Soc., 77 (1979), 329–335.

[22] R. A. Usmani, *Finite difference methods for computing eigenvalues of fourth order boundary value problems*, Int. J. Math. Math. Sci., 9 (1986), 137–143.

[23] J. R. L. Webb, G. Infante and D. Franco, *Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A, 138 (2008), 427–446.

[24] X. Xu, D. Jiang and C. Yuan, *Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation*, Nonlinear Anal., 71 (2009), 4676–4688.

[25] B. Yang, *Positive solutions for a fourth order boundary value problem*, Electron. J. Qual. Theory Differ. Equ., 3 (2005), 1–17.

[26] Y. Yang, *Fourth-order two-point boundary value problems*, Proc. Amer. Math. Soc., 104 (1988), 175–180.

[27] Y. Zoua, Y. Cui, *Uniqueness result for the cantilever beam equation with fully nonlinear term*, J. Nonlinear Sci. Appl., 10 (2017), 4734–4740.

[28] D.A. Quang Dang, T.Q. Ngo, *Existence results and iterative method for solving the cantilever beam equation with fully nonlinear term*, Nonlinear Anal:Real World Applications, 36 (2017) 56–68.

Email address: `twatana@edu.tuis.ac.jp`