Smooth homotopy 4-sphere (research announcement)

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1 Introduction

This paper is a research announcement of the paper [15] in a 4D topology research project [11, 12, 13, 14, 15, 16].

As general conventions throughout this paper, a compact connected oriented smooth r-dimensional manifold for $r \geq 2$ is called an *r-manifold* and a smooth embedding and a smooth isotopy from an r-manifold into an r'-manifold are called an *embedding* and an *isotopy*, respectively, unless otherwise stated. An *m-punctured manifold* of an r-manifold X is an r-manifold $X^{m(0)}$ obtained from X by removing the interiors of m mutually disjoint r-balls in the interior of X. The r-manifold $X^{1(0)}$ is denoted by $X^{(0)}$, where choices of the 4-balls are independent of the diffeomorphism type of X.

By this convention, a *homotopy 4-sphere* is a 4-manifold M homotopy equivalent to the 4-sphere S^4, and a *homotopy 4-ball* is a 1-punctured manifold $M^{(0)}$ of a homotopy 4-sphere M.

The main purpose of this paper is to show that every homotopy 4-sphere is diffeomorphic to the 4-sphere S^4, so that every homotopy 4-ball is diffeomorphic to the 4-ball D^4. For a positive integer n, the *stable 4-sphere of genus* n is the connected sum 4-manifold

$$\Sigma = \Sigma(n) = S^4 \# n(S^2 \times S^2) = S^4 \#_{i=1}^n S^2 \times S_i^2,$$

which is the union of an n-punctured manifold $(S^4)^{n(0)}$ of the 4-sphere S^4 and 1-punctured manifolds $(S^2 \times S_i^2)^{(0)}$ $(i = 1, 2, \dots, n)$ of the 2-sphere products $S^2 \times S_i^2$ $(i = 1, 2, \dots, n)$ pasting the boundary 3-spheres of $(S^4)^{n(0)}$ to the boundary 3-spheres of $(S^2 \times S_i^2)^{(0)}$ $(i = 1, 2, \dots, n)$.

For this purpose, a concept of a trivial surface-knot in the 4-space in [11] is used by observing that the stable 4-sphere Σ of genus n is the double branched covering space $S^4(F)_2$ of the 4-sphere S^4 branched along a trivial surface F of genus n.

An *orthogonal 2-sphere pair* or simply an *O2-sphere pair* of the stable 4-sphere Σ is a pair (S, S') of 2-spheres S and S' embedded in Σ meeting transversely at a point with the intersection numbers $\mathrm{Int}(S, S) = \mathrm{Int}(S', S') = 0$ and $\mathrm{Int}(S, S') = +1$.

A *pseudo-O2-sphere basis* of the stable 4-sphere Σ of genus n is the system (S_*, S'_*) of n mutually disjoint O2-sphere pairs (S_i, S'_i) $(i = 1, 2, \dots, n)$ in Σ. Let $N(S_i, S'_i)$ be a regular neighborhood of the union $S_i \cup S'_i$ of the O2-sphere pair (S_i, S'_i) in Σ such that $N(S_i, S'_i)$ $(i = 1, 2, \dots, n)$ are mutually disjoint and diffeomorphic to a 1-punctured manifold $(S^2 \times S^2)^{(0)}$ of the sphere product $S^2 \times S^2$. The *region* of a pseudo-O2-sphere basis (S_*, S'_*) in Σ of genus n is a 4-manifold $\Omega(S_*, S'_*)$ in Σ obtained from the 4-manifolds $N(S_i, S'_i)\,(i = 1, 2, \ldots, n)$ by connecting them by mutually disjoint 1-handles $h^1_j\,(j = 1, 2, \ldots, n-1)$ in Σ. Since Σ is a simply connected 4-manifold, the region $\Omega(S_*, S'_*)$ in Σ does not depend on any choices of the 1-handles $h^1_j\,(j = 1, 2, \ldots, n-1)$ and is uniquely determined by the pseudo-O2-sphere basis (S_*, S'_*) up to isotopies of Σ (see [9]). The *residual region*

$$\Omega^c(S_*, S'_*) = \mathrm{cl}(\Sigma \setminus \Omega(S_*, S'_*))$$

of the region $\Omega(S_*, S'_*)$ in Σ is always a homotopy 4-ball, which is shown by van Kampen theorem and a homological argument. An *O2-sphere basis* of the stable 4-sphere Σ of genus n is a pseudo-O2-sphere basis (S_*, S'_*) of Σ such that the residual region $\Omega^c(S_*, S'_*)$ is diffeomorphic to the 4-ball. The following result is basically the main result.

Theorem 1.1. For any two pseudo-O2-sphere bases (R_*, R'_*) and (S_*, S'_*) of the stable 4-sphere Σ of any genus $n \geq 1$, there is an orientation-preserving diffeomorphism $h : \Sigma \to \Sigma$ sending (R_i, R'_i) to (S_i, S'_i) for all $i\,(i = 1, 2, \ldots, n)$.

The stable 4-sphere Σ of genus n admits an O2-sphere basis. If (R_*, R'_*) is an O2-sphere basis of Σ and (S_*, S'_*) is the image of (R_*, R'_*) by an orientation-preserving diffeomorphism $f : \Sigma \to \Sigma$, then (S_*, S'_*) is also an O2-sphere basis. Thus, the following corollary is directly obtained from Theorem 1.1.

Corollary 1.2. Every pseudo-O2-sphere basis of the stable 4-sphere Σ of any genus $n \geq 1$ is an O2-sphere basis of Σ.

In this paper, an O2-handle pair $(D \times I, D' \times I)$ on a trivial surface-knot F in S^4 in [11] is discussed (see Section 2 for an explanation). A system $(D_* \times I, D'_* \times I)$ of n mutually disjoint O2-handle pairs $(D_i \times I, D'_i \times I)\,(i = 1, 2, \ldots, n)$ on F of genus n in S^4 is called an *O2-handle basis* of F.

It is shown in Lemma 2.3 that the lift $(S(D_*), S(D'_*))$ of the core system (D_*, D'_*) of any O2-handle basis $(D_* \times I, D'_* \times I)$ of F to $S^4(F)_2 = \Sigma$ is an O2-sphere basis of Σ. Also, in Corollary 5.3, it is shown that every O2-sphere basis of Σ is isotopic to such an O2-sphere basis $(S(D_*), S(D'_*))$ in Σ.

The following result which is called 4D Smooth Poincaré Conjecture is a direct consequence of Corollary 1.2.

Corollary 1.3. Any homotopy 4-sphere M is diffeomorphic to the 4-sphere S^4.

In the topological category, it is well-known by Freedman [3] (see also [4]) that the corresponding result of Corollary 1.3 holds (i.e., every topological 4-manifold homotopy equivalent to the 4-sphere is homeomorphic to the 4-sphere). In the piecewise-linear category, it can be shown by using the piecewise-linear versions of this argument (see Hudson [10], Rourke-Sanderson [18]) that the corresponding result of Corollary 1.3 holds (i.e., every piecewise-linear 4-manifold homotopy equivalent to the 4-sphere is piecewise-linearly homeomorphic to the 4-sphere).

It is known by Wall in [19] that for every closed simply connected signature-zero spin 4-manifold M with the second Betti number $\beta_2(M;\mathbf{Z}) = 2m > 0$, there is a diffeomorphism

$$\kappa : M\#\Sigma(n) \to \Sigma(m+n)$$

for a positive integer n and by Freedman [3] (see also [4]) that there is a homeomorphism from W to $\Sigma(m)$. However, a technique used for the proof of Theorem 1.1 cannot be generalized to this case. In fact, it is known by Akhmedov-Park in [1] that there is a closed simply connected signature-zero spin 4-manifold M with a large second Betti number $\beta_2(M;\mathbf{Z}) = 2m$ such that M is not diffeomorphic to $\Sigma(m)$. What can be said in this paper is the following corollary.

Corollary 1.4. Let M and M' be any closed (not necessarily simply connected) 4-manifolds with the same second Betti number $\beta_2(M;\mathbf{Z}) = \beta_2(M';\mathbf{Z})$. Then an embedding $u : M^{(0)} \to M'$ extends to a diffeomorphism $u^+ : M \to M'$ if and only if the embedding $u : M^{(0)} \to M'$ induces a fundamental group isomorphism

$$u_\# : \pi_1(M^{(0)}, x) \to \pi_1(M', u(x)).$$

The following corollary is obtained by combining Corollary 1.3 with the triviality condition of an S^2-link in S^4 in [11, 12, 13].

Corollary 1.5. Every closed 4-manifold M such that the fundamental group $\pi_1(M,x)$ is a free group of rank n and $H_2(M;\mathbf{Z}) = 0$ is diffeomorphic to the connected sum 4-manifold

$$S^4\#n(S^1 \times S^3) = S^4\#_{i=1}^n S^1 \times S_i^3.$$

The following corollary which is called 4D Smooth Schoenflies Conjecture is also obtained.

Corollary 1.6. Any smoothly embedded 3-sphere S^3 in the 4-sphere S^4 splits S^4 into two components of 4-manifolds which are both diffeomorphic to the 4-ball.

The paper is organized as follows: In Section 2, the stable 4-sphere Σ is identified with the double branched covering space $S^4(F)_2$ of S^4 branched along a trivial surface-knot F. In Section 3, a homological version of Theorem 1.1 is given. Throughout Section 4, outline of the proof of Theorem 1.1 is given. In Section 5, an isotopic deformation of an orientation-preserving diffeomorphism of the stable 4-sphere Σ is studied by combining the proof of Theorem 1.1 with Gabai's 4D light-bulb theorem in [5]. In fact, Theorem 5.1 says that *every orientation-preserving diffeomorphism of Σ is isotopic to the lift of an equivalence of a trivial surface-knot F in S^4 to Σ modulo a diffeomorphism of Σ with a support of a 4-ball disjoint from the lift of F.*

2 The stable 4-sphere as the double branched covering space of the 4-sphere branched along a trivial surface-knot

A *surface-knot* of genus n in the 4-sphere S^4 is a closed surface F of genus n embedded in S^4. It is also called a 2-*knot* if $n = 0$, i.e., F is the 2-sphere S^2. Two surface-knots F and F' in S^4 are *equivalent* by an *equivalence* f if F is sent to F' orientation-preservingly by an orientation-preserving diffeomorphism $f : S^4 \to S^4$.

A *trivial* surface-knot of genus n in S^4 is a surface-knot F of genus n which is the boundary of a handlebody of genus n embedded in S^4, where a handlebody of genus n means a 3-manifold which is a 3-ball for $n = 0$, a solid torus for $n = 1$ or a boundary-disk sum of n solid tori. A *surface-link* in S^4 is a union of disjoint surface knots in S^4, and a *trivial surface-link* is a surface-link bounding disjoint handlebodies in S^4. A trivial surface-link in S^4 is determined regardless of the embeddings and unique up to isotopies (see [9]).

A *symplectic basis* of a closed surface F of genus n is a system (x_*, x'_*) of element pairs $(x_j, x'_j)\,(j = 1, 2, \ldots, n)$ of $H_1(F; \mathbf{Z})$ with the intersection numbers $\mathrm{Int}(x_j, x_{j'}) = \mathrm{Int}(x'_j, x'_{j'}) = \mathrm{Int}(x_j, x'_{j'}) = 0$ for all distinct j, j' and $\mathrm{Int}(x_j, x'_j) = +1$ for all j. By an argument on the intersection form

$$\mathrm{Int} : H_1(F; \mathbf{Z}) \times H_1(F; \mathbf{Z}) \to \mathbf{Z},$$

any pair (x_1, x'_1) with $\mathrm{Int}(x_1, x'_1) = +1$ is extended to a symplextic basis (x_*, x'_*) of F. It is well-known that every symplectic basis $(x_*, x'_*) = \{(x_j, x'_j)|\, j = 1, 2, \ldots, n\}$ is realized by a system of oriented simple loop pairs $(e_*, e'_*) = \{(e_j, e'_j)|\, j = 1, 2, \ldots, n\}$ of F such that the geometric intersections $e_j \cap e_{j'} = e'_j \cap e'_{j'} = e_j \cap e'_{j'} = \emptyset$ for all distinct j, j' and the geometric intersection $e_j \cap e'_j$ is a point for all j, which is called a *loop basis* of F.

For a surface-knot F in S^4, an element $x \in H_1(F; \mathbf{Z})$ is said to be *spin* if the $\mathbf{Z}_2$-reduction $[x]_2 \in H_1(F; \mathbf{Z}_2)$ of x has $\eta([x]_2) = 0$ for the $\mathbf{Z}_2$-quadratic function

$$\eta : H_1(F; \mathbf{Z}_2) \to \mathbf{Z}_2$$

associated with a surface-knot F in S^4. For a simple loop e in F bounding a surface D_e in S^4 with $D_e \cap F = e$, the $\mathbf{Z}_2$-self-intersection number $\mathrm{Int}(D_e, D_e) \pmod 2$ with respect to the F-framing is defined to be the value $\eta([e]_2)$.

For every surface-knot F in S^4, there is a spin basis of F (see [7]). This means that any spin pair (x_1, x'_1) with $\mathrm{Int}(x_1, x'_1) = +1$ is extended to a spin symplectic basis (x_*, x'_*) of F by an argument of Arf invarinat of the $\mathbf{Z}_2$-quadratic function $\eta : H_1(F; \mathbf{Z}_2) \to \mathbf{Z}_2$. In particular, any spin pair (x_1, x'_1) is realized by a spin loop pair (e_1, e'_1) of F extendable to a spin loop basis (e_*, e'_*) of F.

A 2-*handle* on a surface-knot F in S^4 is a 2-handle $D \times I$ on F embedded in S^4 such that $(D \times I) \cap F = (\partial D) \times I$, where I denotes a closed interval with 0 as the center and $D \times 0$ is called the *core* of the 2-handle $D \times I$ and identified with D. For a 2-handle $D \times I$ on F in S^4, the loop ∂D of the core disk D is a spin loop in F since $\eta([\partial D]_2) = 0$.

To save notation, if an embedding $h : D \times I \cup F \to X$ is given from a 2-handle $D \times I$ on a surface F to a 4-manifold X, then the 2-handle image $h(D \times I)$ and the core image $h(D)$ on $h(F)$ are denoted by $hD \times I$ and hD, respectively.

An *orthogonal 2-handle pair* or simply an *O2-handle pair* on a surface-knot F in S^4 is a pair $(D \times I, D' \times I)$ of 2-handles $D \times I$ and $D' \times I$ on F which *meet orthogonally* on F, that is, which meet F only at the attaching annuli $(\partial D) \times I$ and $(\partial D') \times I$ so that the loops ∂D and $\partial D'$ meet transversely at just one point q and the intersection $(\partial D) \times I \cap (\partial D') \times I$ is diffeomorphic to the square $Q = \{q\} \times I \times I$ (see [11]).

An O2-*handle basis* of a trivial surface-knot F of genus n in S^4 is a system $(D_* \times I, D'_* \times I)$ of mutually disjoint O2-handle pairs $(D_i \times I, D'_i \times I)\,(i = 1, 2, \dots, n)$ on F in S^4 such that the loop system $(\partial D_*, \partial D'_*) = \{(\partial D_i, \partial D'_i)|\, i = 1, 2, \dots, n\}$ forms a spin loop basis of F.

Every trivial surface F in S^4 is taken as the boundary of a standard handlebody in the equatorial 3-sphere S^3 of the 4-sphere S^4. A *standard* O2-handle basis of F is an O2-handle basis of F which is taken in S^3 and a *standard* loop basis of F is a loop basis of F determined by the attaching part of a standard O2-handle basis of F.

For any given spin loop basis of a trivial surface-knot F of genus n in S^4, there is an O2-handle basis $(D_* \times I, D'_* \times I)$ of a trivial surface-knot F in S^4 such that the loop basis $(\partial D_*, \partial D'_*)$ coincides with the given spin loop basis of F. This is because there is an equivalence $f : (S^4, F) \to (S^4, F)$ sending the standard spin loop basis to the given spin loop basis of F by [8, 11] and hence there is an O2-handle basis of F which is the image of the standard O2-handle basis of F.

Let $p : S^4(F)_2 \to S^4$ be the double branched covering projection branched along F. The non-trivial covering involution of the double branched covering space $S^4(F)_2$ is denoted by α. The preimage $p^{-1}(F)$ in Σ of F which is the fixed point set of α and diffeomorphic to F is also written by the same notation as F. The following result is a standard result.

Lemma 2.1. For a standard O2-handle basis $(D_* \times I, D'_* \times I)$ of a trivial surface-knot F of genus n in S^4, there is an orientation-preserving diffeomorphism

$$f : S^4(F)_2 \to \Sigma$$

sending the 2-sphere pair system

$$(S(D_*), S(D'_*)) = \{(S(D_i), S(D'_i))|\, i = 1, 2, \dots, n\}$$

to the standard O2-sphere basis $(S^2 \times 1_*, 1 \times S^2_*)$ of the stable 4-sphere Σ of genus n. In particular, the 2-sphere pair system $(S(D_*), S(D'_*))$ is an O2-sphere basis of Σ.

The identification of $S^4(F)_2 = \Sigma$ is fixed by an orientation-preserving diffeomorphism $f : S^4(F)_2 \to \Sigma$ given in Lemma 2.1. Using a result of [11, 12], we have the following corollary.

Corollary 2.2. For any two O2-handle bases $(D_* \times I, D'_* \times I)$ and $(E_* \times I, E'_* \times I)$ of a trivial surface-knot F of genus n in S^4, there is an orientation-preserving α-equivariant diffeomorphisn $\tilde{f}$ of Σ sending the 2-sphere pair system $(S(D_*), S(D'_*))$ to the 2-sphere pair system $(S(E_*), S(E'_*))$. In particular, the 2-sphere pair system $(S(D_*), S(D'_*))$ for every O2-handle basis $(D_* \times I, D'_* \times I)$ is an O2-sphere basis of Σ.

An *n-rooted disk family* is the triplet (d, d_*, b_*) where d is a disk, d_* is a system of n mutually disjoint disks $d_i\,(i = 1, 2, \ldots, n)$ in the interior of d and b_* is a system of n mutually disjoint bands $b_i\,(i = 1, 2, \ldots, n)$ in the n-punctured disk $\mathrm{cl}(d \setminus d_*)$ such that b_i spans an arc in the loop ∂d_i and an arc in the loop ∂d. Let $\hat{b}_*$ denote the centerline system of the band system b_*.

In the following lemma, it is shown that there is a canonical n-rooted disk family (d, d_*, b_*) associated with an O2-handle basis $(D_* \times I, D'_* \times I)$ of a trivial surface-knot F of genus n in S^4.

Lemma 2.3. Let $(D_* \times I, D'_* \times I)$ be an O2-handle basis of a trivial surface-knot F of genus n in S^4, and (d, d_*, b_*) an n-rooted disk family. Then there is an embedding

$$\varphi : (d, d_*, b_*) \times I \to (S^4, D_* \times I, D'_* \times I)$$

such that

(1) the surface F is the boundary of the handlebody V of genus n given by

$$V = \varphi(\mathrm{cl}(d \setminus d_*) \times I),$$

(2) there is an identification

$$\varphi(d_* \times I, d_*) = (\varphi(d_*) \times I, \varphi(d_*)) = (D_* \times I, D_*)$$

as 2-handle systems on F and

(3) there is an identification

$$\varphi(b_* \times I, \hat{b}_* \times I) = (D'_* \times I, D'_*)$$

as 2-handle systems on F.

In other words, Lemma 2.3 says that the 2-handle system $D_* \times I$ attaches to the handlebody V along a longitude system of V and the 2-handle system $D'_* \times I$ attaches to V along a meridian system of V.

The embedding φ in Lemma 2.3 is called a *bump embedding*. The 3-ball $B = \varphi(D \times I)$, the handlebody V in Lemma 2.3 and the pair (B, V) are respectively called a *bump 3-ball*, a *bump handlebody* and a *bump pair* of F in S^4.

For a bump embedding

$$\varphi : (d, d_*, b_*) \times I \to (S^4, D_* \times I, D'_* \times I),$$

there is an embedding $\tilde{\varphi} : d \times I \to S^4(F)_2$ with $p\tilde{\varphi} = \varphi$. The images $\tilde{\varphi}(d_* \times I)$ and $\tilde{\varphi}(b_* \times I)$ are respectively considered as 2-handle systems $\tilde{D}_* \times I$ and $\tilde{D}'_* \times I$ on F in $S^4(F)_2$ by the rules of Lemma 2.3 (1)-(3), so that $(\tilde{D}_* \times I, \tilde{D}'_* \times I)$ is an O2-handle basis of F in $(S^4)(F)_2$ with $p(\tilde{D}_* \times I, \tilde{D}'_* \times I) = (D_* \times I, D'_* \times I)$. The induced embedding

$$\tilde{\varphi} : (d, d_*, b_*) \times I \to (S^4(F)_2, \tilde{D}_* \times I, \tilde{D}'_* \times I)$$

has $p\tilde{\varphi} = \varphi$ and is called a *lifting bump embedding* of the bump embedding φ. The bump 3-ball $\tilde{\varphi}(d \times I)$ and the bump handlebody $\tilde{\varphi}(\mathrm{cl}(d \setminus d_*) \times I)$ are respectively denoted by B and V in $S^4(F)_2$ unless confusion might occur.

The composite embedding

$$\alpha\tilde{\varphi} : (d, d_*, b_*) \times I \to (S^4(F)_2, \alpha\tilde{D}_* \times I, \alpha\tilde{D}'_* \times I)$$

is another lifting bump embedding of the bump embedding φ. The bump 3-ball $\alpha\tilde{\varphi}(d \times I) = \alpha(B)$ and the bump handlebody $\alpha\varphi(\mathrm{cl}(d \setminus d_*) \times I) = \alpha(V)$ are respectively denoted by $\bar{B}$ and $\bar{V}$ in $S^4(F)_2$. Then we have

$$V \cap \bar{V} = B \cap \bar{B} = F \quad \text{in} \quad S^4(F)_2.$$

For an O2-handle basis $(D_* \times I, D'_* \times I)$ of a trivial surface-knot F in S^4, the lifting O2-handle bases $(\tilde{D}_* \times I, \tilde{D}'_* \times I)$ and $(\alpha\tilde{D}_* \times I, \alpha\tilde{D}'_* \times I)$ of F in $S^4(F)_2$ are respectively denoted by

$$(D_* \times I, D'_* \times I), \quad \text{and} \quad (\bar{D}_* \times I, \bar{D}'_* \times I).$$

Note that the unions $S(D_i) = D_i \cup \bar{D}_i$ and $S(D'_i) = D'_i \cup \bar{D}'_i$ are 2-spheres in $S^4(F)_2$ such that $(S(D_i), S(D'_i))$ is an O2-sphere pair in $S^4(F)_2$.

For a lifting bump embedding, we have the following lemma.

Lemma 2.4. Let $\tilde{\varphi} : (d, d_*, b_*) \times I \to (\Sigma, D_* \times I, D'_* \times I)$ be a lifting bump embedding. Let $u : \Sigma^{(0)} \to \Sigma$ be an embedding. Assume that the image $\tilde{\varphi}(d \times I)$ is in the interior of $\Sigma^{(0)}$ to define the composite embedding $u\tilde{\varphi} : (d, d_*, b_*) \times I \to (\Sigma, uD_* \times I, uD'_* \times I)$. Then there is a diffeomorphism $g : \Sigma \to \Sigma$ which is isotopic to the identity such that the composite embedding $gu\tilde{\varphi} : (d, d_*, b_*) \times I \to (\Sigma, guD_* \times I, guD'_* \times I)$ is identical to the lifting bump embedding $\tilde{\varphi} : (d, d_*, b_*) \times I \to (\Sigma, D_* \times I, D'_* \times I)$.

In Lemma 2.4, note that any disk interior of the disk systems $gu\bar{D}_*$ and $gu\bar{D}'_*$ does not meet the bump 3-ball $B = \tilde{\varphi}(d \times I)$ in Σ.

In fact, since gu defines an embedding from $B \cup \bar{B}$ with $B \cap \bar{B} = F$ into Σ and we have $gu(B, F) = (B, F)$, the complement $gu(\bar{B}) \setminus F$ of F in the 3-ball $gu(\bar{B})$ does not meet the bump 3-ball B, which means that any disk interior of the disk systems $gu\bar{D}_*$ and $gu\bar{D}'_*$ does not meet the bump 3-ball B.

Unless $\Sigma^{(0)}$ and Σ have the same genus n, this property cannot be guaranteed.

3 Outline of the proof of a homological version of Theorem 1.1

The following lemma is related to the intersection numbers of the lifting O2-sphere bases between two O2-handle bases of F in S^4.

Lemma 3.1. Let $(D_* \times I, D'_* \times I)$ be an O2-handle basis of a trivial surface-knot F of genus n in S^4, and $(\ell_*, \ell'_*) = (\partial D_*, \partial D'_*)$ a spin loop basis of F. For a 2-handle $E \times I$ on

F in S^4, assume that the homology class $[e] \in H_1(F;\mathbf{Z})$ of the loop $e = \partial E$ is written as

$$[e] = \sum_{j=1}^{n} k_j[\ell_j] + \sum_{j=1}^{n} s_j[\ell'_j]$$

in $H_1(F;\mathbf{Z})$ for some integers k_j, s_j $(j = 1, 2, \ldots, n)$. Then the homology class $[S(E)] \in H_2(\Sigma;\mathbf{Z})$ is written as

$$[S(E)] = \sum_{j=1}^{n} k_j[S(D_j)] + \sum_{j=1}^{n} s_j[S(D'_j)].$$

The following lemma is a homological version of Theorem 1.1, which is obtained by a base change of an O2-handle basis of a trivial surface-knot F of genus n in S^4 by [8, 11].

Lemma 3.2. For any pseudo-O2-sphere bases (R_*, R'_*) and (S_*, S'_*) of the stable 4-sphere Σ of genus n, there is an α-invariant orientation-preserving diffeomorphism $\tilde{f} : \Sigma \to \Sigma$ which induces an isomorphism

$$\tilde{f}_* : H_2(\Sigma;\mathbf{Z}) \to H_2(\Sigma;\mathbf{Z})$$

such that

$$[\tilde{f}R_i] = [S_i] \quad \text{and} \quad [\tilde{f}R'_i] = [S'_i]$$

for all i.

4 Outline of the proof of Theorem 1.1

Throughout this section, the proof of Theorem 1.1 is done. As well as the proof of Lemma 3.2, it suffices to show this theorem when (R_*, R'_*) is an O2-sphere basis of Σ with $(R_*, R'_*) = (S(D_*), S(D'_*))$ for an O2-handle basis $(D_* \times I, D'_* \times I)$ of a trivial surface-knot F of genus n in S^4.

Let $\Omega(S_*, S'_*)$ be the region of the pseudo-O2-sphere basis (S_*, S'_*) of Σ. The 4-manifold obtained from $\Omega(S_*, S'_*)$ by adding a 4-ball D^4 in place of the residual region $\Omega^c(S_*, S'_*)$ is diffeomorphic to Σ. This means that there is an orientation-preserving embedding

$$u : \Sigma^{(0)} \to \Sigma$$

such that

$$(uS(D_*), uS(D'_*)) = (S_*, S'_*).$$

By Lemma 3.2, after applying an α-invariant orientation-preserving diffeomorphism $\tilde{f} : \Sigma \to \Sigma$, we assume that the homology classes $[uS(D_i)] = [S_i]$ and $[uS(D'_i)] = [S'_i]$ are identical to the homology classes $[R_i] = [S(D_i)$ and $[R'_i] = [S(D'_i)]$ for all i, respectively. Let (B, V) be a bump pair of the O2-handle basis $(D_* \times I, D'_* \times I)$ of F in S^4 defined soon after Lemma 2.3. Recall that the two lifts of (B, V) to Σ under the double branched covering projection $p : S^4(F)_2 \to S^4$ are denoted by (B, V) and $(\bar{B}, \bar{V})$.

For the proof of Theorem 1.1, we provide with three lemmas. The first lemma is as follows.

Lemma 4.1. There is a diffeomorphism g of Σ which is isotopic to the identity such that the composite embedding

$$gu : \Sigma^{(0)} \to \Sigma$$

preserves the bump pair (B, V) in Σ identically and has the property that every disk interior in the disk systems $gu\bar{D}_*$ and $gu\bar{D}'_*$ meets every disk in the disk systems $\bar{D}_*$ and $\bar{D}'_*$ only with the intersection number 0.

By Lemma 4.1, we can assume that the orientation-preserving embedding

$$u : \Sigma^{(0)} \to \Sigma$$

sends the bump pair (B, V) to itself identically and has the property that every disk interior in the disk systems $u\bar{D}_*$ and $u\bar{D}'_*$ meets every disk in the disk systems $\bar{D}_*$ and $\bar{D}'_*$ only with intersection number 0. Then we have the following lemma:

Lemma 4.2. There is a diffeomorphism g of Σ which is isotopic to the identity such that the composite embedding

$$gu : \Sigma^{(0)} \to \Sigma$$

sends the disk systems D_* and D'_* identically and the disk interiors of the disk systems $gu\bar{D}_*$, $gu\bar{D}'_*$ to be disjoint from the disk systems $\bar{D}_*$ and $\bar{D}'_*$ in Σ.

For the O2-sphere basis $(S(D_*), S(D'_*))$ of Σ, let

$$q_* = \{q_i = S(D_i) \cap S(D'_i) \,|\, i = 1, 2, \ldots, n\}$$

be the transverse intersection point system between $S(D_*)$ and $S(D'_*)$.

The diffeomorphism g of Σ in Lemma 4.2 is deformed so that the disks guD_i and D_i are separated, and then the disks guD'_i and D'_i are separated while leaving the transverse intersection point q_i. By this deformation, we obtain a pseudo-O2-sphere basis $(guS(D_*), guS(D'_*))$ of Σ which meets the O2-sphere basis $(S(D_*), S(D'_*))$ at just the transverse intersection point system q_*.

Next, the diffeomorphism g of Σ is deformed so that a disk neighborhood system of q_* in $guS(D_*)$ and a disk neighborhood system of q_* in $S(D_*)$ are matched, and then a disk neighborhood system of q_* in $guS(D'_*)$ and a disk neighborhood system of q_* in $S(D'_*)$ are matched.

Thus, there is a diffeomorphism g of Σ which is isotopic to the identity such that the meeting part of the pseudo-O2-sphere basis $(guS(D_*), guS(D'_*))$ and the O2-sphere basis $(S(D_*), S(D'_*))$ is just a disk neighborhood pair system (d_*, d'_*) around the transverse intersection point system q_*.

Now, assume that for an embedding $u : \Sigma^{(0)} \to \Sigma$, the meeting part of the pseudo-O2-sphere basis $(uS(D_*), uS(D'_*))$ and the O2-sphere basis $(S(D_*), S(D'_*))$ is just a disk neighborhood pair system (d_*, d'_*) of q_*. Then we have the following lemma:

Lemma 4.3. There is an orientation-preserving diffeomorphism h of Σ such that the composite embedding

$$hu : \Sigma^{(0)} \to \Sigma$$

preserves the O2-sphere basis $(S(D_*), S(D'_*))$ identically.

Since $(uS(D_*), uS(D'_*)) = (S_*, S'_*)$ and $(R_*, R'_*) = (S(D_*), S(D'_*))$, an outline of the proof of Theorem 1.1 is completed by Lemma 4.3.

The proof of Lemma 4.3 is obtained from Lemma 4.4 (called Framed Light-bulb Diffeomorphism Lemma) which is easily proved in comparison with an isotopy version of this lemma using Gabai's 4D light-bulb theorem [5] stated in Section 6.

A *4D solid torus* is a 4-manifold Y in S^4 which is diffeomorphic to $S^1 \times D^3$. A *boundary fiber circle* of the 4D solid torus Y is a fiber circle of the S^1-bundle $\partial Y \cong S^1 \times S^2$. Let $Y^c = \mathrm{cl}(S^4 \setminus Y)$. Let Y_* be a system of mutually disjoint 4D solid tori Y_i, $(i = 1, 2, \ldots, n)$ in S^4, and Y_*^c the system of the 4-manifolds Y_i^c, $(i = 1, 2, \ldots, n)$. Let

$$\cap Y_*^c = \cap_{i=1}^n Y_i^c.$$

Then Lemma 4.4 is stated as follows.

Lemma 4.4 (Framed Light-bulb Diffeomorphism Lemma). Let $D_* \times I$ be a system of mutually disjoint framed disks $D_i \times I\,(i = 1, 2, \ldots, n)$ in $\cap Y_*^c$ such that ∂D_i is a boundary fiber circle of Y_i and

$$(D_* \times I) \cap \partial Y_i^c = (\partial D_i) \times I$$

for all i. If $E_* \times I$ is any system of mutually disjoint framed disks $E_i \times I\,(i = 1, 2, \ldots, n)$ in $\cap Y_*^c$ such that

$$(E_* \times I) \cap \partial Y_i^c = (\partial E_i) \times I = (\partial D_i) \times I$$

for all i, then there is an orientation-preserving diffeomorphism $h : S^4 \to S^4$ sending Y_* identically such that

$$h(D_* \times I, D_*) = (E_* \times I, E_*).$$

5 A classification of orientation-preserving diffeomorphisms of the stable 4-sphere

Let $\mathrm{Diff}^+(D^4, \mathrm{rel}\ \partial)$ be the orientation-preserving diffeomorphism group of the 4-ball D^4 keeping the boundary ∂D^4 by the identity. An *identity-shift* of a 4-manifold Σ is a diffeomorphism $\iota : \Sigma \to \Sigma$ obtained from the identity map $1 : \Sigma \to \Sigma$ by replacing the identity on a 4-ball in Σ disjoint from F with an element of $\mathrm{Diff}^+(D^4, \mathrm{rel}\ \partial)$.

The following result is obtained by using Framed Light-bulb Isotopy Lemma (explained soon) based on Gabai's 4D light-bulb theorem in [5] instead of Framed Light-bulb Diffeomorphism Lemma.

Theorem 5.1. Let Σ be the stable 4-sphere Σ of any genus $n \geq 1$, and $h : \Sigma \to \Sigma$ any orientation-preserving diffeomorphism. Then there is an α-equivariant diffeomorphism $\tilde{f} : \Sigma \to \Sigma$ such that the composite diffeomorphism $\tilde{f}h : \Sigma \to \Sigma$ induces the identity isomorphism

$$(\tilde{f}h)_* = 1 : H_2(\Sigma; \mathbf{Z}) \to H_2(\Sigma; \mathbf{Z}).$$

Further for any such α-equivariant diffeomorphism $\tilde{f} : \Sigma \to \Sigma$, the composite diffeomorphism $\tilde{f}h : \Sigma \to \Sigma$ is isotopic to an identity-shift ι of Σ.

To prove Theorem 5.1 in the process of the proof of Theorem 1.1, we need to show that the diffeomorphism h of Σ in Lemma 4.3 can be replaced by a diffeomorphism of Σ which is isotopic to the identity. For this purpose, we need the following lemma (Framed Light-bulb Isotopy Lemma), coming from Gabai's 4D light-bulb theorem in [5, Theorem 10.4]. Note that the assumption of Framed Light-bulb Isotopy Lemma adds an additional condition to the assumption of Framed Light-bulb Diffeomorphism Lemma (Lemma 4.4).

Lemma 5.2 (Framed Light-bulb Isotopy Lemma). Let Y_* be a system of mutually disjoint 4D solid tori $Y_i\,(i = 1, 2, \ldots, n)$ in S^4. Let $D_* \times I$ be a system of mutually disjoint framed disks $D_i \times I\,(i = 1, 2, \ldots, n)$ in $\cap Y_*^c$ such that ∂D_i is a boundary fiber circle of Y_i and

$$(D_* \times I) \cap \partial Y_i^c = (\partial D_i) \times I$$

for all i. If $E_* \times I$ is any system of mutually disjoint framed disks $E_i \times I\,(i = 1, 2, \ldots, n)$ in $\cap Y_*^c$ such that

$$(E_* \times I) \cap \partial Y_i^c = (\partial E_i) \times I = (\partial D_i) \times I$$

for all i and the unions $D_i \cup E_i\,(i = 1, 2, \ldots, n)$ are mutually disjoint, then there is a diffeomorphism $h : S^4 \to S^4$ which is Y_*-relatively isotopic to the identity such that

$$h(D_* \times I, D_*) = (E_* \times I, E_*).$$

The identity-shift ι in Theorem 5.1 is needed because at present it appears unknown whether $\pi_0(\mathrm{Diff}^+(D^4, \mathrm{rel}\,\partial)$ is trivial or not. However, it is known that the identity-shift ι is concordant to the identity since $\Gamma_5 = 0$ (see Kervaire [17]), so that *every orientation-preserving diffeomorphism $h : \Sigma \to \Sigma$ for the stable 4-sphere Σ of any genus n is smoothly concordant to an α-equivariant orientation-preserving diffeomorphism $\tilde{h} = \tilde{f}^{-1} : \Sigma \to \Sigma$.*

In the piecewise-linear category, the notion of an identity-shift is not needed since every piecewise-linear orientation-preserving homeomorphism of the 4-disk D^4 is piecewise-linearly isotopic to the identity. Thus, we have that *every piecewise-linear orientation-preserving homeomorphism $h' : \Sigma \to \Sigma$ for the stable 4-sphere Σ of any genus n is piecewise-linearly isotopic to an α-equivariant orientation-preserving piecewise-linear homeomorphism $\tilde{h}' : \Sigma \to \Sigma$.*

The following result is a consequence of Theorem 5.1.

Corollary 5.3. Every O2-sphere basis of Σ is isotopic to the O2-sphere basis $(S(D_*), S(D'_*))$ constructed from an O2-handle basis $(D_* \times I, D'_* \times I)$ of a trivial surface-knot F of genus n in S^4.

This result says that every O2-sphere basis of Σ up to isotopies comes from an O2-handle basis $(D_* \times I, D'_* \times I)$ of a trivial surface-knot F of genus n in S^4 which is unique up to orientation-preserving diffeomorphisms of S^4 by [11, 12].

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References

[1] A. Akhmedov and B. D. Park, Geography of simply connected spin symplectic 4-manifolds, Math. Res. Lett., 17 (2010), 483-492.

[2] J. Cerf, Sur les difféomorphismes de la sphere de dimension trois ($\Gamma_4 = 0$), Springer Lecture Notes in Math. 53 (1968).

[3] M. Freedman, The disk theorem for four-dimensional manifolds, Proc. Internat. Congr. Math. (Warsaw, Poland) (1983), 647-663.

[4] M. H. Freedman and F. Quinn, Topology of 4-manifolds, Princeton Univ. Press (1990).

[5] D. Gabai, The 4-dimensional light-bulb theorem, preprint, arXiv:1705.09989v2 (2017).

[6] A. Hatcher, A proof of the Smale conjecture, Diff(S^3) $\simeq$ O(4), Ann. of Math., 117 (1983), 553-607.

[7] J. A. Hillman and A. Kawauchi, Unknotting orientable surfaces in the 4-sphere, J. Knot Theory Ramifications 4(1995), 213-224.

[8] S. Hirose, On diffeomorphisms over surfaces trivially embedded in the 4-sphere, Algebraic and Geometric Topology 2(2002), 791-824.

[9] F. Hosokawa and A. Kawauchi, Proposals for unknotted surfaces in four-space, Osaka J. Math. 16 (1979), 233-248.

[10] J. F. P. Hudson, Piecewise-linear topology, Benjamin (1969).

[11] A. Kawauchi, Ribbonness of a stable-ribbon surface-link, I. A stably trivial surface-link, Topology and its Applications (to appear). arXiv:1804.02654

[12] A. Kawauchi, Supplement to ribbonness of a stable-ribbon surface-link, I: A revised proof of uniqueness of an orthogonal 2-handle pair on a surface-link. arXiv:1804.02654v11

[13] A. Kawauchi,Triviality of a surface-link with meridian-based free fundamental group. arXiv:1804.04269

[14] A. Kawauchi, Ribbonness of a stable-ribbon surface-link, II. General case. arXiv:1907.09713

[15] A. Kawauchi, Smooth homotopy 4-sphere. arXiv:1911.11904

[16] A. Kawauchi, Classical Poincaré conjecture via 4D topology. arXiv:2103.16001

[17] M. Kervaire, Groups of homotopy spheres: I, Ann. of Math. 77 (1963) 504-537.

[18] C. Rourke and B. Sanderson, Introduction to piecewise-linear topology, Ergebnesse der Math., 69 (1972), Springer-Verlag.

[19] C. T. C. Wall, On simply-connected 4-manifolds, J. London Math. Soc., 39 (1964), 141-149.

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