On invariants for handlebody-knots and spatial surfaces

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Part I
Handlebody-knots

In this part, we introduce the f-twisted Alexander invariant for handlebody-links. A *handlebody-link* [7] is a disjoint union of handlebodies embedded in the 3-sphere S^3. A *handlebody-knot* is a one component handlebody-link. In this paper, we assume that every component of a handlebody-link is of genus at least 1. Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. The f-twisted Alexander invariant is an invariant for handlebody-links derived from a linear extension of a multiple conjugation quandle, which is an algebra whose axioms are motivated from Reidemeister moves for handlebody-links. As an application, we demonstrate that our invariant detects 4-move equivalence classes of handlebody-links. This is a joint work with Atsushi Ishii.

1 Multiple conjugation quandles and MCQ Alexander pairs

A *quandle* [12, 14] is a non-empty set Q equipped with a binary operation $\triangleleft : Q \times Q \to Q$ satisfying the following axioms:

(Q1) For any $a \in Q$, $a \triangleleft a = a$.

(Q2) For any $a \in Q$, the map $\triangleleft a : Q \to Q$ defined by $\triangleleft a(x) = x \triangleleft a$ is bijective.

(Q3) For any $a, b, c \in Q$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

We denote $(\triangleleft a)^n : Q \to Q$ by $\triangleleft^n a$ for $n \in \mathbb{Z}$.

Let G be a group and n an integer. We define a binary operation $\triangleleft$ on G by $a \triangleleft b = b^{-1}ab$. Then, $\operatorname{Conj} G := (G, \triangleleft)$ is a quandle, called the *conjugation quandle* of G. We define another binary operation $\triangleleft$ on G by $a \triangleleft b = ba^{-1}b$. Then, $\operatorname{Core} G := (G, \triangleleft)$ is a quandle, called the *core quandle* of G. For a positive integer n, we denote by $\mathbb{Z}_n$ the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order n. We define a binary operation $\triangleleft$ on $\mathbb{Z}_n$ by $a \triangleleft b = 2b - a$. Then, $R_n := (\mathbb{Z}_n, \triangleleft)$ is a quandle, called the *dihedral quandle* of order n. Let M be an $R[t^{\pm 1}]$-module for a commutative ring R. We define a binary operation $\triangleleft$ on M by $a \triangleleft b = ta + (1 - t)b$. Then M is a quandle, called an *Alexander quandle*.

We define the *type* of a quandle Q by

$$\operatorname{type} Q = \min\{n \in \mathbb{Z}_{>0} \mid x \lhd^n y = x \text{ (for any } x, y \in Q)\},$$

where we set $\min \emptyset := \infty$ for the empty set $\emptyset$. We note that $(Q, \lhd^i)$ is also a quandle for any $i \in \mathbb{Z}$, and any finite quandle is of finite type.

Let $(Q, \lhd)$ be a quandle and R a ring. Throughout this paper, we assume that every ring has the multiplicative identity $1 \neq 0$. The pair of maps $f_1, f_2 : Q \times Q \to R$ is *Alexander pair* [10] if f_1 and f_2 satisfy the following conditions:

- For any $a \in Q$, $f_1(a, a) + f_2(a, a) = 1$.
- For any $a, b \in Q$, $f_1(a, b)$ is invertible.
- For any $a, b, c \in Q$,

$$\begin{aligned} f_1(a \lhd b, c) f_1(a, b) &= f_1(a \lhd c, b \lhd c) f_1(a, c), \\ f_1(a \lhd b, c) f_2(a, b) &= f_2(a \lhd c, b \lhd c) f_1(b, c), \\ f_2(a \lhd b, c) &= f_1(a \lhd c, b \lhd c) f_2(a, c) + f_2(a \lhd c, b \lhd c) f_2(b, c). \end{aligned}$$

An Alexander pair is a dynamical cocycle [1] corresponding to a linear extension of a quandle. Many examples of Alexander pairs are given in [10].

Definition 1.1 ([8])**.** A *multiple conjugation quandle (MCQ)* X is a disjoint union of groups $G_\lambda (\lambda \in \Lambda)$ with a binary operation $\lhd : X \times X \to X$ satisfying the following axioms:

- For any $a, b \in G_\lambda$, $a \lhd b = b^{-1}ab$.
- For any $x \in X$ and $a, b \in G_\lambda$, $x \lhd e_\lambda = x$ and $x \lhd (ab) = (x \lhd a) \lhd b$, where e_λ is the identity of G_λ.
- For any $x, y, z \in X$, $(x \lhd y) \lhd z = (x \lhd z) \lhd (y \lhd z)$.
- For any $x \in X$ and $a, b \in G_\lambda$, $(ab) \lhd x = (a \lhd x)(b \lhd x)$, where $a \lhd x, b \lhd x \in G_\mu$ for some $\mu \in \Lambda$.

We remark that an MCQ itself is a quandle. For two MCQs $X_1 = \bigsqcup_{\lambda \in \Lambda} G_\lambda$ and $X_2 = \bigsqcup_{\mu \in M} G_\mu$, an *MCQ homomorphism* $f : X_1 \to X_2$ is defined to be a map from X_1 to X_2 satisfying $f(x \lhd y) = f(x) \lhd f(y)$ for any $x, y \in X_1$ and $f(ab) = f(a)f(b)$ for any $\lambda \in \Lambda$ and $a, b \in G_\lambda$. An MCQ homomorphism $\rho : X_1 \to X_2$ is also called an *MCQ representation* of X_1 to X_2. We denote by $\operatorname{Hom}(X_1, X_2)$ the set of MCQ homomorphisms from X_1 to X_2. We call a bijective MCQ homomorphism an *MCQ isomorphism.* When there exists an MCQ isomorphism from X_1 to X_2, we call that X_1 and X_2 are *isomorphic*, denoted by $X_1 \cong X_2$. Let $\rho_1 : X_1 \to Y$ and $\rho_2 : X_2 \to Y$ be MCQ representations. We say (X_1, ρ_1) and (X_2, ρ_2) are *isomorphic*, denoted by $(X_1, \rho_1) \cong (X_2, \rho_2)$, if there exists an MCQ isomorphism $f : X_1 \to X_2$ such that $\rho_1 = \rho_2 \circ f$.

Let Q be a quandle. Then $Q \times \mathbb{Z}_{\operatorname{type} Q} = \bigsqcup_{x \in Q} (\{x\} \times \mathbb{Z}_{\operatorname{type} Q})$ is an MCQ, called the *associated MCQ* of Q, with

$$(x, a) \lhd (y, b) := (x \lhd^b y, a), \qquad (x, a)(x, b) := (x, a + b)$$

for any $x, y \in Q$ and $a, b \in \mathbb{Z}_{\operatorname{type} Q}$, where we put $\mathbb{Z}_\infty := \mathbb{Z}$.

Definition 1.2 ([15]). Let $X = \bigsqcup_{\lambda\in\Lambda} G_\lambda$ be an MCQ and R a ring. The pair (f_1, f_2) of maps $f_1, f_2 : X \times X \to R$ is an *MCQ Alexander pair* if f_1 and f_2 satisfy the following conditions:

- For any $a, b \in G_\lambda$,

$$f_1(a,b) + f_2(a,b) = f_1(a, a^{-1}b). \tag{1-i}$$

- For any $a, b \in G_\lambda$ and $x \in X$,

$$f_1(a,x) = f_1(b,x), \tag{2-i}$$
$$f_2(ab,x) = f_2(a,x) + f_1(b \triangleleft x, a^{-1} \triangleleft x) f_2(b,x). \tag{2-ii}$$

- For any $x \in X$ and $a, b \in G_\lambda$,

$$f_1(x, e_\lambda) = 1, \tag{3-i}$$
$$f_1(x, ab) = f_1(x \triangleleft a, b) f_1(x, a), \tag{3-ii}$$
$$f_2(x, ab) = f_1(x \triangleleft a, b) f_2(x, a). \tag{3-iii}$$

- For any $x, y, z \in X$,

$$f_1(x \triangleleft y, z) f_1(x, y) = f_1(x \triangleleft z, y \triangleleft z) f_1(x, z), \tag{4-i}$$
$$f_1(x \triangleleft y, z) f_2(x, y) = f_2(x \triangleleft z, y \triangleleft z) f_1(y, z), \tag{4-ii}$$
$$f_2(x \triangleleft y, z) = f_1(x \triangleleft z, y \triangleleft z) f_2(x, z) + f_2(x \triangleleft z, y \triangleleft z) f_2(y, z). \tag{4-iii}$$

By using the following proposition, we can construct MCQ Alexander pairs from Alexander pairs.

Proposition 1.3. *Let $(Q, \triangleleft)$ be a quandle and assume $k := \operatorname{type} Q < \infty$. Let R be a ring and let (f_1, f_2) be an Alexander pair of maps $f_1, f_2 : Q \times Q \to R$ satisfying*

$$\prod_{i=1}^{k} f_1(x \triangleleft^{k-i} y, y) = 1 \qquad \text{and} \qquad \sum_{i=1}^{k} f_1(x, x)^i = 0$$

for any $x, y \in Q$. Let $X := Q \times \mathbb{Z}_k$ be the associated MCQ of Q. We define maps $\widetilde{f_1}, \widetilde{f_2} : X \times X \to R$ by

$$\widetilde{f_1}((x,a),(y,b)) = \prod_{i=1}^{\bar{b}} f_1(x \triangleleft^{b-i} y, y),$$
$$\widetilde{f_2}((x,a),(y,b)) = \left(\prod_{i=1}^{\overline{b-1}} f_1(x \triangleleft^{b-i} y, y) \right) \sum_{j=1}^{\bar{a}} f_1(x \triangleleft y, x \triangleleft y)^{j-a} f_2(x, y),$$

where for any $l \in \mathbb{Z}$, we denote by $\bar{l}$ the integer such that $1 \leq \bar{l} \leq k$ and $l \equiv \bar{l} \mod k$. Then the pair $(\widetilde{f_1}, \widetilde{f_2})$ is an MCQ Alexander pair.

2 MCQ presentations and the fundamental MCQ of a handlebody-link

In this section, we review the notions of MCQ presentations and the fundamental MCQ of a handlebody-link. For details see [9].

We denote by $F_{\mathsf{Grp}}(S)$ the free group on a set S. Let $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ be a set of pairwise disjoint sets, and put $S := \bigcup_{\lambda\in\Lambda} S_\lambda$. For $(a,x),(b,y) \in \bigcup_{\lambda\in\Lambda} F_{\mathsf{Grp}}(S_\lambda) \times F_{\mathsf{Grp}}(S)$, we write $(a,x) \sim (b,y)$ if there exists $c \in \bigcup_{\lambda\in\Lambda} F_{\mathsf{Grp}}(S_\lambda)$ such that $b = cac^{-1}$ and $y = cx$. Then $\sim$ is an equivalence relation on $\bigcup_{\lambda\in\Lambda} F_{\mathsf{Grp}}(S_\lambda) \times F_{\mathsf{Grp}}(S)$. We define $F_{\mathsf{MCQ}}(S_\Lambda) := \bigcup_{\lambda\in\Lambda} F_{\mathsf{Grp}}(S_\lambda) \times F_{\mathsf{Grp}}(S)/\sim$ and

$$F_{\mathsf{Grp}}(S;S_\lambda) := \left\{ a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n} \in F_{\mathsf{Grp}}(S) \;\middle|\; \begin{array}{l} n \geq 0,\ \varepsilon_1,\ldots,\varepsilon_n \in \{1,-1\}, \\ a_1 \in S - S_\lambda,\ a_2,\ldots,a_n \in S, \\ \text{If } a_i = a_{i+1}, \text{ then } \varepsilon_i = \varepsilon_{i+1} \text{ for each } i. \end{array} \right\}.$$

Set $\overline{\Lambda} := \bigcup_{\lambda\in\Lambda}(\{\lambda\} \times F_{\mathsf{Grp}}(S;S_\lambda))$, and we define

$$F_{\mathsf{Grp}}(S_\lambda) \triangleleft x := \{[(a,x)] \in F_{\mathsf{MCQ}}(S_\Lambda) \mid a \in F_{\mathsf{Grp}}(S_\lambda)\}$$

for $x \in F_{\mathsf{Grp}}(S)$. Then we have $F_{\mathsf{MCQ}}(S_\Lambda) = \bigsqcup_{(\lambda,x)\in\overline{\Lambda}} F_{\mathsf{Grp}}(S_\lambda) \triangleleft x$, which is an MCQ with

$$\begin{aligned} &[(a,x)] \triangleleft [(b,y)] := [(a, xy^{-1}by)] && ([(a,x)],[(b,y)] \in F_{\mathsf{MCQ}}(S_\Lambda)), \\ &[(a,x)][(b,x)] := [(ab,x)] && (a,b \in F_{\mathsf{Grp}}(S_\lambda), x \in F_{\mathsf{Grp}}(S)). \end{aligned}$$

We call $F_{\mathsf{MCQ}}(S_\Lambda)$ the *free multiple conjugation quandle (free MCQ)* on S_Λ. By the injection $\iota : S \to F_{\mathsf{MCQ}}(S_\Lambda)$ defined by $\iota(a) = [(a, 1_{F_{\mathsf{Grp}}(S)})]$, we regard $S \subset F_{\mathsf{MCQ}}(S_\Lambda)$ and often denote $[(a, 1_{F_{\mathsf{Grp}}(S)})]$ by a. Then any element $[(a, a_1^{\varepsilon_1}\cdots a_n^{\varepsilon_n})]$ in $F_{\mathsf{MCQ}}(S_\Lambda)$ can be represented as $(\cdots(a \triangleleft a_1^{\varepsilon_1}) \triangleleft \cdots) \triangleleft a_n^{\varepsilon_n}$.

For any MCQ X, there exist a set of pairwise disjoint sets $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$ and $R \subset F_{\mathsf{MCQ}}(S_\Lambda) \times F_{\mathsf{MCQ}}(S_\Lambda)$ such that X is isomorphic to $\langle S_\Lambda \mid R\rangle$, which is also denoted $\langle S_\lambda\ (\lambda \in \Lambda) \mid R\rangle$ [9]. We then call $\langle S_\Lambda \mid R\rangle$ a *presentation* of X, S_Λ the *generating set* of $\langle S_\Lambda \mid R\rangle$ and an element of R a *relator* of $\langle S_\Lambda \mid R\rangle$. A relator (a,b) is also written as $a = b$. A presentation $\langle S_\Lambda \mid R\rangle$ is called a *finite presentation* if $\bigcup_{\lambda\in\Lambda} S_\lambda$ and R are finite. For a finitely presented MCQ, we often write

$$\begin{aligned} &\langle x_{1,1},\ldots,x_{1,n_1};\ldots;x_{l,1},\ldots,x_{l,n_l} \mid r_1,\ldots,r_m\rangle \\ &:= \langle \{x_{1,1},\ldots,x_{1,n_1}\},\ldots,\{x_{l,1},\ldots,x_{l,n_l}\} \mid \{r_1,\ldots,r_m\}\rangle. \end{aligned}$$

In the following, we recall the fundamental MCQ of a handlebody-link and its Wirtinger presentation. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph whose regular neighborhood is the handlebody-link, where a spatial trivalent graph is a finite trivalent graph embedded in S^3. In this paper, a trivalent graph may contain circle components. Let D be a diagram of a handlebody-link. A *Y-orientation* of D is a collection of orientations of all edges of D without sources and sinks with respect to the orientation as shown in Figure 1, where an edge of D is a piece of a curve each of whose endpoints is a vertex. In this paper, a circle component of D is also regarded as an

Figure 1: Y-orientations.

edge of D. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation counterclockwise by $\pi/2$ on a diagram. It is known that every diagram has a Y-orientation.

Let H be a handlebody-link represented by a Y-oriented diagram D. We denote by $C(D)$, $V(D)$ and $\mathcal{A}(D)$ the sets of crossings, vertices and arcs of D, respectively. We denote by $u_c, v_c, w_c, u_\tau, v_\tau, w_\tau$ the arcs around $c \in C(D)$ and $\tau \in V(D)$ as illustrated in Figure 2, respectively. We denote by $\mathcal{A}^{\sqcup}(D)$ the quotient set of $\mathcal{A}(D)$ by the equivalence relation generated by $\bigcup_{\tau \in V(D)} \{u_\tau, v_\tau, w_\tau\}^2$. For each $c \in C(D)$ and $\tau \in V(D)$, we denote by r_c and r_τ the relators $(u_c \triangleleft v_c, w_c)$ and $(u_\tau v_\tau, w_\tau)$, respectively. Then we define

$$MCQ(D) := \langle \mathcal{A}^{\sqcup}(D) \,|\, r_c, r_\tau \ (c \in C(D), \tau \in V(D)) \rangle \,.$$

The isomorphism class of $MCQ(D)$ does not depend on the choice of a diagram D of H and its Y-orientation [9]. We then define $MCQ(H) := MCQ(D)$ and call it the *fundamental MCQ* of H. This presentation is called the *Wirtinger presentation of $MCQ(H)$ with respect to D.*

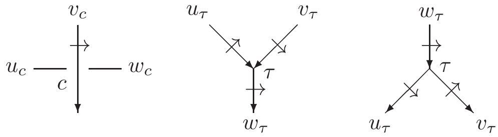

Figure 2: Notations of arcs.

Let D be a Y-oriented diagram of a handlebody-link H and let X be an MCQ. An *X-coloring* of D is a map $C : \mathcal{A}(D) \to X$ satisfying the conditions

$$C(u_c) \triangleleft C(v_c) = C(w_c) \qquad \text{and} \qquad C(u_\tau)C(v_\tau) = C(w_\tau)$$

for each $c \in C(D)$ and $\tau \in V(D)$. We denote by $\mathrm{Col}_X(D)$ the set of X-colorings of D. An X-coloring of D can be regarded as an MCQ representation of $MCQ(H)$ to X, that is, we can identify $\mathrm{Col}_X(D)$ with $\mathrm{Hom}(MCQ(H), X)$.

Let D be a Y-oriented diagram of a handlebody-link H and D' a Y-oriented diagram of H obtained by changing the Y-orientation of D. We then obtain the MCQ isomorphism $f_{(D,D')} : MCQ(D) \to MCQ(D')$ sending x into $x^{\varepsilon(x)}$ for any $x \in \mathcal{A}(D)$, where $\varepsilon(x) = 1$ if the Y-orientations of D and D' coincide on x; otherwise $\varepsilon(x) = -1$. Moreover, let D'' a Y-oriented diagram of H obtained by applying one of Reidemeister moves preserving the Y-orientation to D once. We then obtain a unique MCQ isomorphism $f_{(D,D'')} : MCQ(D) \to MCQ(D'')$ sending x into x for any $x \in \mathcal{A}(D \cap D'')$, where $\mathcal{A}(D \cap D'')$ denotes the set of arcs in the outside of the disk where the move is applied. Let H and H' be handlebody-links represented by Y-oriented diagrams D and D', respectively. Let $\rho : MCQ(D) \to X$

and $\rho' : MCQ(D') \to X$ be MCQ representations. Then (H, ρ) and (H', ρ') are *equivalent*, denoted by $(H, \rho) \cong (H', \rho')$, if there exists a sequence $D = D_1 \leftrightarrow \cdots \leftrightarrow D_n = D'$ of Reidemeister moves and Y-orientation changes such that $\rho' = \rho \circ f^{-1}_{(D_1,D_2)} \circ \cdots \circ f^{-1}_{(D_{n-1},D_n)}$. If H and H' are equivalent, then for any MCQ representation $\rho : MCQ(D) \to X$, there exists a unique MCQ representation $\rho' : MCQ(D') \to X$ such that $(H, \rho) \cong (H', \rho')$.

3 f-twisted Alexander invariants for handlebody-links

Let $S_\Lambda = \{S_\lambda \,|\, \lambda \in \Lambda\}$ be a finite set of pairwise disjoint finite sets, and put $S := \bigcup_{\lambda\in\Lambda} S_\lambda = \{x_1, \dots, x_n\}$. Let $X = \langle S_\Lambda \,|\, \{r_1, \dots, r_m\}\rangle$ be a finitely presented MCQ. Let $F_{\mathsf{MCQ}}(S_\Lambda)$ be the free MCQ on S_Λ and $\mathrm{pr} : F_{\mathsf{MCQ}}(S_\Lambda) \to X$ be the canonical projection. We often omit "pr" to represent $\mathrm{pr}(x)$ as x. Let $f = (f_1, f_2)$ be an MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$, and put $\overline{\Lambda} := \bigcup_{\lambda\in\Lambda}(\{\lambda\} \times F_{\mathsf{Grp}}(S; S_\lambda))$. For any $(\lambda, x) \in \overline{\Lambda}$, we set $G_{\lambda,x} := F_{\mathsf{Grp}}(S_\lambda) \triangleleft x$, that is, $F_{\mathsf{MCQ}}(S_\Lambda) = \bigsqcup_{(\lambda,x)\in\overline{\Lambda}} G_{\lambda,x}$.

Definition 3.1. For $j \in \{1, \dots, n\}$, the *f-derivative with respect to x_j* is a map $\frac{\partial_f}{\partial x_j} : F_{\mathsf{MCQ}}(S_\Lambda) \to R$ satisfying

$$\begin{aligned}
&\frac{\partial_f}{\partial x_j}(x \triangleleft y) = f_1(x, y)\frac{\partial_f}{\partial x_j}(x) + f_2(x, y)\frac{\partial_f}{\partial x_j}(y),\\
&\frac{\partial_f}{\partial x_j}(ab) = \frac{\partial_f}{\partial x_j}(a) + f_1(a, a^{-1})\frac{\partial_f}{\partial x_j}(b),\\
&\frac{\partial_f}{\partial x_j}(x_i) = \delta_{ij}
\end{aligned}$$

for any $x, y \in F_{\mathsf{MCQ}}(S_\Lambda)$, $a, b \in G_{\lambda,x}$ and $i \in \{1, \dots, n\}$, where δ_{ij} denotes the Kronecker delta.

Let R be a ring. We denote by $M(m, n; R)$ the set of $m \times n$ matrices over R. We say that two matrices A_1 and A_2 over R are *equivalent*, denoted by $A_1 \sim A_2$, if they are related by a finite sequence of the following transformations:

- $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_i, \dots, \boldsymbol{a}_j, \dots, \boldsymbol{a}_n) \leftrightarrow (\boldsymbol{a}_1, \dots, \boldsymbol{a}_i + \boldsymbol{a}_j r, \dots, \boldsymbol{a}_j, \dots, \boldsymbol{a}_n)$ $(r \in R)$,
- $\begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} \leftrightarrow \begin{pmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_i + r\boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_j \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix}$ $(r \in R)$,
- $A \leftrightarrow \begin{pmatrix} A \\ \boldsymbol{0} \end{pmatrix}$,
- $A \leftrightarrow \begin{pmatrix} A & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}$.

Let R be a commutative ring, and let $A \in M(m, n; R)$. A *k-minor* of A is the determinant of a $k \times k$ submatrix of A. For any $d \in \mathbb{Z}_{\geq 0}$, the *$d$-th elementary ideal* $E_d(A)$ of

A is the ideal of R generated by all $(n-d)$-minors of A if $n-m \leq d < n$, and

$$E_d(A) := \begin{cases} 0 & \text{if } d < n-m, \\ R & \text{if } n \leq d. \end{cases}$$

Suppose that R is a GCD domain. Then the *d-th Alexander invariant* $\Delta_d(A)$ of A is the greatest common divisor of all $(n-d)$-minors of A if $n-m \leq d < n$, and

$$\Delta_d(A) := \begin{cases} 0 & \text{if } d < n-m, \\ 1 & \text{if } n \leq d. \end{cases}$$

We remark that $\Delta_d(A)$ coincides with the greatest common divisor of generators of $E_d(A)$ and is determined up to unit multiple. If $A \sim B$, then $E_d(A) = E_d(B)$ and $\Delta_d(A) \doteq \Delta_d(B)$, where "$\doteq$" means "is equal to, up to multiplication by a unit". See [6] for more details.

For an MCQ representation $\rho : X \to Y$ and an MCQ Alexander pair $f = (f_1, f_2)$ of maps $f_1, f_2 : Y \times Y \to R$, we set $f \circ (\rho \times \rho) := (f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho))$. Then $f \circ (\rho \times \rho)$ is also an MCQ Alexander pair. For a relator $r = (r_1, r_2)$, we define

$$\frac{\partial_f}{\partial x_j}(r) := \frac{\partial_f}{\partial x_j}(r_1) - \frac{\partial_f}{\partial x_j}(r_2).$$

Definition 3.2. Let H be a handlebody-link and

$$MCQ(D) = \langle x_1, \ldots, x_k; \ldots; x_l, \ldots, x_n \,|\, r_1, \ldots, r_m \rangle$$

the Wirtinger presentation of $MCQ(H)$ with respect to a Y-oriented diagram D of H. Let $\rho : MCQ(H) = MCQ(D) \to X$ be an MCQ representation, which can be regarded as an X-coloring of D. Let $f = (f_1, f_2)$ be an MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$. Then we define the *$f$-twisted Alexander matrix of (H, ρ) (with respect to D)* by

$$A(H, \rho; f_1, f_2) = \begin{pmatrix} \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_1}(r_1) & \cdots & \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_1}(r_m) & \cdots & \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_n}(r_m) \end{pmatrix},$$

the *d-th f-twisted elementary ideal of (H, ρ)* by

$$E_d(H, \rho; f_1, f_2) = E_d(A(H, \rho; f_1, f_2))$$

if R is a commutative ring, and the *d-th f-twisted Alexander invariant of (H, ρ)* by

$$\Delta_d(H, \rho; f_1, f_2) = \Delta_d(A(H, \rho; f_1, f_2))$$

if R is a GCD domain.

Theorem 3.3. *Let H and H' be handlebody-links. Let $\rho : MCQ(H) \to X$ and $\rho' : MCQ(H') \to X$ be MCQ representations. Let (f_1, f_2) be an MCQ Alexander pair of maps $f_1, f_2 : X \times X \to R$. If $(H, \rho) \cong (H', \rho')$, then we have*

$$A(H, \rho; f_1, f_2) \sim A(H', \rho'; f_1, f_2).$$

Especially, we have

$$E_d(H,\rho;f_1,f_2) = E_d(H',\rho';f_1,f_2)$$

if R is a commutative ring, and we have

$$\Delta_d(H,\rho;f_1,f_2) \doteq \Delta_d(H',\rho';f_1,f_2)$$

if R is a GCD domain.

4 Examples and applications

In this section, we give a calculation example of the f-twisted Alexander invariant of the genus g trivial handlebody-knot for any MCQ Alexander pair f, where a handlebody-knot is *trivial* if its exterior is a handlebody. Furthermore, we introduce k-moves for handlebody-links and show that an f-twisted Alexander invariant detects 4-move equivalence classes of handlebody-links for some MCQ Alexander pair f.

Proposition 4.1. *Let O_g be the trivial handlebody-knot of genus g. For any MCQ representation $\rho : MCQ(O_g) \to X$ and MCQ Alexander pair (f_1, f_2) of maps $f_1, f_2 : X \times X \to R$, we have*

$$A(O_g,\rho;f_1,f_2) \sim \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} \in M(1,g;R).$$

Especially, we have

$$E_d(O_g,\rho;f_1,f_2) = \begin{cases} 0 & \text{if } d < g, \\ R & \text{if } g \leq d \end{cases}$$

if R is a commutative ring, and we have

$$\Delta_d(O_g,\rho;f_1,f_2) = \begin{cases} 0 & \text{if } d < g, \\ 1 & \text{if } g \leq d \end{cases}$$

if R is a GCD domain.

A *k-move* is a local move on handlebody-links as illustrated in Figure 3. Two handlebody-links are *k-move equivalent* if they are related by a finite sequence of k-moves and isotopies of S^3. In this section, we focus on 4-moves for handlebody-links. Behavior of 4-moves for classical links has been studied in, for example, [2, 3, 4, 11, 16, 17, etc.].

Proposition 4.2. *Let R_4 be the dihedral quandle and $X := R_4 \times \mathbb{Z}_2$ the associated MCQ of R_4, where we regard R_4 as the core quandle* $\mathrm{Core}\langle t \mid t^4 \rangle$. *Let $f = (f_1, f_2)$ be the MCQ Alexander pair of maps $f_1, f_2 : X \times X \to \mathbb{Z}_4[t^{\pm 1}]/(t^2+1)$ defined by*

$$f_1((x,a),(y,b)) = \begin{cases} 1 & \text{if } b = 0, \\ -yx^{-1} & \text{otherwise}, \end{cases}$$

$$f_2((x,a),(y,b)) = \begin{cases} 0 & \text{if } a = 0, \\ -1 - xy^{-1} & \text{if } a = 1 \text{ and } b = 0, \\ 1 + yx^{-1} & \text{if } a = 1 \text{ and } b = 1. \end{cases}$$

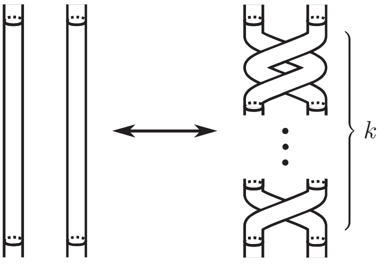

Figure 3: A k-move for a handlebody-link.

Then for any handlebody-link H, the multiset

$$\{\Delta_d(H,\rho;f_1,f_2)\,|\,\rho\in \mathrm{Hom}(MCQ(H),X)\}$$

is an invariant under 4-moves for H for each $d\in\mathbb{Z}_{\geq 0}$.

Example 4.3. Let H be the three component handlebody-link represented by the Y-oriented diagram D depicted in Figure 4. Let X and $f=(f_1,f_2)$ be the MCQ and MCQ Alexander pair that are the same as Proposition 4.2, respectively. Let $\rho: MCQ(H)\to X$ be the MCQ representation depicted in Figure 4. The Wirtinger presentation of $MCQ(H)$ with respect to D is given by

$$\left\langle \begin{array}{c} x_1,x_2,x_3;x_4; \\ x_5,x_6,x_7;x_8;x_9 \end{array} \,\middle|\, \begin{array}{c} x_6\triangleleft x_1=x_7,\ x_1\triangleleft x_7=x_2,\ x_8\triangleleft x_3=x_8,\ x_4\triangleleft x_8=x_3, \\ x_9\triangleleft x_4=x_9,\ x_5\triangleleft x_9=x_4,\ x_3x_1=x_2,\ x_7x_5=x_6 \end{array} \right\rangle.$$

Hence we have

$$\begin{aligned} A(H,\rho;f_1,f_2) &= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1-t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1-t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -t^{-1} & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \end{pmatrix} \\ &\sim \begin{pmatrix} 2+2t & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and $\Delta_3(H,\rho;f_1,f_2)=2+2t$. On the other hand, let H_0 be the three component trivial handlebody-link consisting of one genus 2 component and two genus 1 components. Then for any MCQ representation $\rho_0: MCQ(H_0)\to X$, we have

$$A(H_0,\rho_0;f_1,f_2)\sim\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\Delta_3(H_0,\rho_0;f_1,f_2)=0$. Consequently, H is not 4-move equivalent to the trivial handlebody-link by Proposition 4.2.

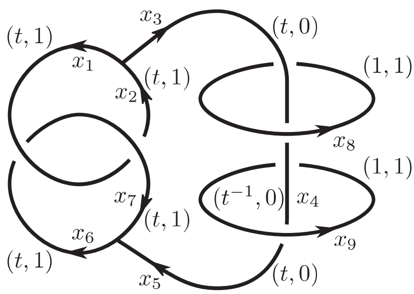

Figure 4: A Y-oriented diagram D of the three component handlebody-link H.

Part II

Spatial surfaces

A *spatial surface* [13] is a compact surface embedded in the 3-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Two (oriented) spatial surfaces F and F' are *equivalent*, denoted by $F \cong F'$, if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other (with its orientation). In this part, unless otherwise stated, we suppose the following conditions:

- Each component of a spatial surface has non-empty boundary.
- A spatial surface has no disk components.

In this part, we introduce a coloring invariant for oriented spatial surfaces by using a multiple group rack, which is an algebra whose axioms are motivated from Reidemeister moves for oriented spatial surfaces. Further, we distinguish some oriented spatial surfaces by using the coloring invariants. This is a joint work with Atsushi Ishii and Shosaku Matsuzaki.

5 Oriented spatial surfaces

In figures of this part, the front side and the back side of an oriented spatial surface are colored by light gray and dark gray as illustrated in Figure 5, respectively.

Figure 5: The front side and the back side.

Let D be a diagram of a spatial trivalent graph. We obtain a spatial surface F from D by taking a regular neighborhood of D in $\mathbb{R}^2$ and perturbing it around all crossings of D, according to its over/under information. Then we give F an orientation so that the front side of F faces into the positive direction of the z-axis of $\mathbb{R}^3$ as illustrated in Figure 6. We then call D a *diagram* of the oriented spatial surface F. We remark that any oriented spatial surface is equivalent to an oriented spatial surface obtained by this process [13].

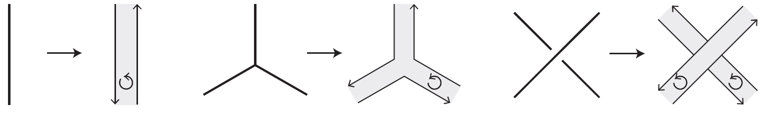

Figure 6: The process for obtaining an oriented spatial surface.

We introduce elementary methods to distinguish oriented spatial surfaces and show some examples. Let F be an oriented spatial surface. The regular neighborhood $N(F)$ of F in S^3 is a handlebody-link. We denote by ∂F the boundary of F, where we assume that ∂F is oriented so that the orientation is coherent with that of F.

Remark 5.1. If oriented spatial surfaces F_1 and F_2 are equivalent, then we have the following:

1. Two oriented links ∂F_1 and ∂F_2 are equivalent.
2. Two handlebody-links $N(F_1)$ and $N(F_2)$ are equivalent.

Example 5.2. For $i, j, k \in \mathbb{Z}$, let $F(i, j, k)$ be the oriented spatial surface as illustrated in Figure 7, where the integers i, j and k indicate i, j and k full-twists, respectively. We note that a negative integer indicates the reverse twists. Then, the following conditions are equivalent:

1. The multiset $\{i, j, k\}$ coincides with the multiset $\{i', j', k'\}$.
2. $F(i, j, k)$ and $F(i', j', k')$ are equivalent.
3. $\partial F(i, j, k)$ and $\partial F(i', j', k')$ are equivalent as oriented links.

Example 5.3. Let F_1 and F_2 be the spatial surfaces illustrated in Figure 8. Both ∂F_1 and ∂F_2 are trivial knots. It is easy to see that $N(F_1)$ is the genus 2 trivial handlebody-knot, but $N(F_2)$ is not. Hence F_1 and F_2 are not equivalent.

Remark 5.4. Let $\widetilde{F_1}$ and $\widetilde{F_2}$ be the spatial closed surfaces illustrated in Figure 9, from which F_1 and F_2 in Figure 8 are obtained by removing a disk, respectively. Then $\widetilde{F_1}$ and $\widetilde{F_2}$ are not equivalent, since $\widetilde{F_1}$ splits S^3 into two solid tori, but $\widetilde{F_2}$ does not. This also implies that F_1 and F_2 are not equivalent by [13].

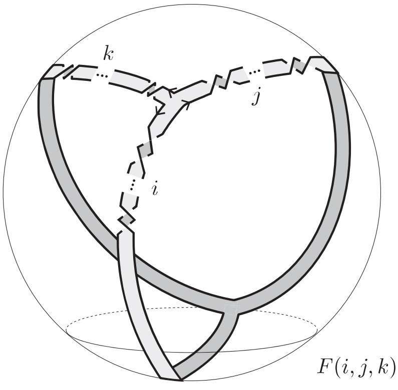

Figure 7: An oriented spatial surface $F(i, j, k)$ whose boundary has three components.

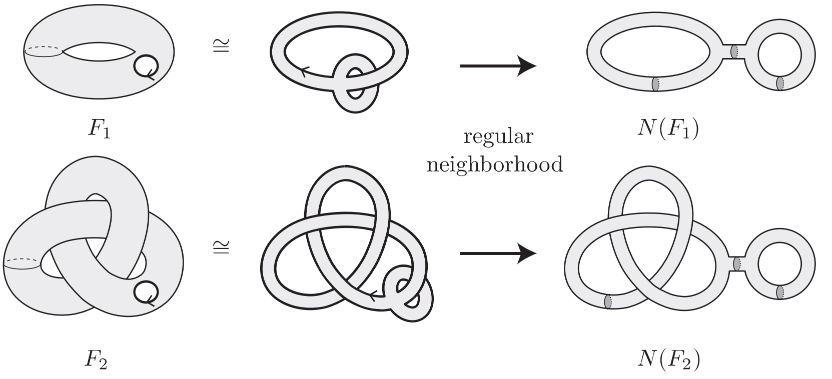

Figure 8: Inequivalent spatial surfaces.

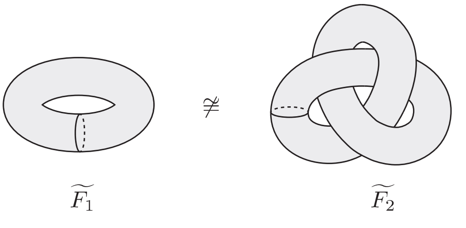

Figure 9: Inequivalent spatial closed surfaces.

The oriented spatial surfaces F_1 and F_2 depicted in Figure 11 can not be distinguished by using these elementary methods. In Section 7, we demonstrate that they are not equivalent by using coloring invariants.

6 Multiple group racks and a coloring invariant for oriented spatial surfaces

A *rack* [5] is a non-empty set X with a binary operation $\triangleleft : X \times X \to X$ satisfying the following axioms:

- For any $a \in X$, the map $\triangleleft a : X \to X$ defined by $\triangleleft a(x) = x \triangleleft a$ is bijective.
- For any $a, b, c \in X$, $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$.

We denote $(\triangleleft a)^n : X \to X$ by $\triangleleft^n a$ for $n \in \mathbb{Z}$.

We remark that a quandle is a rack. For a positive integer n, we define a binary operation $\triangleleft$ on $\mathbb{Z}_n$ by $a \triangleleft b = a + 1$. Then, $C_n := (\mathbb{Z}_n, \triangleleft)$ is a rack, called the *cyclic rack* of order n. Let R be a ring and M a left $R[t^{\pm 1}, s]/(s(t+s-1))$-module. We define a binary operation $\triangleleft$ on M by $x \triangleleft y = tx + sy$. Then, $(M, \triangleleft)$ is a rack, called the (t, s)-*rack*.

Proposition 6.1. *Let X be a rack. Fix $e_1, \ldots, e_m \in \mathbb{Z}$ and $i_1, \ldots, i_m \in \{1, \ldots, n\}$. Then X^n is a rack with the binary operation defined by*

$$\begin{aligned}&(x_1, \ldots, x_n) \triangleleft (y_1, \ldots, y_n)\\ &= ((\cdots (x_1 \triangleleft^{e_1} y_{i_1}) \triangleleft^{e_2} \cdots) \triangleleft^{e_m} y_{i_m}, \ldots, (\cdots (x_n \triangleleft^{e_1} y_{i_1}) \triangleleft^{e_2} \cdots) \triangleleft^{e_m} y_{i_m}).\end{aligned}$$

We define the *type* of a rack X by

$$\operatorname{type} X = \min\{n \in \mathbb{Z}_{>0} \mid x \triangleleft^n y = x \text{ (for any } x, y \in X)\},$$

where we set $\min \emptyset := \infty$ for the empty set $\emptyset$. We note that $(X, \triangleleft^i)$ is also a rack for any $i \in \mathbb{Z}$, and any finite rack is of finite type.

Definition 6.2. A *multiple group rack (MGR) X* is a disjoint union of groups G_λ $(\lambda \in \Lambda)$ with a binary operation $\triangleleft : X \times X \to X$ satisfying the following axioms:

- For any $x \in X$ and $a, b \in G_\lambda$, $x \triangleleft e_\lambda = x$ and $x \triangleleft (ab) = (x \triangleleft a) \triangleleft b$, where e_λ is the identity of G_λ.
- For any $x, y, z \in X$, $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$.
- For any $x \in X$ and $a, b \in G_\lambda$, $(ab) \triangleleft x = (a \triangleleft x)(b \triangleleft x)$, where $a \triangleleft x, b \triangleleft x \in G_\mu$ for some $\mu \in \Lambda$.

We remark that an MGR is a rack, and an MCQ is an MGR. Let X be a rack. Then $X \times \mathbb{Z}_{\operatorname{type} X} = \bigsqcup_{x \in X}(\{x\} \times \mathbb{Z}_{\operatorname{type} X})$ is an MGR, called the *associated MGR* of X, with

$$(x, a) \triangleleft (y, b) := (x \triangleleft^b y, a), \qquad (x, a)(x, b) := (x, a + b)$$

for any $x, y \in X$ and $a, b \in \mathbb{Z}_{\operatorname{type} X}$, where we put $\mathbb{Z}_\infty := \mathbb{Z}$.

Let D be a Y-oriented diagram of an oriented spatial surface and let X be an MGR. We denote by $\mathcal{A}(D)$ the set of arcs of D. We denote by $u_c, v_c, w_c, u_\tau, v_\tau, w_\tau$ the arcs around each crossing c and vertex τ as illustrated in Figure 2, respectively. An X-*coloring* of D is a map $C : \mathcal{A}(D) \to X$ satisfying the conditions

$$C(u_c) \triangleleft C(v_c) = C(w_c) \qquad \text{and} \qquad C(u_\tau)C(v_\tau) = C(w_\tau)$$

for each crossing c and vertex τ. We denote by $\mathrm{Col}_X(D)$ the set of X-colorings of D. Then we have the following theorem.

Theorem 6.3. *Let X be an MGR. Let D and D' be Y-oriented diagrams of an oriented spatial surface F. Then there is a one-to-one correspondence between* $\mathrm{Col}_X(D)$ *and* $\mathrm{Col}_X(D')$. *In particular, the cardinality* $\#\mathrm{Col}_X(D)$ *is an invariant of F.*

7 Coloring examples

In this section, we give examples to distinguish oriented spatial surfaces by using MGR coloring invariants.

Let F be an oriented spatial surface. The *reverse* of F, denoted by $-F$, is the oriented spatial surface obtained by reversing the orientation of F. An oriented spatial surface F is *reversible* if F and its reverse are equivalent. Let h be an orientation-reversing self-homeomorphism of S^3. The oriented spatial surface $h(F)$ is called the *mirror image* of F and denoted by F^* (see Figure 10).

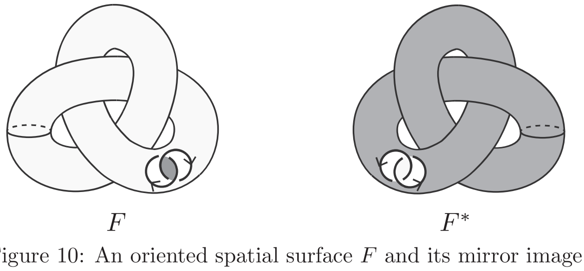

Figure 10: An oriented spatial surface F and its mirror image F^*.

Example 7.1. Let F_1 and F_2 be the oriented spatial surfaces illustrated in Figure 11, where F_2 is the reverse of F_1. Let D_1 and D_2 be the Y-oriented diagrams of F_1 and F_2 illustrated in the figure, respectively. Let R_3 be the dihedral quandle. Then R_3^3 is a rack with

$$(x_1, x_2, x_3) \triangleleft (y_1, y_2, y_3) = (((x_1 \triangleleft y_1) \triangleleft y_2) \triangleleft y_3, ((x_2 \triangleleft y_1) \triangleleft y_2) \triangleleft y_3, ((x_3 \triangleleft y_1) \triangleleft y_2) \triangleleft y_3).$$

Let $R_3^3 \times \mathbb{Z}_2$ be the associated MGR of R_3^3. Then we have $\#\mathrm{Col}_{R_3^3\times\mathbb{Z}_2}(D_1) = 144$ and $\#\mathrm{Col}_{R_3^3\times\mathbb{Z}_2}(D_2) = 72$. Therefore, F_1 and F_2 are not equivalent, that is, they are not reversible.

Example 7.2. Let F_1 and F_2 be the oriented spatial surfaces illustrated in Figure 12. We note that F_1 and F_2 are Seifert surfaces of the same oriented knot. Let D_1 and D_2 be the Y-oriented diagrams of F_1 and F_2 illustrated in the figure, respectively. Let $X := \mathbb{Z}_3[t^{\pm 1}]/(t^2 + t + 1)$ be the (t, s)-rack ($s = t + 2$), and let $X \times \mathbb{Z}_3$ be the associated MGR of X. Then we have $\#\mathrm{Col}_{X\times\mathbb{Z}_3}(D_1) = 45$ and $\#\mathrm{Col}_{X\times\mathbb{Z}_3}(D_2) = 33$. Therefore, F_1 and F_2 are not equivalent.

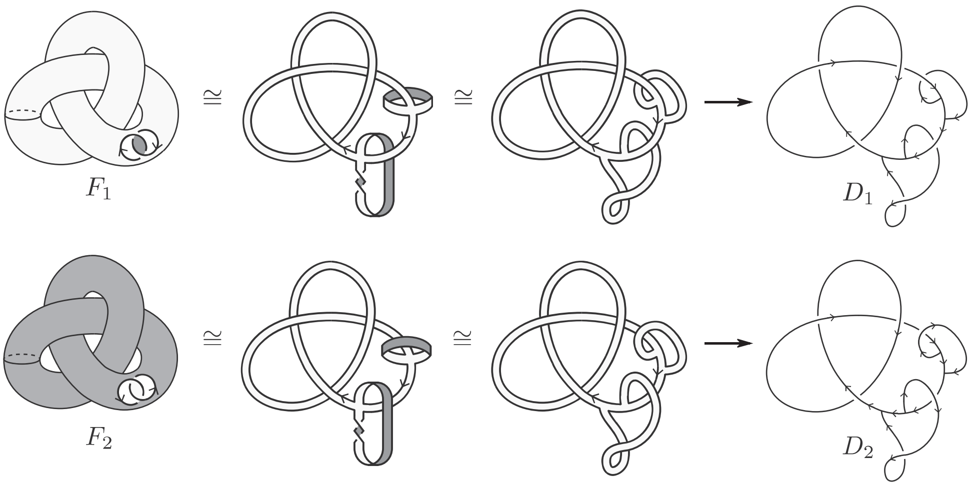

Figure 11: Oriented spatial surfaces F_1, F_2 and their diagrams D_1, D_2.

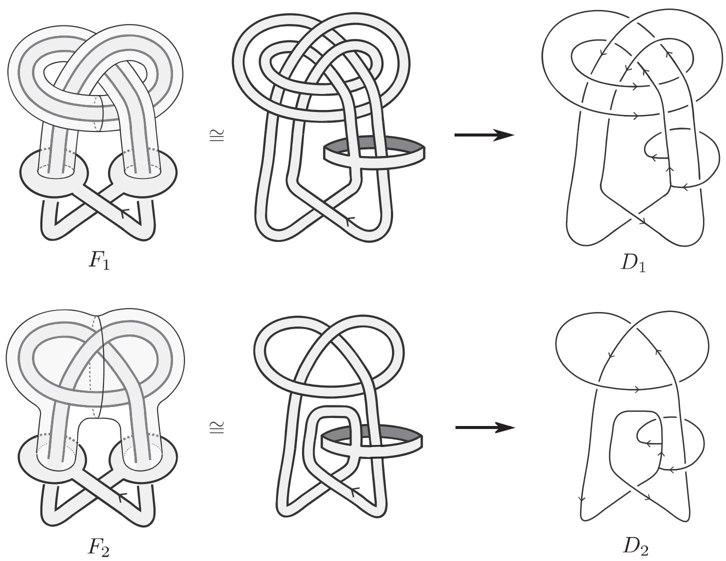

Figure 12: Two Seifert surfaces F_1, F_2 of the same oriented knot and their diagrams D_1, D_2.

Example 7.3. Let $\widetilde{F_1}$ and $\widetilde{F_2}$ be the oriented spatial closed surfaces illustrated in Figure 13, where we note that they do not bound handlebodies in S^3. Let F_1 and F_2 be the oriented spatial surfaces obtained by removing a disk from $\widetilde{F_1}$ and $\widetilde{F_2}$ as shown in the figure, respectively. By [13], $\widetilde{F_1}$ and $\widetilde{F_2}$ are equivalent if and only if F_1 and F_2 are equivalent. Let D_1 and D_2 be Y-oriented diagrams of F_1 and F_2, respectively. Let R_3 be the dihedral quandle, and let $R_3 \times \mathbb{Z}_2$ be the associated MGR of R_3. Then we have $\#\mathrm{Col}_{R_3\times\mathbb{Z}_2}(D_1) = 96$ and $\#\mathrm{Col}_{R_3\times\mathbb{Z}_2}(D_2) = 144$. Therefore, F_1 and F_2 are not equivalent, that is, $\widetilde{F_1}$ and $\widetilde{F_2}$ are not equivalent.

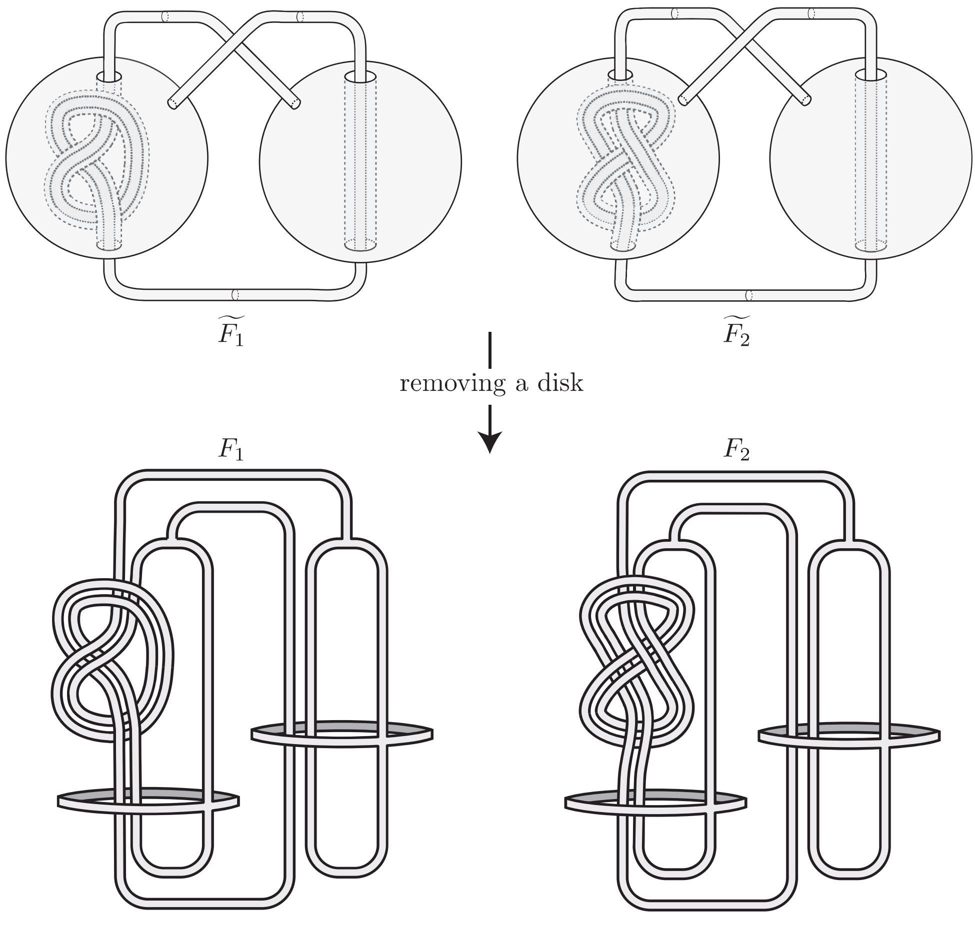

Figure 13: Two oriented spatial closed surfaces $\widetilde{F_1}$ and $\widetilde{F_2}$.

Proposition 7.4. *Let X be an MGR. Let F be an oriented spatial surface and let D and $-D^*$ be Y-oriented diagrams of F and $-F^*$, respectively. Then there is a one-to-one correspondence between* $\mathrm{Col}_X(D)$ *and* $\mathrm{Col}_X(-D^*)$.

By the above proposition, We can not distinguish an oriented spatial surface and its mirror image of its reverse by using MGR coloring numbers.

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