Degenerations of skein algebras and pants decomposition

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1 Introduction

This report is a rough explanation of the paper [KL] with Thang T. Q. Lê.

The (Kauffman bracket) skein algebras of surfaces were introduced by Przytycki [Pr2] and Turaev [Tu] independently. These have many applications and connections to interesting and important objects, e.g. topological quantum field theory [BHMV] and character varieties [Bu], [BFK], [PS1], [Tu]. Clarifying the structures of skein algebras helps us understand such applications and connections more. One way to clarify is to construct embeddings of skein algebras into well-known algebras.

For a surface with an ideal triangulation, Bonahon and Wong [BW] showed that the skein algebra of the surface can be embedded into a quantum torus, where a quantum torus is a non-commutative algebra with nice properties, e.g. it is a Noetherian domain. From their result, we would like to know whether a similar claim for closed surfaces holds. Note that it is known that the skein algebra of the torus can be embedded into a quantum torus by Frohman and Gelca [FG]. Hence, we would like to consider the the following problem:

Problem 1. *For the closed surface Σ_g of genus $g \geq 2$, construct an embedding of $\mathscr{S}(\Sigma_g)$ into a quantum torus.*

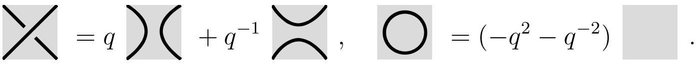
While the question is natural and simple, the problem is still open since Bonahon–Wong's proof uses an ideal triangulation essentially and any closed surfaces have no ideal triangulations. As a joint work with Thang T. Q. Lê, we approached this problem. In this article, we introduce one of our theorems; the associated graded algebra of the skein algebra of a closed surface with respect to a certain filtration can be embedded into a quantum torus.

2 Notations

Throughout of the report, suppose $g \geq 2$, let Σ_g be the closed surface of genus g and let $\mathbb{N}$ and $\mathbb{Z}$ be the set of non-negative integers and the set of integers respectively.

3 Skein module/algebra

Let $\mathcal{R}$ be a commutative domain with an identity and a distinguished invertible element $q^{1/2}$ and let M be an oriented 3-manifold. The (Kauffman bracket) *skein module* $\mathscr{S}(M)$ of M, introduced by Przytycki [Pr2] and Turaev [Tu] independently, is the $\mathcal{R}$-module spanned by all the isotopy classes of framed unoriented links in M subject to the following two relations, where, in each relation, the framed links are identical except where shown:



For an oriented surface Σ, consider the case of $\Sigma \times (0,1)$. Then, a product $\alpha_1\alpha_2$ of two framed links α_1 and α_2 in $\Sigma \times (0,1)$ is defined by stacking, i.e. we rescale α_1 and α_2, and then regard $\alpha_1 \subset \Sigma \times (1/2, 1)$ and $\alpha_2 \subset \Sigma \times (0, 1/2)$. With this product, $\mathscr{S}(\Sigma \times (0,1))$ has an algebraic structure. We call the algebra the *skein algebra* of Σ and denote it by $\mathscr{S}(\Sigma)$.

We often consider framed links in $\Sigma_g \times (0,1)$ as their diagrams. Fix a framed link L in $\Sigma \times (0,1)$. We isotope a framed link so that the framing are vertical, i.e. the framing at each point is parallel to the $(0,1)$-factor pointing to 1. Then, we project the framed link with a vertical framing on $\Sigma \times \{1/2\}$ and give over/under information at each double point. The result is called a farmed link diagram of L. It is known that any two framed link diagrams of L are related by a finite sequence of framed Reidemeister moves and isotopy on Σ.

A *multicurve* on Σ is a disjoint union of simple closed curves on Σ. Note that, by assigning a vertical framing to a multicurve, we obtain a framed link diagram. In the sense, we regard a multicurve as an element of $\mathscr{S}(\Sigma)$.

A multicurve on Σ is *simple* if it has no component which bounds an embedded disk in Σ. It is known that the set of the isotopy classes of simple multicurves on Σ is a basis of $\mathscr{S}(\Sigma)$ as an $\mathcal{R}$-module by Przytycki [Pr1]. For the closed surface Σ_g, let B denote the set of the isotopy classes of simple multicurves on Σ_g.

4 Pants decomposition and dual graph

In this section, we review a pants decomposition of the closed surface Σ_g of genus $g \geq 2$.

First, we review the definition of a pair of pants. A *pair of pants* is a surface diffeomorphic to S^2 minus small open neighborhoods of distinct three points. For later convenience, we give an alternative definition. Consider two oriented hexagons whose edges are labeled as in Figure 1. A pair of pants is a surface diffeomorphic to the result obtained from the

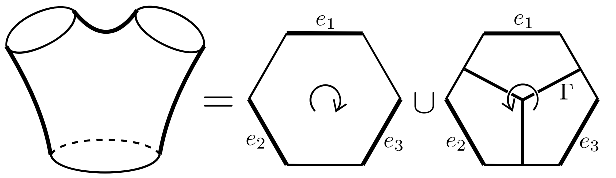


Fig.1: A pair of pants obtained from two hexagons

hexagons by gluing the alternating edges e_i $(i = 1, 2, 3)$ so that the orientations of the hexagons are compatible.

Consider the graph Γ in Figure 1, a uni-trivalent graph with only one trivalent. A *dual graph* of a pair of pants is the image of Γ in the pair of pants, also denoted by Γ.

Consider finitely many disjoint simple closed curves on Σ_g such that any two of them are not homotopic and each connected component of the result obtained from Σ_g by removing small open neighborhood of the curves is a pair of pants. Then, there are $3g-3$ numbers of simple closed curves and $2g-2$ pairs of pants. Let $\mathcal{C} = \{C_i\}_{i=1}^{3g-3}$ be the set of such simple closed curves. We call $\mathcal{C}$ a *pants decomposition* of Σ_g.

Fix a pants decomposition $\mathcal{C}$ of Σ_g. Then, we have pairs of pants as above. A *dual graph* of Σ_g with respect to $\mathcal{C}$ is an embedded trivalent graph Γ in Σ_g such that the intersection of Γ and each pair of pants is a dual graph of the pair of pants.

5 Dehn–Thurston coordinate and modification

In this section, we review the Dehn–Thurston coordinate following [PH].

Consider a pairs of pants P with a dual graph Γ. Let C_1, C_2, C_3 be the connected boundary components of P in clockwise with respect to Γ. A *standard arc* on P is an embedded arc on $P \setminus (\partial P \cap \Gamma)$ satisfying either of the following conditions:

• it connects different boundary components of P and it does not intersect with Γ,

• both of its vertices are on C_i and it intersects with Γ only once on the edge of Γ attached to C_{i+1}, where $C_4 = C_1$. In particular, we call the standard arc whose vertices are on C_i the *returning arc* with respect to C_i. In Figure 2, there are examples of standard arcs.

Fix a pants decomposition $\mathcal{C} = \{C_i\}_{i=1}^{3g-3}$ of Σ_g and a dual graph Γ with respect to $\mathcal{C}$.

Recall that B is the set of the isotopy classes of simple multicurves on Σ_g. A representative α' of $\alpha \in B$ is *good* if α' satisfies the following three conditions:

(i) $\alpha' \cap C_i \cap \Gamma = \emptyset$,

(ii) $|\alpha' \cap C_i| = n_i(\alpha)$ for any i,

(iii) each component of the intersection of α' and $\Sigma_g \setminus \sqcup_{i=1}^{3g-3} N(C_i)$ is a standard arc,

where $n_i(\alpha)$ is the geometric intersection number of α and C_i, i.e. $n_i(\alpha) =$

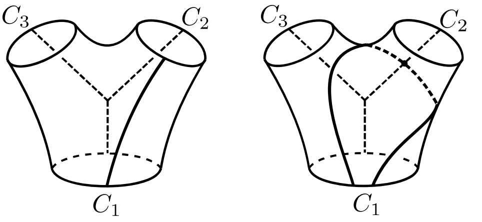


Fig.2: Left: a standard arc connecting C_1 and C_2
Right: a standard arc whose vertices are on C_1

$min\{\#(\bar{\alpha} \cap C_i) | \bar{\alpha}$ is a representative of $\alpha\}$. When we emphasize the small open neighborhoods $N(C_i)$, we call α' a good representative with respect to $N(C_i)$ $(i = 1, \dots, 3g-3)$.

To define twisting number $t_i(\alpha)$ of $\alpha \in B$, we fix a good representative α' of α with respect to small open tubular neighborhoods $N(C_i)$ $(i = 1, \dots, 3g-3)$. Let $\overline{N(C_i)}$ be the closure of $N(C_i)$. Let m_i be the minimum of $\#(\overline{\alpha} \cap \overline{N(C_i)} \cap \Gamma)$ over all embedded curves $\overline{\alpha}$ in $\overline{N(C_i)}$ isotopic to $\alpha' \cap \overline{N(C_i)}$ fixing $\partial\overline{N(C_i)}$. Note that $\alpha' \cap \overline{N(C_i)}$ is isotopic to right-hand twists, left-hand twists or parallel copies of C_i fixing $\partial\overline{N(C_i)}$. Then, we have the following well-defined value:

$$t_i(\alpha) = \begin{cases} m_i & \text{If } \alpha' \cap \overline{N(C_i)} \text{ is isotopic to right-hand twists or parallel copies of } C_i, \\ -m_i & \text{if } \alpha' \cap \overline{N(C_i)} \text{ is isotopic to left-hand twists.} \end{cases}$$

For $\alpha \in B$, the coordinate $(n_1(\alpha), \dots, n_{3g-3}(\alpha), t_1(\alpha), \dots, t_{3g-3}(\alpha))$ is called the *Dehn–Thurston coordinate* of α with respect to the pants decomposition $\mathcal{C} = \{C_i\}_{i=1}^{3g-3}$.

Note that the Dehn–Thurston coordinate is not compatible with the multiplication of $\mathscr{S}(\Sigma_g)$; see e.g. [PS2]. To construct an algebra embedding, we will modify the Dehn–Thurston coordinate, especially twisting numbers $t_i(\alpha)$.

Recall that, for each $C_i \in \{C_j\}_{j=1}^{3g-3}$, there are (possibly the same) two pairs of pants, say P_j and P_k, containing at least one of the connected components of $\partial\overline{N(C_i)}$. Let $P_{\ell 1}, P_{\ell 2}, P_{\ell 3}$ be the connected boundary components of P_ℓ $(\ell = j, k)$ and $P_{\ell 1}, P_{\ell 2}, P_{\ell 3}$ are in clockwise with respect to $\Gamma \cap P_\ell$. Suppose that $\partial\overline{N(C_i)} = P_{j1} \sqcup P_{k1}$ Then, modified twisting number $s_i(\alpha)$ is defined by

$$\begin{aligned} s_i(\alpha) := t_i(\alpha) &+ \#(\text{returning arcs in } P_j \text{ with respect to } P_{j2}) \\ &+ \#(\text{returning arcs in } P_k \text{ with respect to } P_{k2}). \end{aligned}$$

We call the coordinate $(n_1(\alpha), \dots, n_{3g-3}(\alpha), s_1(\alpha), \dots, s_{3g-3}(\alpha))$ the *modified Dehn–Thurston coordinate* of $\alpha \in B$.

Consider the map $B \to \mathbb{N}^{3g-3} \times \mathbb{Z}^{3g-3}$ which maps a simple multicurve to its modified

Dehn–Thurston coordinate. Let Λ denote the image of the map. It is known that the map is injective, i.e. B and Λ are in one-to-one correspondence.

6 Filtration and associated graded algebra

Let $(n_1(\alpha), \ldots, n_{3g-3}(\alpha), s_1(\alpha), \ldots, s_{3g-3}(\alpha))$ denote the modified Dehn–Thurston coordinate of $\alpha \in B$. For $\mathbf{m} \in \mathbb{N} \times \mathbb{Z}$, let $F_{\le \mathbf{m}}$ be the $\mathcal{R}$-submodule of the skein algebra $\mathscr{S}(\Sigma_g)$ spanned by

$$\{\alpha \in B \mid (\sum_{i=1}^{3g-3} n_i(\alpha), \sum_{i=1}^{3g-3} s_i(\alpha)) \le \mathbf{m}\}, \tag{1}$$

where the inequality is defined by the lexicographic order on $\mathbb{N} \times \mathbb{Z}$. Then, $\{F_{\le \mathbf{m}}\}_{\mathbf{m} \in \mathbb{N} \times \mathbb{Z}}$ forms a filtration of $\mathscr{S}(\Sigma_g)$.

Consider the associated graded algebra

$$\operatorname{Gr} \mathscr{S}(\Sigma_g) := \bigoplus_{\mathbf{m} \in \mathbb{N} \times \mathbb{Z}} F_{\le \mathbf{m}} / F_{< \mathbf{m}},$$

where $F_{<\mathbf{m}}$ is the $\mathcal{R}$-submodule of $\mathscr{S}(\Sigma_g)$ spanned by the set obtained from (1) by replacing $\le$ with $<$. We embed $\operatorname{Gr} \mathscr{S}(\Sigma_g)$ into a quantum torus later.

7 Quantum torus

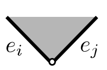
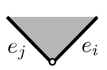
For an anti-symmetric $r \times r$ integer matrix $Q = (Q_{ij})$, we have the non-commutative algebra

$$\mathbb{T}(Q) = \mathbb{T}(Q; \mathcal{R}) = \mathcal{R}\langle x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_r^{\pm 1} \rangle / (x_i x_j = q^{Q_{ij}} x_j x_i),$$

called the quantum torus associated to Q. Note that quantum tori are Noetherian domains. We often denote a monomial $x_1^{n_1} \ldots x_r^{n_r} \in \mathbb{T}(Q)$ by $x^{\mathbf{n}}$ with $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$.

For a submonoid $\Lambda \subset \mathbb{Z}^r$, then the $\mathcal{R}$-submodule $A(Q; \Lambda) \subset T(Q)$ spanned by $\{x^{\mathbf{r}} \mid \mathbf{r} \in \Lambda\}$ forms an $\mathcal{R}$-subalgebra of $T(Q)$, called the Λ-*monomial subalgebra* of $T(Q)$.

For a dual graph Γ with respect to a pants decomposition $\mathcal{C}$ of Σ_g, consider an open tubular neighborhood $N(\Gamma)$ of Γ in Σ_g. Then, $N(\Gamma)$ and $\mathcal{C}$ give an ideal triangulation of $N(\Gamma)$ whose edges are $N(\Gamma) \cap \mathcal{C}$. For the ideal triangulation, consider an anti-symmetric $r \times r$-matrix $Q = (Q_{ij})$ defined by

$Q_{ij} = \#$  $- \#$  (2)

for any $i \neq j \in \{1, 2, \ldots, r\}$ and $Q_{ii} = 0$ for any $i \in \{1, 2, \ldots, r\}$, where $r = |\mathcal{C}|$ and there are no extra edges incident to the ideal vertex between e_i and e_j. Then, we have the quantum torus $\mathbb{T}(Q)$ as the above.

For the anti-symmetric integer $2r \times 2r$ matrix

$$\bar{Q} = \begin{pmatrix} Q & -2I_r \\ 2I_r & O \end{pmatrix},$$

consider the quantum torus $\mathbb{T}(\bar{Q})$ containing $\mathbb{T}(Q)$, where I_r and O are the identity matrix of size r and the zero matrix of size r respectively. This is the target space of our algebra embedding.

8 Result

The following is one of our theorems.

Theorem 8.1 (Karuo–Le [KL]). *Let $\mathcal{R}$ be a commutative domain with a distinguished invertible element q and Σ_g be the closed surface of genus $g \geq 2$. Let $\mathrm{Gr}\,\mathscr{S}(\Sigma_g)$ be the associated graded algebra of the skein algebra $\mathscr{S}(\Sigma_g)$ with respect to the filtration $\{F_{\leq \mathbf{m}}\}_{\mathbf{m} \in \mathbb{N} \times \mathbb{Z}}$ defined in Section 6. Then, $\mathrm{Gr}\,\mathscr{S}(\Sigma_g)$ is isomorphic to the Λ-monomial subalgebra $A(\bar{Q};\Lambda)$, where $A(\bar{Q};\Lambda)$ is defined in Section 7.*

The theorem implies that, since $A(\bar{Q};\Lambda) \subset \mathbb{T}(\bar{Q})$, $\mathrm{Gr}\,\mathscr{S}(\Sigma_g)$ can be embedded into the quantum torus $\mathbb{T}(\bar{Q})$.

9 Acknowledgement

The author was supported by JSPS KAKENHI Grant Number JP20J10108.

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